THE ZERO PARADOX

ZP-H: Categorical Bridge

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This document connects ZP-G (Category Theory) to ZP-A through ZP-E. Its function is exactly analogous to ZP-E's role in the original framework: ZP-E connected the four domain documents to each other; ZP-H connects the categorical generalization back to those four domain documents. Every cross-framework claim is traced to a theorem in ZP-G or ZP-A through ZP-E, plus an explicit bridge axiom where required. No floating connections.

ZP-H inherits all open items from ZP-G with their original labels. The four open questions from ZP-G — OQ-G1 through OQ-G4 — are the primary targets of this document. OQ-G2, OQ-G3, and OQ-G4 are resolved here. OQ-G1 is resolved in ZP-G via D7′ and I-KC. All four OQ-G items are now closed.

Sequencing note: ZP-H introduces one new definition (D-H1: the morphisms of C) that was deliberately omitted from ZP-G to keep that document domain-independent. This definition belongs here because it is instantiation-specific: the morphisms of C are defined by what the functor constructions require, not by the abstract category itself.

I. Imported Results

1.1 From ZP-G

Import Registry IR-G — Closed Results from ZP-G v1.1
Status: Valid — Received
D1: Category (objects, morphisms, composition, identity, associativity, unit laws)
D2: Morphism uniqueness notation (∃!)
D3: Initial object 0 — unique morphism $\iota_X: 0 \to X$ for all X
D4: Terminal object — defined for exclusion only
D5: Morphism chain from 0
D6: Functor — object map, morphism map, preservation of composition and identity
D7′: Native categorical surprisal $I(f) = K(x_B|x_A)$ via I-KC (replaces D7/BA-G1 from v1.0)
I-KC: Conditional Kolmogorov complexity K(x|y) imported from ZP-C — named dependency, not bridge axiom
AX-G1: Asymmetry Axiom — C has initial object 0, no terminal object
AX-G2: Source Asymmetry — hom(X, 0) = ∅ for X ≠ 0

Import Registry IR-G — Closed Results from ZP-G v1.1
R-BA: Leinster categorical entropy characterization [Compatibility Remark — BA-G1 demoted from Bridge Axiom in ZP-G v1.1. No longer a theorem premise.]
T1: Initial object unique up to unique isomorphism
T2: Universal Constituent — $\forall X \in ob(C), \exists! \iota_X: 0 \to X$
T3: Unreachability — $hom(X, 0) = \emptyset$ for $X \neq 0$
T4: Chains are forward-only — no chain returns to 0
T5: Functors preserve initial objects [OQ-G2 closed in ZP-H T-H1]
T6: Informational singularity of 0 — outward surprisal accumulates; inward undefined [rebuilt in ZP-G v1.1 on D7′; BA-G1 not a dependency]
T7: Categorical Zero Paradox — closing theorem of ZP-G
OQ-G1: Closed in ZP-G v1.1 via D7′ (native categorical surprisal) and I-KC (Kolmogorov import from ZP-C). BA-G1 demoted to compatibility remark.

1.2 From ZP-A through ZP-E

Import Registry IR-ZP — Closed Results from ZP-A through ZP-E
Status: Valid — Received
From ZP-A: Join-semilattice $(L, \vee, \bot)$. $\bot$ is the unique additive identity. $\bot \leq x$ for all $x \in L$ (T2). State sequences are monotone (T3). No subtraction operator.
From ZP-B: Q_2 with ultrametric d. AX-B1 (binary existence). MP-1 (minimality principle). $p = 2$ derived (T0). Every ball is clopen (T2). Q_2 totally disconnected (T5). Clopen separation of 0 from all nonzero elements (T3). Snap irreversibility (C3).
From ZP-C: Incompressibility threshold P_0 (D1). RP-1 (representation principle). State representations derived from AX-B1 (T1). JSD = 1 bit (T1b). Discrete surprisal operator DF (D5, D6). Non-conservatism of DF on infinite sequences approaching 0 (T2). L-RUN ($c_0 \neq c_1$, so the step from $\bot$ of MachinePhase (c_0) to c_1 is a state change off it). TQ-IH (the same fact read from c_1). T-BUF / T-SNAP: Binary Snap derived as theorem.
From ZP-D: $H = \mathbb{C}^n$ (D1). T: $Q_2 \to H$ constructed by basis assignment (T2). T unique up to unitary equivalence (T3). Snap produces orthogonal shift (T4). Monotone sequences map to non-decreasing norms, all equal to 1 by D2(v) (T5). DP-1 (orthogonality design commitment).

Import Registry IR-ZP — Closed Results from ZP-A through ZP-E

From ZP-E: Universal constituent cross-framework (T1). Hamming/JSD consistency (T2). Landauer bridge BA-1. Processing bounds T3. Unified Snap description T4. Iterative forcing T5. State representations T6. Zero Paradox T7. T-SNAP: the shape of the Binary Snap is a derived theorem — AX-1 is retired, and that the Snap occurs is stated separately: it follows from the occurrence commitment together with DA-1 (closed given DP-2). The occurrence commitment is stated in two charts, execution and slot, in the header of ZeroParadox/Order/Snap.lean (indexed in ZeroParadox/Order/SnapCannotBe.lean § II and CLAIMS.md), the slot chart as the hypothesis hw of t_snap_of_fires_once, that the tape fires at some stage, beside the theorem trials_fires_once that, for independent stages each firing with probability p, $0 < p \leq 1$, firing at some stage has probability one, which names no tape (ZeroParadox/Information/CrossingTrials.lean § VIII).

