THE ZERO PARADOX

ZP-F: The Counterexamples

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This document is self-contained within ordered field theory. No p-adic topology, no information theory, and no Hilbert space machinery is required. All results follow from the axioms of a linearly ordered field alone.

Illustrated Companion: A paired ZP-F Illustrated Companion provides plain-language explanation and motivation for the results here. Examples and intuitions are kept separate from the formal layers to distinguish illustrative material from proofs.

I. Purpose and Scope

The Zero Paradox framework separates a derived SHAPE from a committed first step. The shape, the Binary Snap, is T-SNAP: in any ZPSemilattice (ZP-A's join-semilattice with least element, axioms A1–A4), a run that starts at the semilattice's ⊥ and whose first transition goes to a different state (S 1 ≠ S 0) meets that shape (t_snap_given, with both as hypotheses). What is derived is that shape, not that the transition is taken: a join-semilattice in which T-SNAP holds and nothing moves is exhibited in the Lean source (tsnap_holds_but_nothing_moves). A first step above the carrier's ⊥, an element covering it (HasFirstStep), is AX-B1's form, and it is committed, not derived. This document establishes the negative result for that commitment: in no linearly ordered field does 0 have a first step above it, AX-B1's form. T-SNAP's shape does not need a first step: t_snap_given holds at a step taken in a densely ordered join-semilattice ($\mathbb{Q}_{\geq 0}$ under max, whose ⊥ is 0; ZeroParadox/Order/Snap.lean). The real numbers $\mathbb{R}$ are the canonical and most familiar instance; the theorem applies to any field carrying a compatible linear order.

The blocking mechanism is elementary: in any such field, for any positive element ε, the element ε/2 exists, is positive, and is strictly smaller. There is no least positive element, so 0 has no first step above it, AX-B1's form, in any linearly ordered field. That a run's first transition is a cover is a hypothesis beyond T-SNAP's shape (t_snap_given_cover), and the shape itself (t_snap_given) does not need it.

This result contextualises the choice of Q_2 in ZP-B. Among all completions of $\mathbb{Q}$, Ostrowski's theorem identifies exactly two kinds: Archimedean completions ($\mathbb{R}$, a densely ordered field, so no element has a first step) and non-Archimedean completions (Q_p, where $v_p(0) = +\infty$ while every nonzero element has a finite valuation). Q_2 is the non-Archimedean completion at $p = 2$, the prime fixed by ZP-B Theorem T0 from binary existence (AX-B1) together with the minimality principle MP-1, a design commitment (T0 in Lean: t0_mp1_faithful_nonredundant and t0_mp1_least_prime_unique in ZeroParadox/Valuation/Padic.lean take MP-1 as hypotheses, in two readings). In both kinds zero is a metric limit of nonzero elements (for Q_2, floor_reach_separates_mu_nu), so being a limit point does not separate them. The topology does: Q_2 is totally disconnected (ZP-B, t5_totallyDisconnected), while $\mathbb{R}$ is connected (place_topological_signature), so not totally disconnected (real_no_snap). The order separates them globally, not at zero: at 0 neither has a first step ($\mathbb{R}$ in its field order, Q_2 in its norm preorder), and globally $\mathbb{R}$ is densely ordered (axb1_fails_everywhere_iff_dense), in its norm preorder as in its field order, while Q_2's norm preorder is not dense (the § ROLE examples in ZeroParadox/BottomCannotBe.lean). That zero has valuation $+\infty$ is set by definition, not produced by the completion: Mathlib's Padic.addValuationDef assigns ⊤ to 0 by an explicit case. What holds in Q_2 is that only 0 has infinite valuation (padic_addVal_eq_top_iff).

II. The General Result

The following results hold for any type F carrying a field structure, a linear order, and the IsStrictOrderedRing property — the standard Mathlib replacement for the deprecated LinearOrderedField typeclass (deprecated October 2025). All proofs are machine-verified in Lean 4.

Theorem F-DENSITY

$\forall\, \varepsilon : F,\ 0 < \varepsilon \to 0 < \varepsilon/2 \wedge \varepsilon/2 < \varepsilon$

For any positive element ε in a linearly ordered field F, the element ε/2 is also positive and strictly smaller than ε.

Proof: half_pos gives $0 < \varepsilon/2$; half_lt_self gives $\varepsilon/2 < \varepsilon$. Both follow from the ordered field axioms. Status: DERIVED. ✓

Theorem F-NO-MIN

$\neg\exists\, \varepsilon : F,\ 0 < \varepsilon \wedge \forall\, \delta : F,\ 0 < \delta \to \varepsilon \leq \delta$

No linearly ordered field has a minimal positive element.

