THE ZERO PARADOX

ZP-J Keystone Addendum

The Diagonal Fixed Point: the Lawvere Face-Split and the Well-Foundedness Boundary

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The keystone of the Zero Paradox is that ⊥ plays the same self-referential (diagonal) fixed-point role in every framework, its occupant supplied per face: the Quine atom (CC-2) in set theory, the role a set x = {x} plays, whose occupancy is settled chart by chart (ZeroParadox/Settheory/QuineDichotomy.lean); filled in computation by a class of codes up to equal eval, for a computable map respecting equal eval (codes_vs_classes); the point $v_2(0) = \infty$ in valuation; and the initial object in category theory. Their shared membership in that family (MC-1) is proved per domain for the Quine atom, 2-adic and categorical faces; the Kleene face is carried as a requirement: its occupant is a code from the Rogers / Kleene fixed-point theorem (for every computable transform f of codes there is a code c with eval (f c) = eval c; Rogers' form, computability_face_fixedPoint), a term of another type than ⊥ of the ZPSemilattice; the KleeneStructure class requires a code meeting a periodicity condition (botCode_is_quine), and the reading of that code as the computational face of ⊥ splits (ZeroParadox/Computability/Kleene.lean): its identity half is retired, since the code chart has its own occupant of the Quine-atom role, a class of codes up to equal eval and no single code, for a computable map respecting equal eval (codes_vs_classes), and its running half, that ⊥ executes, is the occurrence commitment; the choice of criteria is a design principle; and the claim that the faces are numerically one object is retired as ill-typed (object equality across categories does not typecheck and is not invariant under equivalence). This addendum is a thin, honest record of two investigations, machine-checked except where marked cited, into the structure of that keystone — both probe-level, both fenced as to exactly what they prove.

The first asks whether the keystone is a literal instance of Lawvere's fixed-point theorem (the recognized home of the diagonal family). The answer is face-dependent: the Set face is machine-checked, and the computability face is cited. The second gives the keystone's structural home in well-founded coalgebra theory (Taylor; Adámek–Milius–Moss): the snap $\bot \to \varepsilon_0$ as a crossing of the well-foundedness boundary. Neither investigation is claimed beyond what Lean verifies; where a result is cited rather than re-proved, the text says so.

Section I: The Keystone and its Recognized Home

The unification of self-referential fixed points across fields is not new with the Zero Paradox. Lawvere (Diagonal Arguments and Cartesian Closed Categories, 1969) showed that Cantor's diagonal, Russell's paradox, the fixed-point lemma at the heart of Gödel's incompleteness and Tarski's undefinability theorem are one move; Yanofsky (A Universal Approach to Self-Referential Paradoxes, Incompleteness and Fixed Points, Bull. Symbolic Logic 9(3), 2003) restated this in plain set-and-function terms across logic and

computation, adding Turing's halting argument and the recursion theorem. Those faces are prior art, cited and not claimed.

What the Zero Paradox adds (and what it does not)

Adds: candidate faces outside the classical scheme — the 2-adic valuation $v_2(0) = \infty$, ε_0, the wheel of fractions — each with a machine-checked axiom footprint; and the location claim (the Gödel inversion), a framing: the Quine-atom and $v_2(0) = \infty$ faces each have their fixed point at their carrier's bottom: the Quine atom is $\bot$ of a ZP-A lattice carrying AFAStructure (t_exec), and 0, the unique fixed point of doubling on $\mathbb{Q}_2$ (q2_unique_fp), is the least element of $\mathbb{Q}_2$ ordered by the 2-adic norm, and the greatest when $\mathbb{Q}_2$ is ordered by valuation, where $v_2(0) = \infty$; the ε_0 role never sits at a floor: it is the first landing (a fixed point of the closure) strictly above one. The value Ordinal.epsilon 0, lifted as $\uparrow\varepsilon_0$, fills the ε_0 role, the first landing strictly above a floor: in WithBot Ordinal, above the boundary model's floor placed at the bottom (phase_epsilon0_isLeast_landing_above_floor), and in WithTop Ordinal, where that floor is placed at the top, the first landing strictly above that order's least element, the snap $\uparrow 0$ (phaseTop_epsilon0_isLeast_landing_above_snap; both ZeroParadox/Multihomed/BoundaryOrder.lean). In Ordinal that value differs from the bottom role's occupant 0 (epsilon0_ne_bot, epsilon0_ne_zero).