Inherited Labels Registry IR-2 — Open Items Carried into ZP-H

Status: Structural — labels preserved without laundering

OQ-G1 [Closed in ZP-G v1.1]: Native derivation of categorical surprisal without importing Shannon entropy. Closed by D7′ (conditional Kolmogorov complexity K(B|A)) and I-KC. BA-G1 demoted to compatibility remark R-BA.

OQ-G2 [Resolved in Section IV]: Left adjoint verification for instantiation functors.

OQ-G3 [Resolved in Section III]: Explicit construction of four instantiation functors.

OQ-G4 [Resolved in Section V]: Reconciliation of categorical and ZP-C singularity characterizations.

II. The Morphisms of C — New Definition

ZP-G defined C abstractly: objects, morphisms, composition, identity, satisfying the category axioms plus AX-G1 and AX-G2. The morphisms of C were left unspecified because the abstract results (T1 through T7) require only the categorical axioms, not a concrete description of what morphisms are. That abstraction was correct for ZP-G.

ZP-H requires concreteness. To construct the instantiation functors, we must specify what morphisms of C are, so that we can say what the functor maps them to. That specification belongs here, not in ZP-G.

Definition D-H1 — Morphisms of C — State Transitions

Status: Definition — instantiation-specific

A morphism $f: A \to B$ in C is a state transition: a structure-preserving map from state A to state B satisfying:

(i) Forward direction: Every morphism $f: A \to B$ represents a net addition of informational content. No morphism reduces state.

(ii) Compatibility with AX-G2: For any non-initial object $X \neq 0$, no morphism $X \to 0$ exists. D-H1(i) and AX-G2 are consistent: a net-additive transition cannot return to the minimal state 0.

(iii) Composition: Composition of morphisms corresponds to sequential state accumulation. $f: A \to B$ followed by $g: B \to C$ produces a state at least as large as B.

Definition D-H1 — Morphisms of C — State Transitions

Status note: D-H1 is a design commitment, not a derivation. It is the choice that makes the instantiation functors well-defined. A different choice of morphism structure would yield different functors. D-H1 is the choice that maps correctly to all four ZP domain documents.

III. Construction of the Instantiation Functors [OQ-G3: PDF Constructions Complete]

ZP-G Remark R3 claimed the existence of four functors F_A, F_B, F_C, F_D without constructing them. This section constructs all four. Each construction must specify the object map, the morphism map, and verify preservation of composition and identity. Each must also show that AX-G1 and AX-G2 are respected.

3.1 F_A: C → SLat (Join-Semilattices)

Construction C-H1 — Functor FA: C → SLat

Status: Derived — OQ-G3 partially closed

Object map: F_A sends each object $X \in ob(C)$ to the state $S_X \in L$ in the join-semilattice $(L, \vee, \bot)$ of ZP-A. The initial object 0 maps to $\bot$: $F_A(0) = \bot$.

Morphism map: F_A sends each morphism $f: A \to B$ to the join operation that witnesses the transition: $F_A(f) = (S_A \vee \alpha)$ for some $\alpha \in L$ such that $S_A \vee \alpha = S_B$. By ZP-A D2, every valid state transition is a join.

Preservation of composition: For $f: A \to B$ and $g: B \to C$, $F_A(g \circ f) = S_A \vee \alpha_f \vee \alpha_g = F_A(g) \circ F_A(f)$ by associativity of $\vee$ (ZP-A A1). ✓

Preservation of identity: $F_A(id_A) = S_A \vee \bot = S_A$ by ZP-A A4 (additive identity). ✓

AX-G1 respected on a carrier with no top element: $F_A(0) = \bot$ is the global minimum of L (ZP-A T2). Having no top element is the optional ZP-A hypothesis HasNoTop (ZeroParadox/Order/Lattice.lean), not a property of every carrier: ZP-A's two-state carriers have a top. Where it holds, every element has something strictly above it, so no element is terminal (the last example of § I-b in ZeroParadox/Category/Category.lean). The Lean witness is $\mathbb{N}$ with $\leq$, which has no greatest element (natZPCategory, ZeroParadox/Multihomed/CategoricalBridge.lean). ✓

AX-G2 respected: No join operation in L can return to $\bot$ from a strictly larger state (ZP-A T3, monotonicity). Therefore F_A sends no non-initial morphism to a map terminating at $\bot$. Note: this verification depends on CC-1 / DA-2 for F_A to send C's initial object 0 to L's $\bot$. CC-1 is restated in ZP-J, not forced: given AFAStructure, starting at a Quine atom and starting at $\bot$ are the same condition (cc1_derived, t_exec_iff, Lean 4). The choice of starting point remains a ZP-A modelling commitment, so this check holds given CC-1. ✓

3.2 F_B: C → pTop (p-Adic Topological Spaces)

Construction C-H2 — Functor FB: C → pTop

Status: PDF construction complete — Lean: full functor (fb_functor, sorry-free)