Proof: Suppose such ε exists. By F-DENSITY, $0 < \varepsilon/2 < \varepsilon$. Then $\varepsilon \leq \varepsilon/2$ (by minimality) contradicts $\varepsilon/2 < \varepsilon$. Status: DERIVED. ✓

Theorem F-SNAP-BLOCKED

$\forall\, \varepsilon_0 : F,\ 0 < \varepsilon_0 \to \exists\, \delta : F,\ 0 < \delta \wedge \delta < \varepsilon_0$

Any candidate first step $\varepsilon_0 > 0$ from 0 in a linearly ordered field is blocked: $\varepsilon_0/2$ is a smaller positive element, so ε_0 cannot be a genuine first step.

Proof: Take $\delta = \varepsilon_0/2$. F-DENSITY gives $0 < \delta < \varepsilon_0$. Status: DERIVED. ✓

Theorem F-SNAP-IMPOSSIBLE

$\neg\exists\, \varepsilon_0 : F,\ 0 < \varepsilon_0 \wedge \neg\exists\, \delta : F,\ 0 < \delta \wedge \delta < \varepsilon_0$

F-SNAP-IMPOSSIBLE (f_snap_impossible): no linearly ordered field has a least positive element, so 0 has no first step above it (AX-B1's form). T-SNAP's shape does not need a first step (t_snap_given holds at a step taken in a densely ordered join-semilattice, $\mathbb{Q}_{\geq 0}$ under max, ZeroParadox/Order/Snap.lean).

A first step is a minimal first departure from 0, with nothing strictly between it and 0 (a cover, HasFirstStep). F-SNAP-BLOCKED shows such a step never exists in a linearly ordered field.

Proof: Direct from F-SNAP-BLOCKED. Status: DERIVED. ✓

III. The Real Numbers — Canonical Instance

The following are corollaries of the general results above, instantiated at $F = \mathbb{R}$. They are stated explicitly because $\mathbb{R}$ is the most familiar and intuitive case, and because the ZP-F companion document is written

around $\mathbb{R}$ as its primary example.

Corollary R-DENSITY

$\forall\, \varepsilon : \mathbb{R},\ 0 < \varepsilon \to 0 < \varepsilon/2 \wedge \varepsilon/2 < \varepsilon$

Density at zero for $\mathbb{R}$. Follows from F-DENSITY at $F = \mathbb{R}$. Status: DERIVED (corollary). ✓

Corollary R-NO-MIN

$\neg\exists\, \varepsilon : \mathbb{R},\ 0 < \varepsilon \wedge \forall\, \delta : \mathbb{R},\ 0 < \delta \to \varepsilon \leq \delta$

$\mathbb{R}$ has no minimal positive element. Follows from F-NO-MIN at $F = \mathbb{R}$. Status: DERIVED (corollary). ✓

Corollary R-SNAP-BLOCKED

$\forall\, \varepsilon_0 : \mathbb{R},\ 0 < \varepsilon_0 \to \exists\, \delta : \mathbb{R},\ 0 < \delta \wedge \delta < \varepsilon_0$

Any candidate first step in $\mathbb{R}$ can be halved. Follows from F-SNAP-BLOCKED at $F = \mathbb{R}$. Status: DERIVED (corollary). ✓

Corollary R-SNAP-IMPOSSIBLE

$\neg\exists\, \varepsilon_0 : \mathbb{R},\ 0 < \varepsilon_0 \wedge \neg\exists\, \delta : \mathbb{R},\ 0 < \delta \wedge \delta < \varepsilon_0$

R-SNAP-IMPOSSIBLE (r_snap_impossible): $\mathbb{R}$ has no least positive element, so 0 has no first step above it in $\mathbb{R}$ (AX-B1's form). Follows from F-SNAP-IMPOSSIBLE at $F = \mathbb{R}$. Status: DERIVED (corollary). ✓

IV. Axiom Profile

All eight results (F-DENSITY, F-NO-MIN, F-SNAP-BLOCKED, F-SNAP-IMPOSSIBLE and their $\mathbb{R}$ corollaries) verify under #print axioms with the profile: propext, Classical.choice, Quot.sound. This is the standard profile for Mathlib field and order results. No snap-specific axioms are introduced.

