THE ZERO PARADOX

ZP-H Addendum

The Snap Floor in Native Categories

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ZP-H derives the Binary Snap across four mathematical frameworks and locates the snap floor ⊥ as the categorical bottom in each. In the main document that location is carried by three depth-index proxy categories — copies of ℕ with the order ≤ (Q_2BallDepth, InfoDepth, HilbDimDepth), engineered so that depth 0 is the initial object. This addendum does the stronger thing: it drops the snap floor into each framework's own category, taken directly from Mathlib, where the floor appears in that field's native syntax. A topologist, a functional analyst, and a probabilist each meet the floor in their own category and their own language, in the role that category supports: the zero object, initial and terminal, in ModuleCat ℂ (fD_zero_isInitial, fD_zero_isTerminal); an initial object that is not terminal in the Kleisli category of PMF (fC_zero_isInitial, kleisli_bottom_not_terminal); in TopCat the floor {0} is terminal, since it has one point (padic_bottom_isTerminal), and not initial (padic_bottom_not_initial), and separately the inverse limit of the ball tower (floorConeIsLimit, ZeroParadox/Order/PadicLimitCone.lean). The three full positions are stated together in floor_positions_full (ZeroParadox/Category/TreeSeam.lean: a zero object; initial and not terminal; terminal and not initial). They are members of the bottom family (MC-1), and not one object, since an equation across categories is ill-typed. mc1_correspondence bundles the two initial objects, fC_no_return (no morphism into the floor from any later stage of the functor), and the set equation in Q_2 fB_bottom_is_limit; floorConeIsLimit sits outside it.

Three genuine Lean functors realize the snap chain inside the standard categories: F_B into TopCat (topological spaces), F_D into ModuleCat ℂ (ℂ-modules), and F_C into the Kleisli category of the probability monad PMF (stochastic maps). Each is sorry-free. F_D and F_C send depth 0 to an initial object of their categories (fD_zero_isInitial, fC_zero_isInitial), and F_D's is terminal as well, a zero object (fD_zero_isTerminal); for F_B the floor {0} is the inverse limit of its ball tower in TopCat (floorConeIsLimit), the limit of the functor's diagram rather than an object in its image. The bundled witness is mc1_correspondence (MC1Bridge.lean); its F_B field is the set equation in Q_2 beneath that limit (fB_bottom_is_limit).

Two honesty boundaries hold throughout. The realized claim is correspondence, not identity. What is proved is, category by category, the characterization named above, the floors agreeing on the snap; that the four bottoms are numerically one object is retired as ill-typed (object equality across categories does not typecheck and is not invariant under equivalence); what separates the members is proved property by property (seam_unique_among_named, for the named bottoms, in a lattice with no top). The native categories are the honest target. Mathlib has no bespoke category of p-adic spaces, Hilbert spaces, or information spaces, so the general-purpose categories — topological spaces, ℂ-modules, stochastic maps — are where the floor genuinely lives.

Section I: From Depth Proxies to Native Categories

The proxy categories of the main document are correct but indirect. $\mathbb{Q}_2$BallDepth, InfoDepth, and HilbDimDepth are each a copy of $\mathbb{N}$ with the order $\leq$, wrapped so that the bottom 0 satisfies the universal property of an initial object and the source-asymmetry condition AX-G2 (no morphism returns to 0). They witness the snap floor's categorical role, but inside a structure built for the purpose rather than inside the domain itself.

The native categories are not built for the purpose, and that is the point — they also do not satisfy the ZP categorical axioms. A ZPCategory (ZP-G) requires that no terminal object exist (AX-G1): the snap has nowhere higher to go. TopCat, ModuleCat $\mathbb{C}$ and the Kleisli category all have a terminal object (the one-point space; the zero module; the one-point type, on which the only distribution is the point mass — the example after kleisli_bottom_not_terminal in ZeroParadox/Category/TreeSeam.lean). Of the categories on this page, AX-G1 holds only in the three depth-proxy categories, where the depth index has no maximum. So the snap floor cannot be "the initial object of a ZPCategory" here. The honest statement is the one each category supports natively: in TopCat the floor is terminal, since it has one point, and not initial, and separately the limit of the shrinking system; in ModuleCat $\mathbb{C}$ it is the zero object (initial and terminal), and in the Kleisli category it is initial and not terminal. Each is stated below.

Section II: F_B — The Snap Floor in TopCat

The 2-adic snap floor is $0 \in \mathbb{Q}_2$. Around it sits the descending tower of clopen balls $B(0, 2^{-n})$ — the unit ball, then radius 1/2, then 1/4, and so on — each a genuine topological subspace of $\mathbb{Q}_2$. The inclusions $B(0, 2^{-m}) \subseteq B(0, 2^{-n})$ for $n \leq m$ are continuous, so the tower is a functor out of $\mathbb{N}^{op}$ (the balls shrink as the index grows): a real diagram in TopCat.