Construction C-H2 — Functor FB: C → pTop
Object map: F_B sends each object $X \in ob(C)$ to an element $x \in Q_2$. The initial object 0 maps to the element $0 \in Q_2$: $F_B(0) = 0 \in Q_2$. pTop is the informal name of the target: in Lean, F_B is realized in TopCat, the category of all topological spaces (fB_functor, ZeroParadox/Valuation/TopFunctor.lean), and its ℕ-indexed stand-in is Q_2BallDepth (fb_functor, ZeroParadox/Multihomed/CategoricalBridge.lean).
Morphism map: F_B sends each morphism $f: A \to B$ to the discrete jump from x_A to x_B in Q_2. By ZP-B T2, any two distinct elements of Q_2 lie in disjoint clopen balls. The jump is across a clopen boundary — a well-defined topological transition.
Preservation of composition: Sequential discrete jumps in Q_2 compose by transitivity of the ball structure. $x_A \to x_B \to x_C$ is a valid sequence of clopen transitions. ✓
Preservation of identity: $F_B(id_A)$ is the trivial jump $x_A \to x_A$, which is the identity on Q_2. ✓
In Q_2, $F_B(0) = 0$ is clopen-separated from all nonzero elements (ZP-B T3).
AX-G1 respected in the stand-in category: the Lean functor fb_functor lands in Q_2BallDepth, the ℕ-indexed ball-depth category, which has no terminal object because the depth index has no maximum: $n + 1 > n$ (q2BallZPCat, ZeroParadox/Multihomed/CategoricalBridge.lean). ✓
AX-G1 fails in TopCat: TopCat has a terminal object, the one-point space (the example after kleisli_bottom_not_terminal in ZeroParadox/Category/TreeSeam.lean). The TopCat realization fB_functor presents 0 as the limit of the shrinking balls, and that floor {0} is itself a one-point space, so it is terminal and not initial (padic_bottom_isTerminal and padic_bottom_not_initial, ZeroParadox/Multihomed/TreeObstructions.lean).
AX-G2 respected in the stand-in category: in Q_2BallDepth a morphism $m \to 0$ means $m \leq 0$, so $m = 0$ by antisymmetry (q2BallZPCat). ZP-B C3, that no continuous path in Q_2 returns to 0 from any non-zero element, is the domain analogue of this stand-in, proved in its own layer. ✓
AX-G2 in TopCat is stated at the initial object, which is not the floor {0}. TopCat's initial object is the empty space, and it satisfies the AX-G2 shape: a space with a morphism into the empty space is empty, so it is isomorphic to it (the example after padic_bottom_isTerminal in ZeroParadox/Multihomed/TreeObstructions.lean).

3.3 F_C: C → InfoSp (Information-Theoretic Spaces)

Construction C-H3 — Functor FC: C → InfoSp
Status: PDF construction complete — Lean: full functor (fc_functor, sorry-free)
Object map: F_C sends each object $X \in ob(C)$ to a probability distribution P_X over {0, 1}. The initial object 0 maps to the point mass at non-existence: $F_C(0) = P = (1, 0)$ (derived from AX-B1 and RP-1 in ZP-C T1). InfoSp is the informal name of the target: in Lean, F_C is realized in KleisliCat PMF, the Kleisli category of the probability monad (fC_functor, ZeroParadox/Multihomed/InfoFunctor.lean), and its ℕ-indexed stand-in is InfoDepth (fc_functor, ZeroParadox/Multihomed/CategoricalBridge.lean).
Morphism map: F_C sends each morphism $f: A \to B$ to the informational transition from P_A to P_B, with informational work $E = JSD(P_A \| P_B) \geq 0$. The fundamental transition (0 → first non-initial object) maps to $JSD(P \| Q) = 1$ bit (ZP-C T1b).

Construction C-H3 — Functor FC: C → InfoSp

Preservation of composition: In the binary framework (two outcome states), only two point-mass distributions exist: P = (1, 0) and Q = (0, 1). The snap f: 0 → ε_0 is the unique morphism mapping P → Q at cost JSD(P ‖ Q) = 1 bit. All successor morphisms g: $\varepsilon_n \to \varepsilon_{n+1}$ map Q → Q (Q-stability of the post-snap codomain), giving $F_C(g)$ = JSD(Q ‖ Q) = 0. Therefore $F_C(g \circ f) = 1$ bit = $F_C(g) + F_C(f) = 0 + 1$ bit. Composition is preserved exactly by Q-stability, not by subadditivity. (Note: JSD subadditivity is an inequality and does not establish this — the equality holds here as a structural consequence of the binary framework.) ✓

Preservation of identity: $F_C(\mathrm{id}_A) = \mathrm{JSD}(P_A \,\|\, P_A) = 0$. No informational work is done by a trivial transition. ✓

The point masses P = (1, 0) and Q = (0, 1) are exactly the distributions of zero entropy (Mathlib Real.binEntropy_eq_zero: binEntropy p = 0 ↔ p = 0 ∨ p = 1).

AX-G1 respected in the stand-in category: the Lean functor fc_functor lands in InfoDepth, the ℕ-indexed surprisal-depth category, which has no terminal object because the depth index has no maximum: n + 1 > n (infoDepthZPCat, ZeroParadox/Multihomed/CategoricalBridge.lean). ✓

AX-G1 fails in KleisliCat PMF: the one-point type is a terminal object (the example after kleisli_bottom_not_terminal in ZeroParadox/Category/TreeSeam.lean). In ZP-A, having no top element is the optional hypothesis HasNoTop (ZeroParadox/Order/Lattice.lean), which ZP-A's two-state carriers do not satisfy.

AX-G2 respected in the stand-in category: in InfoDepth a morphism m → 0 means m ≤ 0, so m = 0 by antisymmetry (infoDepthZPCat). Its domain analogue: JSD ≥ 0, and a transition returning to P = (1, 0) from any Q ≠ P would require JSD(Q ‖ P) = 0, which holds only if Q = P. ✓

In KleisliCat PMF, no stage n > 0 of the realization has a morphism into stage 0 (fC_no_return, ZeroParadox/Multihomed/InfoFunctor.lean): the AX-G2 shape on the functor's image.

Inherited label: The distributions P = (1, 0) and Q = (0, 1) are derived from AX-B1 and RP-1 (ZP-C T1, ZP-E T6). All results depending on F_C inherit the AX-B1 and RP-1 labels.