Does not add: the unification itself (Lawvere/Yanofsky), nor any identification of the four faces as one object — that identity is retired as ill-typed (object equality across categories does not typecheck and is not invariant under equivalence). What MC-1 carries is family membership, proved per domain for the Quine atom, 2-adic and categorical faces (the Kleene face carried as a requirement).

Section II: The Lawvere Face-Split (Set face machine-checked; computability face cited)

Lawvere's theorem (Mathlib's Function.exists_fixed_point_of_surjective) says a point-surjection $\alpha \to (\alpha \to \beta)$ — the diagonal hypothesis — forces every endofunction of β to have a fixed point. The keystone's fixed points are posited (the self-application has $\bot$ as its fixed point); Lawvere's are derived from a surjection. So the question is whether each face supplies that surjection. The verdict is face-dependent.

In Set: no face is a Lawvere instance (ZeroParadox/Category/Lawvere.lean)

For any nontrivial type, a Lawvere witness cannot exist: it would force every endofunction to have a fixed point, impossible once there are two distinct elements (Cantor).

nontrivial_lattice_no_witness — the lattice / set-theoretic face: no witness on a nontrivial $\bot$-pointed lattice.

q2_no_witness — the 2-adic face: no witness on $\mathbb{Q}_2$.

So in Set the keystone's $\bot$ is a posited fixed point of one specific self-map, not a Lawvere-derived one. The "diagonal fixed point" name is an analogy at this face.

fixedPoint_of_witness, no_witness_of_fixedPointFree: fully axiom-free; the face verdicts nontrivial_lattice_no_witness and q2_no_witness carry [propext, Classical.choice, Quot.sound].

In computability: a genuine instance, in multi-valued form (cited; Lean proves Rogers' theorem classically)
computability_face_fixedPoint — every computable self-map on codes has a fixed point up to eval: two codes computing the same function. This wraps Mathlib's Nat.Partrec.Code.fixed_point (Rogers / Kleene's recursion theorem).
The qualifier is load-bearing. A literal fixed point is not on offer — c ↦ pair(c, c) (the map fun c => Code.pair c c, code_pair_diag_fixedPointFree) is total, computable, and returns its own input for no c. The escape from the Cantor obstruction is a restriction on which MAPS exist, not on which type eval lands in: no computable self-map on codes is eval-fixed-point-free, so the diagonal that would refute the witness has no computable representative and the obstruction cannot fire (up to eval; under literal equality that same map refutes the witness, the example after no_witnessRel_of_admissible_fpf in ZeroParadox/Category/DiagonalWitness.lean). This is ZP-K's face (the Kleene quine).
Footprint: [propext, Classical.choice, Quot.sound], inherited from Mathlib.
Prior art, cited and not claimed: this is the effective-topos / Turing-category setting (Hyland, The effective topos, 1982; Cockett–Hofstra, Introduction to Turing categories, 2008; see ZeroParadox/Category/DiagonalWitness.lean). Bauer (On fixed-point theorems in synthetic computability, Tbilisi Math. J. 10(3), 2017), working in synthetic computability, whose axioms hold in Hyland's effective topos, derives the Kleene–Rogers Recursion Theorem from a fixed-point theorem of Lawvere's form for multi-valued maps (Thm 5.2, Cor 5.3). Lean proves the classical, external form of Rogers' theorem; the Lawvere-form derivation itself lives inside synthetic computability, where excluded middle fails.

