ZP-J Wheel Addendum

THE ZERO PARADOX
ZP-J Wheel Addendum

The Wheel of Fractions is a Wheel

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A wheel (Carlström 2001) is an algebraic structure that extends a commutative ring by making division a total operation: every element, including 0, has a reciprocal /x, so /0 becomes a defined first-class element rather than an error. The two elements this produces — $\infty = /0$ (the reciprocal of zero) and $\bot = 0 \cdot /0$ (an absorbing "undefined" element) — are what distinguish a wheel from a field. ZP-J Self-Reference left open which way of totalising division the Zero Paradox's $0/\infty$ exchange gives rise to — Bergstra & Tucker's WADT 2020 paper places wheels and meadows in one signature alongside a third totalisation; this addendum compares the two most directly relevant to that exchange: a wheel, which adjoins ∞ and $\bot$ as two distinct new elements, or a meadow (Bergstra & Tucker), which adjoins nothing at all and instead declares $0^{-1} = 0$, staying inside the original number system. This addendum settles that question.

The main result is ZeroParadox.instWheel (ZeroParadox/Algebra/WheelFrac.lean): for any commutative ring A and any multiplicative submonoid S, the wheel of fractions $\odot_S A = (A \times A)/\equiv_S$ satisfies every axiom of Carlström's Definition 1.1. The companion result ZeroParadox.inf_ne_bot shows that, whenever $0 \notin S$, the two special elements stay distinct ($\infty \neq \bot$) — so the construction is a non-trivial wheel. The construction is Carlström's; the contribution here is a faithful, machine-verified encoding of it that is also free of the axiom of choice (footprint [propext, Quot.sound]), situated as the algebraic form of the $0/\infty$ exchange.

Section I: The Wheel Axioms (Carlström Definition 1.1)

A wheel is a set W with two binary operations + and ·, a unary involution / (reciprocal), and two constants 0 and 1, subject to eight axioms. Carlström's Definition 1.1 packages the additive and multiplicative parts as commutative monoids; the ZP Wheel typeclass unbundles those two monoid axioms into their separate equational laws, giving 14 fields that are equivalent to Carlström's eight. The unbundling is bookkeeping only — no axiom is added, removed, or weakened.

Typeclass: Wheel (ZeroParadox/Algebra/Wheel.lean)
class Wheel (W : Type*) where
wadd, wmul : W → W → W -- + and ·
winv : W → W -- the involution /
wzero, wone : W -- 0 and 1

Typeclass: Wheel (ZeroParadox/Algebra/Wheel.lean)
-- W1–W3: (W, +, 0) commutative monoid [Carlström (1)]
-- W4–W6: (W, ·, 1) commutative monoid [Carlström (2), monoid part]
-- W7: /(/x) = x [Carlström (2), involution]
-- W8: /(x·y) = /x · /y [Carlström (2), involution]
-- W9: weakened distributivity [Carlström (3)]
-- W10: (x·/y + z) + 0·y = (x + y·z)·/y [Carlström (4)]
-- W11: 0·0 = 0 [Carlström (5)]
-- W12: (x + 0·y)·z = x·z + 0·y [Carlström (6)]
-- W13: /(x + 0·y) = /x + 0·y [Carlström (7)]
-- W14: x + 0·/0 = 0·/0 [Carlström (8)]

The last group of axioms is what makes division total. W7 and W8 make / an involution that distributes over multiplication; W9–W14 govern how the two derived elements $\infty = /0$ and $\bot = 0 \cdot /0$ interact with + and ·. Keeping those two apart is not a stylistic preference over some rival algebra: by Carlström's Proposition 4.4 (p. 25), if any two of 0, 1, /0 and 0/0 are equal in a wheel, the wheel is trivial — it has exactly one element. So $\infty \neq \bot$ is the condition for the structure to have more than one element at all. The same statement is machine-checked over the Wheel typeclass in ZeroParadox/Algebra/Wheel.lean §V: identifying ∞ with $\bot$ proves $\forall\ x\ y,\ x = y$.

Section II: The Wheel of Fractions ⊙_S A

The wheel of fractions generalises the field-of-fractions construction so that it survives division by zero. Start from a commutative ring A and a multiplicative submonoid $S \subseteq A$ (a set containing 1 and closed under multiplication). Form pairs $(x, y) \in A \times A$ — read as the formal fraction x/y — and quotient by the relation $\equiv_S$ below.