V. Relationship to the Framework

ZP-F is self-contained and does not feed formally into any other layer. It is the negative complement to ZP-B: where ZP-B establishes that Q_2 removes the topological obstruction to an irreversible step (c3_irreversible: no continuous path in Q_2 from a nonzero point to 0, since its 2-adic topology is totally disconnected, t5_totallyDisconnected, with clopen balls; a case of Mathlib's TotallyDisconnectedSpace.eq_of_continuous: in Q_2 no continuous path joins any two distinct points, so the step out from 0 is blocked as well), ZP-F establishes that in $\mathbb{R}$ — and any linearly ordered field — 0 has no first step above it, AX-B1's form. Neither supplies the first step AX-B1 commits to, a cover of $\bot$ on the state carrier, the set of states a run S moves through, ordered as in ZP-A (HasFirstStep). t_snap_given_cover assumes the run's first transition is a cover (hcov); with CC-1, the commitment that the run starts at $\bot$ ($S\ 0 = \bot$), that transition is a cover of $\bot$. MachinePhase meets the commitment: its state c_1 covers its $\bot$, c_0.

The dependency order of the framework places ZP-F as independent:

Layer	Role	Dependency
ZP-B	p-Adic topology — Q_2 removes the topological obstruction to an irreversible step (c3_irreversible) and supplies no first step	ZP-A
ZP-F	Counterexample — in $\mathbb{R}$ and ordered fields 0 has no first step above it (AX-B1's form)	None (self-contained)
ZP-G	Category theory	Self-contained; ZP-E conceptually

VI. A Philosophical Note: The Wrong Setting

The real numbers are the most natural and familiar number line precisely because they fill every gap. Density — the property established in F-DENSITY: for any $\varepsilon > 0$ there exists δ with $0 < \delta < \varepsilon$ — is not a deficiency in $\mathbb{R}$; it is one of $\mathbb{R}$'s defining virtues. The counterexample above shows that this virtue is exactly what rules out a first step above 0 in $\mathbb{R}$, AX-B1's form; T-SNAP's shape does not need one (t_snap_given). The exclusion is not a matter of degree: density follows from the ordered field axioms, so every linearly ordered field is ruled out by exactly the same argument. Familiarity here misleads: the setting that feels most natural for foundational mathematics is precisely the wrong one.

The exclusion of $\mathbb{R}$ is not an isolated fact about one number system. F-SNAP-IMPOSSIBLE applies to any linearly ordered field. By Ostrowski's theorem, every completion of $\mathbb{Q}$ falls into exactly one of two classes: Archimedean (including $\mathbb{R}$) or non-Archimedean (the p-adic fields Q_p). The theorems of this document eliminate the entire Archimedean class. What remains is the non-Archimedean class, every Q_p, not Q_2 alone. Among these, ZP-B Theorem T0 fixes $p = 2$ from binary existence (AX-B1) together with the minimality principle MP-1, a design commitment. So Q_2 is not chosen against $\mathbb{R}$: ruling out $\mathbb{R}$ and every Archimedean completion leaves the p-adic fields, and T0, under MP-1, then selects Q_2 among them.

The structural difference can be stated directly. It has an order side and a topological side, and the order separates $\mathbb{R}$ from Q_2 globally, not at zero. AX-B1 commits the point the snap leaves to having a first step: an element directly above it with nothing strictly between, a cover (HasFirstStep). A cover of a least element is what order theory calls an atom (Mathlib bot_covBy_iff). An ordered field has no bottom element; its 0, which plays that role here, has no first step (axb1_fails_in_ordered_field). That is one instance of a stronger fact: an order is dense if and only if no element at all has a first step (axb1_fails_everywhere_iff_dense, the corpus restatement of Mathlib's denselyOrdered_iff_forall_not_covBy). The equivalence quantifies over every element, so density rules out a first step at 0, and one element without a first step does not make an order dense. In the metric chart zero is a limit point of the nonzero elements, in $\mathbb{R}$ and equally in Q_2. Q_2's zero is a floor of the limit kind: the doubling orbit of a nonzero element converges to 0 and does not reach it in finite time, while counting down in $\mathbb{N}$ reaches 0 in finitely many steps (floor_reach_separates_mu_nu). In the valuation chart zero sits at the other end: $v(0) = +\infty$, the greatest value, and no nonzero element shares it (padic_addVal_eq_top_iff). Neither fact is a first step. No order instance on Q_2 synthesizes in Mathlib (LT $\mathbb{Q}_{[2]}$ does not). Take instead the norm preorder of the § ROLE examples in ZeroParadox/BottomCannotBe.lean (Preorder.lift of the norm). At zero the order does not separate the two: 0 is least and has no first step, as in $\mathbb{R}$'s field order, since between 0 and any nonzero a lies an element of strictly smaller nonzero norm; that holds in every nontrivially normed field, $\mathbb{R}$'s norm preorder included (the § ROLE example stating it for every such field). Globally the order does: Q_2's norm preorder is not dense, since every nonzero norm is an integer power of 2 (Padic.padicNormE.image) and so 2 is covered by