3.4 F_D: C → Hilb (Hilbert Spaces)

Construction C-H4 — Functor FD: C → Hilb

Status: PDF construction complete — Lean: full functor (fd_functor, sorry-free)

Object map: F_D sends each object of C to a state space: in Lean, with C realized as ℕ, object n goes to depth n of HilbDimDepth (fd_functor.obj n) and to the module StateSpace n of ModuleCat ℂ (fD_functor.obj n). The initial object 0 maps to $F_D(0)$: depth 0 of HilbDimDepth (fd_functor.obj 0) and the zero module $\mathbb{C}^0$ (fD_functor.obj 0). ZP-D's vector model reads a point x of Q_2 as the vector T(x) ∈ H = $\mathbb{C}^n$, through the transition operator T: Q_2 → H constructed in ZP-D (T2); there 0 of Q_2 is read as the unit vector T(0) = e_0 (ZP-D D2(i), met by T2), which $\mathbb{C}^0$ does not contain, so the two models differ at 0. Hilb is the informal name of the target: in Lean, F_D is realized in ModuleCat ℂ, the category of ℂ-modules (fD_functor, ZeroParadox/State/HilbFunctor.lean), and its ℕ-indexed stand-in is HilbDimDepth (fd_functor, ZeroParadox/Multihomed/CategoricalBridge.lean).

Construction C-H4 — Functor FD: C → Hilb

Morphism map: F_D sends each morphism f: A → B to the orthogonal extension from $T(x_A)$ to $T(x_B)$ in H. By ZP-D D2(iii) and D2(iv) (DP-1), points in different cells of the level-k partition P_k map to orthogonal basis vectors under T, and points in one cell share one vector, so the transition is an orthogonal step when x_A and x_B lie in different cells and the trivial step when they share a cell.

Preservation of composition: $F_D(g \circ f) = F_D(g) \circ F_D(f)$ is the functor law map_comp of the Lean functors, fd_functor.map_comp and fD_functor.map_comp. ✓

Remark (ZP-D T5, not composition): every state vector is a unit vector by ZP-D D2(v), so along $T(x_A) \to T(x_B) \to T(x_C)$ the norms are all 1 and ZP-D T5's non-decreasing norms hold with equality.

Preservation of identity: $F_D(id_A)$ is the identity on $F_D(A)$, the functor law map_id of the Lean functors, fd_functor.map_id and fD_functor.map_id; in the vector model it is the trivial extension $T(x_A) \to T(x_A)$. ✓

In the vector model, $T(0) = e_0$ (ZP-D D2(i), met by T2) is the anchor vector from which all other state vectors are orthogonal extensions.

AX-G1 respected in the stand-in category: the Lean functor fd_functor lands in HilbDimDepth, the ℕ-indexed dimension-depth category, which has no terminal object because the depth index has no maximum: n + 1 > n (hilbDimZPCat, ZeroParadox/Multihomed/CategoricalBridge.lean). ✓

AX-G1 fails in ModuleCat ℂ: the realization fD_functor sends 0 to the zero module fD_functor.obj 0, which is a zero object of ModuleCat ℂ, initial and terminal at once (hilbert_bottom_isZero, ZeroParadox/Category/TreeSeam.lean).

AX-G2 respected in the stand-in category: in HilbDimDepth a morphism m → 0 means m ≤ 0, so m = 0 by antisymmetry (hilbDimZPCat). ZP-D T4, that the Snap produces an orthogonal shift (proved in ZP-D), and ZP-A Remark R1 (no subtraction) are the domain analogues of this stand-in. ✓

AX-G2 fails in ModuleCat ℂ: the zero module fD_functor.obj 0 is a zero object, every stage has a morphism into it (the zero map, fD_has_return, ZeroParadox/Category/Heterogeneous.lean) and stage 1 is not a zero object (leaf_not_isZero, ZeroParadox/Category/SeamNotColimit.lean).

Design commitment inherited: DP-1 (orthogonality as representation of clopen separation) is a design commitment in ZP-D v1.2. F_D inherits this label. T4 and T5-b of ZP-D depend on DP-1 as a premise.

IV. Left Adjoint Verification [OQ-G2 Closed]

ZP-G T5 stated that functors preserve initial objects, but marked it conditional on OQ-G2: verifying that the instantiation functors are left adjoints (or otherwise preserve the universal property of 0 fully, not just for objects in their image). This section closes OQ-G2.

Theorem T-H1 — Each Instantiation Functor Preserves the Initial Object

Status: Derived — OQ-G2 closed

Claim: For each functor $F \in \{F_A, F_B, F_C, F_D\}$, F(0) is an initial object in the codomain category (for F_B, in its stand-in).

Theorem T-H1 — Each Instantiation Functor Preserves the Initial Object

Strategy: Rather than proving each functor is a left adjoint in full generality (which would require establishing adjoint pairs for SLat, pTop, InfoSp, and Hilb), we verify the universal property directly for each functor. The universal property of the initial object requires: for every object Y in the codomain, ∃! morphism F(0) → Y.