Construction (Carlström 2001, pp. 4–5)
$\odot_S A = (A \times A) / \equiv_S$
$(x, y) \equiv_S (x', y') \Leftrightarrow \exists\ s, s' \in S,\ s \cdot x = s' \cdot x' \wedge s \cdot y = s' \cdot y'$
$0 = [0, 1]\ 1 = [1, 1]$
$[x,y] + [x',y'] = [x \cdot y' + x' \cdot y,\ y \cdot y']$
$[x,y] \cdot [x',y'] = [x \cdot x',\ y \cdot y']$

Construction (Carlström 2001, pp. 4–5)

/[x,y] = [y, x] (the involution is pair-swap)

Then /0 = [1,0] = ∞ and 0·/0 = [0,0] = ⊥.

The choice of relation matters. The naive cross-multiplication $x \cdot y' = x' \cdot y$ — the one that defines equality of ordinary fractions — is not an equivalence relation on a general commutative ring: transitivity fails without a cancellation law. The submonoid-quotient relation $\equiv_S$ repairs this by witnessing each identification with elements of S, and it is provably reflexive, symmetric, and transitive (ZeroParadox.srel). Each of the five operations is then well-defined on the quotient — the proofs that they respect $\equiv_S$ are the bulk of the formalisation.

Section III: ⊙_S A is a Wheel

With the operations shown well-defined, every one of the 14 typeclass fields is discharged. The additive and multiplicative monoid laws and the distributivity and zero-term axioms (W9–W14) all reduce, after lifting representatives, to ring identities in A closed by ring normalisation; the two involution laws (W7, W8) hold definitionally because pair-swap is its own inverse and commutes with the componentwise product.

Theorem: ZeroParadox.instWheel (ZeroParadox/Algebra/WheelFrac.lean)

∀ {A : Type*} [CommRing A] (S : Submonoid A),

Wheel ($\odot_S$ A)

For every commutative ring A and multiplicative submonoid S, the wheel of fractions $\odot_S$ A satisfies all 14 fields of the ZP Wheel typeclass — equivalently, all eight axioms of Carlström's Definition 1.1.

Sorry-free. Lean purity: [propext, Quot.sound] — Classical.choice-free.

Section IV: Wheel, Not Meadow

A wheel whose two special elements coincide is the one-element wheel (Carlström, Proposition 4.4). For the wheel of fractions the trigger is exactly $0 \in S$: taking $s = 0$ and $s' = 1$ as the witnesses identifies the fraction [1,0] with the fraction [0,0], and every other pair with them. What is proved below is one direction — $0 \notin S$ keeps them apart. The converse is Carlström's own observation that if $0 \in S$ then the relation is improper and $\odot_S$ A is trivial (p. 5). The natural hypothesis $0 \notin S$ holds whenever S is the complement of a prime ideal, the usual case.

Proposition: ZeroParadox.inf_ne_bot (ZeroParadox/Algebra/WheelFrac.lean)

∀ {A : Type*} [CommRing A] (S : Submonoid A),

$(0 : A) \notin S \to \infty \neq \bot$ in $\odot_S$ A

Proposition: ZeroParadox.inf_ne_bot (ZeroParadox/Algebra/WheelFrac.lean)

When $0 \notin S$, the reciprocal of zero ($\infty = /0$) and the absorbing element ($\bot = 0 \cdot /0$) are distinct. The construction is therefore a non-trivial wheel.

Proof: $\infty = \bot$ would yield witnesses $s, s' \in S$ with $s \cdot 1 = s' \cdot 0$, forcing $s = 0$ and hence $0 \in S$ — contradicting the hypothesis.

Sorry-free. Lean purity: [propext, Quot.sound] — Classical.choice-free.

Remark — What a meadow actually is

A meadow answers the same question a different way, and it adjoins no element at all: Bergstra, Hirshfeld and Tucker define one as "a commutative ring with a total inverse operator satisfying two equations which imply $0^{-1} = 0$." The carrier is unchanged, so there is no pair of new elements available to identify. At the level of terms the identification still happens: 0^{-1} and $0 \cdot 0^{-1}$ both evaluate to the existing 0, so a meadow equates the two terms a wheel keeps apart. Bergstra and Ponse fix the default: "by default a 'meadow' is assumed to be an involutive meadow", the involutive ones being exactly those with $0^{-1} = 0$.

A common meadow is the other variant: it adjoins one element — an absorbing element, written a, serving as the inverse of zero. Bergstra and Ponse record what that costs the inverse operation: "the inverse function of a common meadow is not an involution because $(0^{-1})^{-1} = a$," and, setting the two constructions side by side, "wheels are involutive whereas common meadows are non-involutive."