1, its norms being 1/2 and 1 (the § ROLE example stating that cover), while ℝ is densely ordered in its field order and in its norm preorder alike (the § ROLE example stating both). In the standard terms this is a discrete group of norm values against a dense one; Mathlib's name for the dense side is DenselyNormedField, ℝ is an instance (Real.denselyNormedField), and its density lemma is NormedField.exists_lt_norm_lt. In the valuation order 0 is greatest, v(0) = ⊤ (addVal_bot). The first step AX-B1 commits to is a cover of ⊥ on the state carrier, in ZP-A's induced order (HasFirstStep), a commitment. t_snap_given_cover (ZeroParadox/Order/Snap.lean) assumes the run's first transition is a cover (hcov); with CC-1, the run's start at ⊥, that transition is a cover of ⊥. The converse fails: a run can start at a ⊥ that has a cover while its own first transition is not one (ZeroParadox/Order/SnapCannotBe.lean). MachinePhase meets the commitment (the MachinePhase example in ZeroParadox/Order/Snap.lean stating that c_1 covers its ⊥). In neither ordering of Q_2 does its 0 have a first step: in the norm preorder it has none, and in the valuation order nothing lies above it. The first step is not missing in ℝ by accident. It is missing because ℝ is densely ordered, and in a dense order no element has a first step, 0 included (axb1_fails_everywhere_iff_dense; for 0 in any linearly ordered field, f_snap_impossible). The topological difference from Q_2 is that ℝ is connected (place_topological_signature), so not totally disconnected (real_no_snap). That is not a separate fence against a first step: for a linear order carrying its order topology, connectedness implies density (Mathlib denselyOrdered_of_preconnectedSpace).

Remark: The Dual Limit Condition

The results of §II–§III and ZP-L together admit an informal structural account of why a first step above 0, AX-B1's form, is domain-specific.

In any dense ordered field, 0 has no nearest neighbour from above: for any $\varepsilon > 0$ there exists δ with $0 < \delta < \varepsilon$ (F-DENSITY). Zero is a topological limit point from above with no minimum positive element. AX-B1 commits the point the snap leaves to a first step (HasFirstStep); density forbids a first step at every element, 0 included (axb1_fails_everywhere_iff_dense), so the block is global, not located at 0. This is the blocking condition established in this document.

ε_0 of Ordinal (the snap threshold in the ordinal chart, ZP-L) is the least fixed point of $\alpha \mapsto \omega^\alpha$: it is not reachable by any finite iteration of the $\omega^{(\cdot)}$ operation, only by the limit of the tower $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ It has no predecessor in the ordinal hierarchy: no ordinal α satisfies $\alpha + 1 = \varepsilon_0$.

Both cases are limit-type conditions: the field case is a topological limit point with no minimum positive neighbour above 0; the ordinal case is a limit ordinal (specifically, the least fixed point of $\alpha \mapsto \omega^\alpha$) with no predecessor below ε_0. The analogy is structural — both describe a boundary approached but not reachable by finite steps on one side — but it is not a proved identification. The two structures live in different mathematical categories.

In the ordinal chart the snap meets two conditions at once, at opposite ends of the crossing: (1) below the landing, the boundary is a limit with no immediate predecessor (as ε_0 is approached by the tower and no α has $\alpha + 1 = \varepsilon_0$ — what makes the threshold non-arbitrary) and (2) above the point left, the crossing is discrete: that point has a first step, a cover (HasFirstStep). Condition (1) is not required of every host: in the two-state carrier MachinePhase, where T-SNAP holds (t_snap_derived), the landing c_1 covers the bottom c_0 (the MachinePhase example in ZeroParadox/Order/Snap.lean stating that c_1 covers ⊥), so it has an immediate predecessor. Dense ordered fields fail condition (2) at every element (axb1_fails_everywhere_iff_dense). Their 0 also lacks an immediate predecessor, for the same reason, and without (2) that does not help. In the non-standard naturals *ℕ with the order topology, every nonzero element has an immediate predecessor, so there is no limit point at infinity in the order-topology sense; condition (1) fails informally. (This *ℕ observation is informal — not a proved result of this document or ZP-L.)