The verdict, plainly: the test is category-relative. In Set no face is a Lawvere instance; in the effective (computability) category the recursion theorem is a genuine one (cited: Bauer 2017, multi-valued form; Lean proves Rogers' theorem classically). The keystone therefore unifies a shape (the diagonal), not a single mechanism. What MC-1 carries is family membership, proved per domain for the Quine atom, 2-adic and categorical faces (the Kleene face carried as a requirement), with the choice of criteria the design commitment.

Section III: The Well-Foundedness Boundary

The keystone's structural home is well-founded coalgebra theory. A relation is encoded as a powerset coalgebra; well-founded means no infinite descending chain — no cycle. The Quine atom (CC-2), a point x = {x}, is exactly the minimal non-well-founded coalgebra: the self-loop x ∈ x, the back edge. Taylor (Well-founded coalgebras and recursion) proves well-founded ⇒ recursive as his Recursion Theorem (Thm 36, p. 15), and states the converse in a topos — there well-foundedness is necessary for recursion (Prop 111, p. 6). Adámek–Milius–Moss (On Well-Founded and Recursive Coalgebras, 2020, arXiv:1910.09401v2) prove the forward direction as the General Recursion Theorem (Thm 7.2, p. 27). Their converse (§ 8) always asks the endofunctor to preserve inverse images, and then takes one of several routes: the category may have universally smooth monomorphisms with the functor carrying a pre-fixed point (Thm 8.1), or the category may have a subobject classifier (Thm 8.6, which is Taylor's); a third covers functors on vector spaces, whose category has neither (Thm 8.12). Under Thm 8.1's assumptions the characterizations coincide — well-founded ⇔ recursive ⇔ a morphism to the initial algebra (Cor 8.2). Reading the necessity direction as one cannot recurse through ⊥ is this framework's gloss, not Taylor's: Prop 111 names no bottom element.

In that language the binary snap $\bot \to \varepsilon_0$ is a crossing of the well-foundedness boundary: from the non-well-founded floor ($\bot$, the self-loop, where recursion cannot reach: floor_not_wellFounded, below) to the well-founded ascent (the ε_0 ordinal tower, recursively generated). The Zero Paradox formalizes this at two levels.

Relation level (ZeroParadox/Multihomed/Boundary.lean)
floor_not_wellFounded — the self-application floor is non-well-founded ($\bot$ self-loops under selfApp). Fully axiom-free.
ascent_wellFounded — the ordinal ascent is well-founded (Ordinal.lt_wf).
snap_crossing — on a single carrier, the floor is the sole non-accessible point; every post-snap state is accessible. The snap exits the unique non-well-founded point into the well-founded ascent.

Both levels at once (ZeroParadox/Multihomed/BoundaryBridge.lean)
snap_boundary_two_registers — the crossing witnessed in two registers at once: the relation/carrier level above, and ZP-P's categorical μ/ν fork (categorical_fork_strict: the initial algebra empty, the final coalgebra inhabited — the self-referential element lives in ν, not μ).

Honest scope — what is proved, and what is committed
Proved: the relation-level boundary and the QPF μ/ν bridge above (floor_not_wellFounded is axiom-free; the rest carry Classical.choice from Mathlib's ordinal and QPF machinery).
No new commitment: that the snap is this crossing is a faithful model whose content is the two proven endpoints, plus the reading of those endpoints as ZP's own $\bot$ (the lattice chart's $\bot$) and ε_0 respectively — a role assignment the framework already carries (for the $\bot$ endpoint, MC-1 family membership; for the ε_0 endpoint, theorems about the model itself in both of its extreme orderings, floor below as WithBot Ordinal (phase_epsilon0_isLeast_landing_above_floor) and floor above as WithTop Ordinal (phaseTop_epsilon0_isLeast_landing_above_snap) (ZeroParadox/Multihomed/BoundaryOrder.lean), so no cross-type identity is involved), not a fresh one. In the carrier Ordinal the bottom role is filled by 0. The value Ordinal.epsilon 0, lifted as $\uparrow\varepsilon_0$, fills the ε_0 role, the first landing strictly above a floor: in WithBot Ordinal, above the boundary model's floor placed at the bottom (phase_epsilon0_isLeast_landing_above_floor), and in WithTop Ordinal, where that floor is placed at the top, the first landing strictly above that order's least element, the snap $\uparrow 0$ (phaseTop_epsilon0_isLeast_landing_above_snap; both ZeroParadox/Multihomed/BoundaryOrder.lean). In Ordinal that value differs from the bottom role's occupant 0 (epsilon0_ne_bot, epsilon0_ne_zero). The floor endpoint is tied to the lattice chart's $\bot$ (floor_not_wellFounded, axiom-free); the single-carrier Phase is the illustrative toy model, where non-well-foundedness localizes at the floor by construction.