F_A: $F_A(0) = \bot$. For any $S_Y \in L$, the unique morphism $\bot \to S_Y$ is the join $\bot \vee S_Y = S_Y$ (ZP-A A4). Uniqueness: $\bot$ is the global minimum (ZP-A T2), so the only order-preserving map from $\bot$ to S_Y is the join with S_Y. ✓

F_B: $F_B(0) = 0 \in Q_2$. For any $x \in Q_2$, the unique morphism $0 \to x$ is the discrete jump from 0 to x across the clopen boundary. Uniqueness: 0 is clopen-separated from all nonzero elements (ZP-B T3); the only clopen-respecting transition from 0 to x is the direct jump. This leg holds in the stand-in Q_2BallDepth (fb_preserves_initial, ZeroParadox/Multihomed/CategoricalBridge.lean). In TopCat it does not: the floor {0} is a one-point space, so it is terminal and not initial there (padic_bottom_isTerminal and padic_bottom_not_initial, ZeroParadox/Multihomed/TreeObstructions.lean). ✓

F_C: $F_C(0) = P = (1, 0)$. For any distribution P_Y, the unique morphism $P \to P_Y$ is the informational transition with work $E = JSD(P \| P_Y)$. Uniqueness: JSD is symmetric and uniquely determined by P and P_Y; there is exactly one value of informational work for any pair of distributions. ✓

F_D: $F_D(0)$ is depth 0 of HilbDimDepth (fd_functor.obj 0) and the zero module $\mathbb{C}^0$ of ModuleCat $\mathbb{C}$ (fD_functor.obj 0); ZP-D's vector model reads 0 of Q_2 as the base vector $T(0) = e_0$ (ZP-D D2(i)), and reads the morphism from $F_D(0)$ to the object of x as the orthogonal extension from e_0 to e_k (the basis vector assigned to x). Uniqueness of that morphism is a fact about the codomain category, not about T: in the stand-in HilbDimDepth every hom-set out of depth 0 has exactly one element (hilbDimHom0Unique), so fd_functor sends 0 to an initial object (fd_preserves_initial, ZeroParadox/Multihomed/CategoricalBridge.lean). In ModuleCat $\mathbb{C}$ the realization fD_functor sends 0 to the zero module, a zero object and so initial (hilbert_bottom_isZero, ZeroParadox/Category/TreeSeam.lean). ZP-D T3 (T unique up to a unitary) concerns the vectors T(x) up to a unitary, not the morphisms between objects, and is not used here. ✓

Conclusion: OQ-G2 is closed. ZP-G T5 is now unconditional for the four instantiation functors of this framework, for F_B in its stand-in. The universal property of $0 \in ob(C)$ is fully preserved under F_A, F_B, F_C, and F_D (for F_B, in its stand-in). ✓

V. Singularity Reconciliation [OQ-G4 Closed]

ZP-G T6 characterized the informational singularity of 0 as an undefined domain: the categorical surprisal $I(X \to 0)$ is undefined because no such morphism exists. ZP-C established the singularity as infinite accumulation: the discrete surprisal DF diverges along infinite sequences approaching 0 in Q_2. ZP-G R4 claimed these are compatible but did not prove it. This section closes OQ-G4.

Theorem T-H2 — Compatibility of Singularity Characterizations

Status: Derived — OQ-G4 closed

Claim: The categorical singularity (undefined domain, ZP-G T6) and the ZP-C singularity (infinite accumulation, ZP-C T2) hold together: a conjunction, and neither is derived from the other.

Theorem T-H2 — Compatibility of Singularity Characterizations

Setup (motivation only; th2_singularity_compatibility involves neither F_C nor F_B): Let $\{X_n\}$ be a sequence of objects in C such that the corresponding sequence $\{F_C(X_n)\} = \{P_n\}$ in InfoSp approaches $F_C(0) = P = (1, 0)$ in distribution. Under F_B, the corresponding sequence $\{x_n\}$ in Q_2 approaches $0 \in Q_2$.

ZP-C side: By ZP-C T2, the discrete surprisal DF along the infinite sequence $\{x_n\}$ approaching 0 in Q_2 is non-conservative: the accumulated surprisal $\sum DF(x_k \to x_{k+1})$ diverges as $n \to \infty$. The singularity is infinite accumulation.

ZP-G side: By ZP-G T6(ii), $I(X \to 0)$ is undefined for all $X \neq 0$ because $\hom(X, 0) = \varnothing$ (AX-G2). The singularity is undefined domain.

Reconciliation: The two characterizations are compatible: both hold. In ZP-C, we ask: what happens to surprisal as we approach 0 along an infinite sequence? The answer is divergence — the accumulation is unbounded. In ZP-G, we ask: can we reach 0 by a morphism from outside? The answer is no — the morphism does not exist. Read informally, if the accumulation required to reach 0 is infinite, then no finite morphism can accomplish it; that link runs from divergence toward absence and is a reading, not a theorem.

Lean: th2_singularity_compatibility pairs t3_unreachability (the ZP-G side) with t2_diverges (the ZP-C side): the proof term is the pair of these two separately proved lemmas, so neither is derived from the other. OQ-G4 is closed in that sense. ✓

Remark R-FORCING — Structural Forcing vs. Re-Description

The four functors (F_A, F_B, F_C, F_D) established in Sections III-V do not merely translate the Binary Snap into categorical language. Each functor independently grounds the snap in its own domain: F_A shows the lattice join places ε_0 above $\bot$; F_B shows the 2-adic topology makes any departure from 0 irreversible (it does not supply the departure — ZP-B cannot force the snap, and the first step is AX-B1); F_C shows the information-theoretic transition costs exactly 1 bit; F_D shows the Hilbert space snap is an orthogonal shift.

That a transition of the snap's shape appears in each of the four domain categories, with the groundings sorry-free in Lean 4 (the lattice conjunct join $c_0\ c_1 = c_1$ of th3_snap_all_functors for F_A: fb_snap_q2_grounded, fc_snap_info_grounded and fd_snap_hilb_grounded for F_B, F_C and F_D), is more than re-description: in each of the four canonical mathematical languages for state description, the phenomenon's shape is constrained by that domain's own structure. That the four are one structural singularity is the framework's reading, not a theorem. That the transition is taken follows from the occurrence commitment together with DA-1 (closed given DP-2), not from any of the four.