Remark: The Dual Limit Condition

The tower images cnfToZp2(towerNONote n) converge to 0 in $\mathbb{Z}_2$ as $n \to \infty$ (proved in ZP-L), and ε_0 is the minimal snap threshold in the ordinal setting (proved in ZP-L). cnf_bridge_type_boundary (ZeroParadox/Ordinal/CnfBridge.lean) states these as co-witnesses, and not as an identity, in three conjuncts: ε_0 is the least fixed point of $\alpha \mapsto \omega^\alpha$ at or above the bottom of the ordinals; the images cnfToZp2(towerNONote n) converge to the 2-adic 0; and each towerNONote n represents the n-th stage of the ω-tower (towerNONote_repr). cnfToZp2 maps the ω-tower into $\mathbb{Z}_2$. $\varepsilon_0 \neq 0$, with 0 the ordinal zero, is a theorem in the ordinals (epsilon0_ne_zero). An equation between ε_0 and the 2-adic 0 does not type-check, with no coercion between the ordinals and $\mathbb{Z}_2$ in either direction; a Lean guard in that file detects the failure (MC-1 / ZP-P). Under cnfToZp2 the 2-adic 0 is the image of the seed stage towerNONote 0, whose ordinal value is the bottom of the ordinals, and the limit of the stages' images; ε_0 has no image there, since every ordinal notation in the map's domain denotes an ordinal below ε_0 (repr_lt_epsilon0, ZeroParadox/Ordinal/PricedInterface.lean). Through another map ε_0 does land on that 0: the canonical threshold map sends ε_0 to c_1 and snapEmbed sends c_1 to 0 (snap_state_zp2_is_zero, ZP-M), a map value, not an identity (see ZP-L, The Bridge: What Is Formal and What Is Assumed).

This is an observation, not a proved theorem. The blocking result (F-SNAP-IMPOSSIBLE) is proved here; the threshold and convergence results are proved in ZP-L. The claim that both conditions reflect a common structural property is interpretive — it does not follow from their conjunction alone. The *ℕ example above is informal.

The dual-approach structure recurs across the ZP layers: in ZP-L, ε_0 is located from above as the least fixed point of $\alpha \mapsto \omega^\alpha$ and from below as the limit of the CNF tower; in ZP-M, ε_0 is sent through the threshold map and snapEmbed to the 0 of $\mathbb{Z}_2$ that the tower's 2-adic images tend to (zpm_triangle). Reading: this recurrence is not coincidental. The snap is the point where a domain's own measurement framework exhausts itself; no single framework can locate it from inside because the framework runs out exactly there. This suggests a dual approach is required in each case: each framework reaches the boundary from its own direction, and the snap is the point they agree on. No general theorem to this effect is established here.

The surreal numbers (No) are non-Archimedean and contain the limit ordinals, ε_0 among them, so they are the natural boundary test case for any duality theorem. No is a linearly ordered field, so 0 of No has no first step above it, AX-B1's form (ZP-L, Remark R-L.2), and No fails condition (2) at 0 as $\mathbb{R}$ does. (Informal — not a result of this document.)

Development path toward a formal theorem: prove the two conditions as independent Lean results (density impossibility established here; ε_0 a limit ordinal, the Order.IsSuccLimit example in ZeroParadox/Order/SnapCannotBe.lean), then prove they agree on the same structural point.

VII. Validation Status

Component	Status / Notes
F-DENSITY	DERIVED — Lean-verified; general case
F-NO-MIN	DERIVED — Lean-verified; general case
F-SNAP-BLOCKED	DERIVED — Lean-verified; general case
F-SNAP-IMPOSSIBLE	DERIVED — Lean-verified; general case
R-DENSITY	DERIVED — Corollary of F-DENSITY at F = $\mathbb{R}$

Component	Status / Notes
R-NO-MIN	DERIVED — Corollary of F-NO-MIN at $F = \mathbb{R}$
R-SNAP-BLOCKED	DERIVED — Corollary of F-SNAP-BLOCKED at $F = \mathbb{R}$
R-SNAP-IMPOSSIBLE	DERIVED — Corollary of F-SNAP-IMPOSSIBLE at $F = \mathbb{R}$