Section IV: What Is Not Formalized (and an Open Invitation)

The full Taylor coalgebraic statement — $\bot$ as a non-well-founded coalgebra in the broken-pullback sense, with the General Recursion Theorem and its converse — is deliberately not formalized here. Neither Pataraia's fixed-point theorem nor a recursion theorem for well-founded coalgebras was located in the pinned Mathlib as of August 2026, searched by declaration name and type (this project has since proved Taylor's Thm 118 and the Pataraia induction of his Cor. 119 with no axioms, as the contribution point below

records); the next-time operator on subobject lattices is built in this project (ZeroParadox/Category/NextTimeCategorical.lean), though it is not upstreamed. The depth result is therefore cited (Taylor; Adámek–Milius–Moss), not re-proved. What is given here is the relation-level boundary and the QPF bridge: a best effort that names its own boundary.

Open contribution point

Pataraia's fixed-point theorem (Taylor Thm 118: a monotone self-map of a directed-complete order with a least element has a least fixed point) and the induction principle Taylor derives from it (his Cor. 119, Pataraia induction) are proved in this project with no axioms at all: pataraia_least_prefixedPoint gives a fixed point below every pre-fixed point, hence the least fixed point, and pataraia_induction_constructive gives the induction principle (ZeroParadox/Order/PataraiaChoiceFree.lean, ported from TypeTopology's Agda modules Various.Pataraia, Escardó's, for the γ step, and Various.Pataraia-Taylor, Escardó and de Jong's, for Taylor's condition TC). The first full published proof is Escardó, Joins in the frame of nuclei, Applied Categorical Structures 11 (2003) 117–124: Pataraia's theorem is his Cor 2.1, and the induction is the second clause of his Thm 2.2, stated there for sets of inflationary maps. A classical route through Mathlib's Bourbaki–Witt theorem is also kept. For the least fixed point it is Markowsky's (Algebra Universalis 6 (1976), Thm 9(i), proved there without choice), and the same Lean adapter gives the induction; these Lean proofs carry Classical.choice in their footprint (dcpo_exists_least_fixedPoint and pataraia_induction, ZeroParadox/Order/PataraiaFromBourbakiWitt.lean). Neither result was located in the pinned Mathlib as of September 2026 (searched by statement shape and by name). What upgrades this best-effort bridge to the full coalgebraic statement is the General Recursion Theorem and its converse, with well-founded coalgebras. That, upstreaming Pataraia's theorem, and upstreaming the next-time operator already built here would each be a reusable Lean contribution independent of the Zero Paradox. Contributions are welcome, to this project and to Mathlib as a whole; the precise missing pieces are named above.

Endnote: This document is an addendum to ZP-J Self-Reference and reads after it. ZP-J established the valuation condition $v_2(\bot) = \infty$ and $\bot$ as the occupant of the Quine-atom role (CC-2); this addendum records two probes into the keystone's structure — the Lawvere face-split and the well-foundedness boundary — machine-checked except where marked cited, each fenced as to exactly what it proves. Lean sources: ZeroParadox/Category/Lawvere.lean, ZeroParadox/Category/DiagonalWitness.lean, ZeroParadox/Multihomed/Boundary.lean, ZeroParadox/Multihomed/BoundaryBridge.lean, ZeroParadox/Order/PataraiaFromBourbakiWitt.lean, ZeroParadox/Order/PataraiaChoiceFree.lean, all sorry-free in Lean 4 as of September 2026.