VI. The Binary Snap — Categorical Description

Theorem T-H3 — The Binary Snap Under All Four Functors

Status: Derived — Cross-Framework

Setup: The Binary Snap is the transition from $0 \in \text{ob}(C)$ to the first non-initial object $\varepsilon_0 \in \text{ob}(C)$, driven by the incompressibility threshold P_0 (ZP-C D1) under T-SNAP (Binary Snap — Derived Theorem, ZP-E v2.0).

In C (ZP-G): The Snap is the morphism $\iota_{\varepsilon 0}: 0 \to \varepsilon_0$. By D3, this morphism is unique. By T4, no chain returns from ε_0 to 0. The Snap is categorically irreversible.

Theorem T-H3 — The Binary Snap Under All Four Functors
Under F_A (ZP-A): $F_A(\iota_{\varepsilon 0}) = \bot \vee \varepsilon_0 = S_1$. The Snap is the first join. Monotone and irreversible by ZP-A T3. T-SNAP is a derived theorem (ZP-E v2.0); inherited here with derived status.
Under F_B (ZP-B): $F_B(\iota_{\varepsilon 0})$ is the discrete jump $0 \to \varepsilon_0 \in Q_2$ across a clopen boundary (ZP-B T2). Topologically irreversible by ZP-B C3.
Under F_C (ZP-C): $F_C(\iota_{\varepsilon 0})$ is the informational transition $P \to Q$ with $E = JSD(P \parallel Q) = 1$ bit (ZP-C T1b). Irreversible: $JSD(Q \parallel P) = 1$ bit $\neq 0$.
Under F_D (ZP-D): $F_D(\iota_{\varepsilon 0})$ is the orthogonal shift $T(0) = e_0 \to T(\varepsilon_0) = e_1$. $\langle e_0, e_1 \rangle = 0$ (ZP-D T4). Both vectors have norm 1 (ZP-D D2(v)), so ZP-D T5's non-decreasing norms hold with equality. DP-1 is a design premise.
Cross-framework consistency: All four functors agree that the Snap is irreversible. Each irreversibility result is a closed theorem within its own domain document. Their agreement is structural consistency, not circular argument. Independence of null-analog discovery: each framework located its null-analog through its own domain logic — $\bot$ through A4 (ZP-A), $0 \in Q_2$ through T3 (ZP-B), $P = (1,0)$ through D1/RP-1 (ZP-C) — prior to and without reference to the others. ZP-D's e_0 is not independent in this sense: it is reached by transport of ZP-B's 0 along T (ZP-D D2(i), $T(0) = e_0$). No identification of these objects is made, under the functors or otherwise: they live in different types, and an equation across them is ill-typed (MC-1, the bottom family). Where a shared role is checked, it is checked in one framework at a time, and the occupant depends on the step chosen there. In the die (ZeroParadox/Die.lean) the role Motion.SelfOnlyAt is a point that steps to itself and to nothing else. In ZP-A, under a self-application map supplied by AbstractSelfApp, $\bot$ is its only occupant by that class's own fields (selfApp_quine_role_iff, ZeroParadox/Settheory/QuineDichotomy.lean), which the constant map onto $\bot$ also meets. In ZP-B, under doubling, 0 is its only occupant, on the 2-adic integers (z2_double_selfOnlyAt_iff, ZeroParadox/Ordinal/Incompleteness.lean) and on Q_2 (the doubling example after c_t_iz_null_balance, ZeroParadox/Valuation/SemilatticeInstance.lean). For ZP-C's $P = (1,0)$ and ZP-D's e_0 no die statement was located as of 2026-11-03 (searched: every file under ZeroParadox/ for Motion, SelfOnlyAt, NoStepAt and LeavesAt beside those objects). What the categorical bridge checks for F_C and F_D is a different role, initiality: they send the initial object 0 to the empty type Fin 0 and the zero module (fC_zero_isInitial, fD_zero_isInitial). On the empty type no point occupies the die role at all (the control in ZeroParadox/Category/SeamComparisonMap.lean); on the zero module, which has one point, every endomorphism has exactly one point that steps to itself only (hilbert_zero_seam_role_unique), which restates that it is a zero object. This is not the claim that any domain forced the others' choices. Their agreement confirms coherence across mathematical representations, not external replication. T-SNAP (Binary Snap Causality) is a derived theorem in ZP-E v2.0 and is inherited here with derived status. "Causality" in this name refers to the shape of the step, not to its occurrence. DP-1 remains the only design premise. ✓

VII. Full Traceability Register

Every cross-framework claim in ZP-H is traced here to its grounding theorem and any required bridge axiom.

Claim	Grounded In	Bridge Axiom?	Status
$F_A(0) = \bot$	ZP-A T2; C-H1	None	Valid — Derived
F_A preserves composition	ZP-A A1; C-H1	None	Valid — Derived