Functor: fB_functor : ℕ^op → TopCat (TopFunctor.lean)
Object $n \mapsto$ the clopen ball $B(0, 2^{-n}) \subseteq \mathbb{Q}_2$, as a TopCat object.
Morphism ($n \leq m$): the continuous inclusion of the smaller ball into the larger.
fB_bottom_is_limit: $\bigcap_n B(0, 2^{-n}) = \{0\}$
The one point common to every ball is 0: a set equation in $\mathbb{Q}_2$.
floorConeIsLimit (ZeroParadox/Order/PadicLimitCone.lean): the floor {0}, with the inclusions into each ball, is a limit cone, so the snap floor is the inverse limit of the tower in TopCat; fB_limit_iso_floor identifies Mathlib's limit object with it.
Sorry-free. Purity [propext, Classical.choice, Quot.sound].

TopCat has both a terminal object (the one-point space) and an initial object (the empty space). The snap floor {0} is a one-point space, so it is terminal (padic_bottom_isTerminal, ZeroParadox/Multihomed/TreeObstructions.lean) and not initial (padic_bottom_not_initial). It is also the limit of the ball tower (floorConeIsLimit) — the categorical expression a topologist would expect for "the

point everything converges to." This is the honest realization in a category that is not a ZPCategory: not an initial object: terminal, since it has one point, and separately a limit.

Section III: F_D — The Snap Floor in ModuleCat $\mathbb{C}$

The state layer represents the snap in a complex Hilbert space: StateSpace n is the n-dimensional space $\mathbb{C}^n$, with the snap an orthogonal shift between basis states (ZP-D). As $\mathbb{C}$-modules, these spaces and the linear maps between them form a diagram in ModuleCat $\mathbb{C}$.

Functor: fD_functor : $\mathbb{N}$ → ModuleCat $\mathbb{C}$ (HilbFunctor.lean)
Object n ↦ StateSpace n = the $\mathbb{C}$-module $\mathbb{C}^n$.
Morphism (n ≤ m): the isometric $\mathbb{C}$-linear embedding that pads the extra coordinates with zero.
fD_zero_isInitial: StateSpace 0 is the initial object of ModuleCat $\mathbb{C}$.
The snap floor is the zero module — the one-element $\mathbb{C}$-module {0}, which is the categorical zero object (both initial and terminal).
fD_embed_inner: the embeddings preserve the inner product (genuine isometries).
Sorry-free. Purity [propext, Classical.choice, Quot.sound].

Here, unlike TopCat, the floor is the initial object outright: the zero object of ModuleCat $\mathbb{C}$ is StateSpace 0. The isometry lemma records what ModuleCat $\mathbb{C}$ forgets — that the embeddings are not merely linear but inner-product preserving — so the Hilbert-space structure travels alongside the functor rather than being discarded by the choice of category.

Section IV: F_C — The Snap Floor in KleisliCat PMF

The information layer measures the snap as one bit of divergence between distributions (ZP-C, JSD = log 2). Its native category is the Kleisli category of the probability-mass-function monad PMF: its objects are types, finite or not (PMF is defined on any type), and a morphism A → B is a stochastic map (a Markov kernel), assigning to each a ∈ A a probability distribution on B; the functor uses only finite ones, the types Fin n. Mathlib supplies this category for free, since PMF is a lawful monad; it is a Markov category in the sense of Fritz (T. Fritz, A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics, Adv. Math. 370 (2020) 107239, arXiv:1908.07021, Cor. 3.2 and fn. 10).

Functor: fC_functor : $\mathbb{N}$ → KleisliCat PMF (InfoFunctor.lean)
Object n ↦ Fin n, the type of n distinguishable outcomes.
Morphism (n ≤ m): the deterministic embedding (each outcome mapped to its image with certainty).

Functor: fC_functor : ℕ → KleisliCat PMF (InfoFunctor.lean)
fC_zero_isInitial: Fin 0 (the empty type) is the initial object.
fC_no_return: for n > 0 there is NO stochastic map Fin n → Fin 0.
AX-G2 as a theorem: no probability distribution lives on the empty type, so no later stage returns to the floor.
Sorry-free. Purity [propext, Classical.choice, Quot.sound].