Claim	Grounded In	Bridge Axiom?	Status
F_A respects AX-G1, AX-G2	ZP-A T2, T3; natZPCategory; C-H1	HasNoTop (no-terminal half)	Valid on a carrier with no top
$F_B(0) = 0 \in Q_2$	ZP-B T3; C-H2	None	Valid — Derived
F_B preserves composition	ZP-B T2; C-H2	None	Valid — Derived
F_B respects AX-G1, AX-G2	q2BallZPCat (stand-in); ZP-B C3; C-H2	None	Valid in the stand-in — TopCat has a terminal object
$F_C(0) = P = (1,0)$	ZP-C T1; ZP-E T6; C-H3	AX-B1, RP-1	Valid — from AX-B1, RP-1
F_C preserves composition	Q-stability of post-snap codomain; C-H3	None	Valid — Derived (binary framework; Q-stability, not subadditivity)
F_C respects AX-G1, AX-G2	infoDepthZPCat (stand-in); ZP-C T1b; C-H3	AX-B1, RP-1 (for ZP-C T1b; the stand-in proofs use neither)	Valid in the stand-in — KleisliCat PMF has a terminal object
$F_D(0)$ = depth 0, $\mathbb{C}^0$; $T(0) = e_0$ in ZP-D's vector model	ZP-D D2(i), T2; C-H4	None (D2(i), by assignment)	Valid — by construction
F_D preserves composition	fd_functor.map_comp, fD_functor.map_comp; C-H4	None	Valid — Derived (functor law)
F_D respects AX-G1, AX-G2	hilbDimZPCat (stand-in); ZP-D T4; C-H4	DP-1 (for ZP-D T4; the stand-in proofs do not use it)	Valid in the stand-in — both fail in ModuleCat $\mathbb{C}$
T-H1: universal property preserved	ZP-A T2; ZP-B T3; ZP-C T1b; C-H4 (fd_preserves_initial)	None	Valid — OQ-G2 closed; F_B in its stand-in
T-H2: singularity compatibility	ZP-G T6; ZP-C T2; C-H3	None	Valid — OQ-G4 closed
T-H3: Snap under all four functors	C-H1 through C-H4; ZP-E T4; ZP-E T-SNAP	DP-1	Valid — T-SNAP derived (ZP-E v2.0)
D-H1: morphisms of C	ZP-A D2; ZP-B T2; ZP-C; ZP-D	None	Design Commitment

VIII. Open Items Register

Item	Status	Description
OQ-G1	Closed — ZP-G v1.1 D7′, T6	Native derivation of categorical surprisal without importing Shannon entropy. Closed in ZP-G v1.1 by D7′ (native categorical surprisal K(B|A)) and I-KC (Kolmogorov import from ZP-C). BA-G1 demoted to compatibility remark R-BA. No bridge axiom remains as a theorem premise.
OQ-G2	Closed — T-H1	Left adjoint verification for instantiation functors. Resolved in Section IV by direct verification of the universal property for each of the four functors (for F_B, in its stand-in).
OQ-G3	Closed — all four functors (Lean)	PDF-level constructions of all four instantiation functors complete (Section III). All four now have concrete Lean Functor terms (sorry-free): F_A: fa via NatSLat (Functor ℕ ℕ, effectively). F_B: fb_functor (Functor ℕ Q_2BallDepth), snap grounded in C3. F_C: fc_functor (Functor ℕ InfoDepth), snap grounded in T1b. F_D: fd_functor (Functor ℕ HilbDimDepth), snap grounded in T4. Each domain type has a distinct ZPCategory instance and preserves_initial definition. OQ-G3 fully closed.
OQ-G4	Closed — T-H2	Reconciliation of categorical (undefined domain) and ZP-C (infinite accumulation) singularity characterizations. Resolved in Section V. The two characterizations are shown to hold together, a conjunction; neither is derived from the other.
AX-1	Retired — shape: T-SNAP (ZP-E v2.0); the snap occurs given the occurrence commitment and DA-1	Binary Snap Causality. AX-1 is retired. Its content was split in two: the shape of the Snap is proved, as Theorem T-SNAP (from L-RUN, TQ-IH and the bottom law, with no Lean axioms), and that the Snap occurs is stated separately: it follows from the occurrence commitment together with DA-1 (closed given DP-2). (tsnap_holds_but_nothing_moves shows T-SNAP does not carry occurrence.) T-H3 inherits T-SNAP as a derived result. Not a gap.
AX-G1	Axiom — initial half not novel	Asymmetry: initial object 0, no terminal object. Inherited from ZP-G. The initial-object half is not a novel commitment — it is grounded in ⊥ as the bottom element of the ZP-A semilattice. The no-terminal half is ZP-G's own commitment: in ZP-A, having no top element is the optional hypothesis HasNoTop, which ZP-A's two-state carriers do not satisfy. It holds in the ℕ-indexed depth categories ZP-H's Lean functors land in, and fails in TopCat, ModuleCat ℂ and KleisliCat PMF, each of which has a terminal object (the example after kleisli_bottom_not_terminal in ZeroParadox/Category/TreeSeam.lean). Not a gap.

Item	Status	Description
AX-G2	Axiom — not novel where it holds	Source asymmetry: hom(X, 0) = ∅ for X ≠ 0. Inherited from ZP-G. Not a novel commitment where it holds: in a category built from a partial order with a least element, such as ZP-A's semilattice order and the ℕ-indexed depth categories ZP-H's Lean functors land in, a morphism X → 0 means X ≤ 0, so X = 0 by antisymmetry (for the depth categories, the ax_g2 fields of q2BallZPCat, infoDepthZPCat and hilbDimZPCat in ZeroParadox/Multihomed/CategoricalBridge.lean; for a partial order in general, the two antisymmetry examples at the end of ZeroParadox/Category/AxG2Reduce.lean). ZP-B C3 (no continuous path in Q_2 returns to 0) is the domain analogue of the Q_2 stand-in, proved in its own layer. AX-G2 does not hold in every category: in ModuleCat ℂ the zero module fD_functor.obj 0 is a zero object, every stage has a morphism into it (the zero map, fD_has_return, ZeroParadox/Category/Heterogeneous.lean) and stage 1 is not a zero object (leaf_not_isZero, ZeroParadox/Category/SeamNotColimit.lean), so AX-G2 fails there. At an initial object, AX-G2 in its isomorphism form (no morphism into 0 from an object not isomorphic to 0) is equivalent to strict initiality (ax_g2_from_strict_initial and the converse example after it, ZeroParadox/Category/AxG2Reduce.lean). Not a gap.
R-BA	Remark — BA-G1 demoted (ZP-G v1.1)	Leinster Shannon entropy characterization. BA-G1 demoted from Bridge Axiom in ZP-G v1.0 to Compatibility Remark R-BA in ZP-G v1.1. Derivable from D7′ and I-KC via the coding theorem. Not a gap.
D-H1	Design Commitment — intentional	Morphisms of C defined as state transitions. This is a design choice that makes the instantiation functors well-defined. A different morphism structure would yield different functors. Explicitly stated and not laundered as a derivation.
DP-1	Design Commitment — inherited	Orthogonality as representation of clopen separation. Inherited from ZP-D v1.1. F_D and T-H3 depend on it as a premise.

IX. Validation Status

Component	Status / Notes
IR-G: ZP-G v1.1 results imported	Valid — all ZP-G v1.1 closed results received; labels preserved. BA-G1 demoted to R-BA; I-KC noted as named dependency.
IR-ZP: ZP-A through ZP-E imported	Valid — all closed results received from ZP-A through ZP-E; labels preserved. T-SNAP (ZP-E v2.0) noted: AX-1 is retired; its shape is derived, not axiomatic, and the snap occurs given the occurrence commitment and DA-1.
IR-2: Inherited labels preserved	Valid — OQ-G1 closed in ZP-G v1.1; OQ-G2, OQ-G3, OQ-G4 resolved here; BA-G1 updated to R-BA; AX-1 updated to retired, its shape to T-SNAP (Derived) and its occurrence to the occurrence commitment together with DA-1.

Component	Status / Notes
D-H1: Morphisms of C	Design Commitment — explicitly stated; required for functor construction
C-H1: F_A: C → SLat	Valid — Derived. Object map, morphism map, composition, identity all verified. ✓
C-H2: F_B: C → pTop	Valid — all four requirements verified; AX-G1 and AX-G2 checked in the stand-in Q_2BallDepth (TopCat has a terminal object). Lean: full functor (fb_functor, sorry-free). ✓
C-H3: F_C: C → InfoSp	Valid — from AX-B1, RP-1; all four requirements verified; AX-G1 and AX-G2 checked in the stand-in InfoDepth (KleisliCat PMF has a terminal object). Lean: full functor (fc_functor, sorry-free). ✓
C-H4: F_D: C → Hilb	Valid — the object map at 0 by assignment (ZP-D D2(i)), the morphism map from ZP-D D2(iii) and D2(iv), D2(iv) under DP-1; all four requirements verified; AX-G1 and AX-G2 checked in the stand-in HilbDimDepth (both fail in ModuleCat ℂ). Lean: full functor (fd_functor, sorry-free). ✓
T-H1: Universal property preserved	Valid — OQ-G2 closed. ZP-G T5 is now unconditional for all four instantiation functors, for F_B in its stand-in. ✓
T-H2: Singularity reconciliation	Valid — OQ-G4 closed. Categorical (undefined domain) and ZP-C (infinite accumulation) shown to hold together, a conjunction; neither is derived from the other. ✓
T-H3: Snap under all four functors	Valid — Derived; cross-framework. T-SNAP inherited as derived theorem (ZP-E v2.0). DP-1 labelled as design premise. ✓
Traceability register	Valid — Complete. Every cross-framework claim traced to source theorem and bridge axiom. No floating connections.
OQ-G1: Native surprisal derivation	Closed in ZP-G v1.1 via D7′ and I-KC. No bridge axiom remains as a theorem premise.
AX-1 (Binary Snap Causality)	Retired — the shape is derived as T-SNAP in ZP-E v2.0 and is no longer an axiom; T-H3 and all downstream results inherit the derived status for the shape. That the Snap occurs is stated separately: it follows from the occurrence commitment together with DA-1 (closed given DP-2).
AX-G1, AX-G2	AX-G2 and the initial-object half of AX-G1 are grounded, not novel: AX-G2 by antisymmetry in categories built from a partial order with a least element, and it fails in ModuleCat ℂ (see the Open Items register). The no-terminal half of AX-G1 is ZP-G's own commitment; it holds in the ℕ-indexed stand-ins and fails in TopCat, ModuleCat ℂ and KleisliCat PMF. Stated as local axioms in ZP-G for self-containment. Not gaps.
R-BA, D-H1, DP-1	Compatibility Remark / Design Commitments — intentional. Explicitly stated. Not laundered.

End of ZP-H | Four instantiation functors constructed | OQ-G1 through OQ-G4 closed | All four functors have concrete Lean Functor terms (sorry-free): fb_functor (Q_2BallDepth), fc_functor (InfoDepth), fd_functor (HilbDimDepth), F_A via

NatSLat | OQ-G3 fully closed | T-SNAP inherited as derived theorem | T-H3 independence-of-discovery note: the null-analogs of ZP-A, ZP-B and ZP-C located independently, ZP-D's e_0 by transport of ZP-B's 0 | AX-G2 and the initial-object half of AX-G1 are grounded in prior layers where they hold, and the no-terminal half of AX-G1 is ZP-G's own commitment; AX-B1 is the framework's one substantive modelling commitment — discrete Boolean existence rather than a continuum of partial states. The decide proof only checks the two states are distinct GIVEN the two-element type; it does not verify the commitment