The empty type is the snap floor here, and it carries the snap's irreversibility for free. A stochastic map into Fin 0 would be a probability distribution on the empty type, and there is none. So while every object has a unique map out of Fin 0 (it is initial), no later stage Fin n with n > 0 has any map back — AX-G2 (source asymmetry) is not assumed but proved, as fC_no_return. For the same reason Fin 0 is not terminal: a terminal object needs a map into it from every object, and Fin 1 has none (kleisli_bottom_not_terminal). Of the three native categories, this is the one where the framework's irreversibility axiom becomes a theorem of the ambient mathematics.

Section V: The MC-1 Correspondence, Scope, and Purity

The fields below are bundled as a single witness, mc1_correspondence; the TopCat limit cone floorConeIsLimit is named beside them, outside the bundle, and F_A (the join-semilattice ℕ of the main document, where 0 is already the initial object) is not a field of it.

Definition: mc1_correspondence : MC1Correspondence (MC1Bridge.lean)
A record collecting the native-category realizations:
• F_D: StateSpace 0 is initial in ModuleCat ℂ; it is terminal as well, the zero object (fD_zero_isTerminal, outside the bundle).
• F_C: Fin 0 is initial in KleisliCat PMF, with no morphism into it from any later stage of the functor (fC_no_return).
• F_B: the clopen balls meet in exactly {0}, a set equation in Q_2 (fB_bottom_is_limit); the limit cone in TopCat is floorConeIsLimit, outside the bundle.
Sorry-free. Purity [propext, Classical.choice, Quot.sound].

This settles the correspondence half of MC-1: across the native categories, the snap floor is realized by each category's own characterization (the zero object in ModuleCat ℂ; initial and not terminal in the Kleisli category; in TopCat terminal, since it has one point, and not initial, and separately the inverse limit of the ball tower), and the four agree on the snap. The identity half, that the algebraic ⊥, the 2-adic 0, the zero module, and the empty type are numerically one object, is retired as ill-typed (object equality across categories does not typecheck and is not invariant under equivalence); what separates the members is proved property by property (seam_unique_among_named, for the named bottoms, in a lattice with no top).

Scope and Purity

This is a formalisation milestone, not new category theory. TopCat, ModuleCat $\mathbb{C}$, and the Kleisli category of PMF are standard Mathlib categories with standard universal properties. The contribution is realizing the ZP snap floor inside each as a genuine functor, in place of the $\mathbb{N}$-indexed depth proxies of the main document.

Purity: [propext, Classical.choice, Quot.sound]. Unlike the choice-free ZP-J results, these functors inherit Classical.choice from Mathlib's topology, inner-product, and PMF libraries. The dependency is a library inheritance, not a new commitment of the construction; whether it is structurally forced is the open question tracked under Classical.choice necessity (ZP-L / ZP-M).

Lean Source Files

TopFunctor.lean — fB_functor, fB_bottom_is_limit (F_B into TopCat).

PadicLimitCone.lean — floorCone, floorConeIsLimit, fB_limit_iso_floor (the TopCat limit cone).

HilbFunctor.lean — fD_functor, fD_zero_isInitial, fD_embed_inner (F_D into ModuleCat $\mathbb{C}$).

InfoFunctor.lean — fC_functor, fC_zero_isInitial, fC_no_return (F_C into KleisliCat PMF).

MC1Bridge.lean — mc1_correspondence (the bundled witness).

TreeSeam.lean — floor_positions_full (the three full positions: a zero object; initial and not terminal; terminal and not initial), fork_positions_distinct, kleisli_bottom_not_terminal; TreeObstructions.lean — padic_bottom_isTerminal, padic_bottom_not_initial, fD_zero_isTerminal.

All in ZeroParadox/ in the public repository.

Endnote: This document is an addendum to ZP-H Categorical Bridge and reads after it. ZP-H locates the snap floor as the categorical bottom across four frameworks using depth-index proxy categories; this addendum realizes the floor inside each framework's native Mathlib category by that category's own characterization — in TopCat the floor {0}, terminal since it has one point and not initial (padic_bottom_isTerminal, padic_bottom_not_initial), and separately the inverse limit of the ball tower (floorConeIsLimit, outside the bundle); the zero object in ModuleCat $\mathbb{C}$; and an initial object that is not terminal in the Kleisli category of distributions (fC_zero_isInitial, kleisli_bottom_not_terminal); the three full positions are stated together in floor_positions_full; mc1_correspondence bundles the two initial objects, fC_no_return (no morphism into the floor from a later stage), and the set equation in Q_2 fB_bottom_is_limit; floorConeIsLimit sits outside it. The floors are not one object, since an equation across categories is ill-typed. All results sorry-free in Lean 4 as of November 2026, footprint [propext, Classical.choice, Quot.sound].