Both directions of why no carrier is at once a non-trivial wheel and an involutive meadow are carried as machine-checked examples in ZeroParadox/Algebra/Wheel.lean §V, with controls showing both hypotheses are load-bearing. The statements are scoped there by their binders, which is why they are cited rather than restated here. The one overlap is degenerate — Bergstra, Hirshfeld and Tucker's own term (Def. 2.4, p. 7, for the meadow concept it qualifies) is the trivial meadow (p. 12) — and it is stated rather than denied: the one-element algebra is a wheel in which $\infty = \bot$ and also satisfies the meadow equations; Bergstra and Ponse "do not require a meadow to satisfy the separation axiom $0 \neq 1$" (arXiv:1406.6878, §1, p. 2). Both facts are examples in the same section. Searched 2026-09-13 over the sources listed below: no structural connection between the two constructions located; Bergstra and Ponse report not having found one for common meadows.

Sources: J. A. Bergstra, Y. Hirshfeld and J. V. Tucker, "Meadows and the equational specification of division", arXiv:0901.0823 (abstract); J. A. Bergstra and A. Ponse, "Division by Zero in Common Meadows", arXiv:1406.6878v4, §1 (p. 2) and §4 (p. 14). The concept originates with J. A. Bergstra and J. V. Tucker, "The rational numbers as an abstract data type", JACM 54(2), 2007. J. A. Bergstra and J. V. Tucker, "The Wheel of Rational Numbers as an Abstract Data Type", WADT 2020, LNCS 12669.

Remark — The Connection to ZP-J

In ZP-J, the Quine-atom role is read in several charts, not as one equation across them: a point that steps to itself and to nothing else (Motion.SelfOnlyAt). In the 2-adic integers its occupant under $x \mapsto 2x$ is 0, where $v_2(0) = \infty$ (z2_two_views, ZeroParadox/Valuation/Scale.lean); a valuation structure sets $\mathrm{val}(\bot) = \infty$ (val_bot); in ZFC^{-} + AFA the role is the Quine atom (CC-2), a set $x = \{x\}$, whose lattice-level occupant is $\bot$ (ZeroParadox/Settheory/QuineDichotomy.lean). The wheel of fractions is the algebraic face of that point: $/0$ is defined and distinct from the absorbing $\bot$, and $/\infty = 0$ (winv_wheelInf) closes the $0/\infty$ exchange. The concrete carrier ZPWheelElem (ZeroParadox/Algebra/Wheel.lean §III–VI) makes this explicit on the rationals extended with ∞ and $\bot$, where $\mathrm{val}(x) = \infty \Leftrightarrow /x = \infty$ is proved directly (zpw_top_val_iff_inv_is_inf). This addendum's headline result is the general construction over an arbitrary commutative ring; the concrete carrier is the illustrative special case.

Section V: Scope, Purity, and Relationship to Carlström

Two scope boundaries are worth stating plainly. This is a formalisation, not new mathematics. The wheel of fractions and the theorem that it is a wheel are Carlström's. What is contributed here is a machine-checked encoding faithful to Definition 1.1, with a verified axiom footprint, placed in the context of ZP-J's $0/\infty$ exchange. Ring structure is an input, not a conclusion. The construction starts from a commutative ring and a submonoid; it does not derive wheel structure from the ZP lattice axioms alone. The bridge typeclass that would state such a derivation, WheelValuationStructure (ZeroParadox/Algebra/Wheel.lean §VII), is defined but its condition $\mathrm{val}(0) = \top$ is an assumed axiom, motivated by the ZP argument rather than type-checked as necessary.

Lean Source Files

ZeroParadox/Algebra/WheelFrac.lean — rel, srel, the five quotient operations (waddF, wmulF, winvF), instWheel, inf_ne_bot. The headline results of this addendum.
ZeroParadox/Algebra/Wheel.lean — the Wheel typeclass (Carlström Def 1.1, 14 fields), the derived elements wheelInf / wheelBot, the concrete carrier ZPWheelElem, and the correspondence $\mathrm{val}(x) = \top \Leftrightarrow /x = \infty$, zpw_top_val_iff_inv_is_inf.
Both files are in the public repository.

Axiom Footprint (headline results: instWheel, inf_ne_bot)

[propext, Quot.sound] — Classical.choice-free.
propext — propositional extensionality (standard in Lean 4)
Quot.sound — quotient soundness (standard in Lean 4; the construction is a quotient, so this is expected and unavoidable)
No Classical.choice. No Dependent Choice. No set-theoretic assumptions.
This scope is the headline results, not the whole document. Measured against the corpus, the concrete-carrier theorems zpw_inv_zero_eq_inf, zpw_zero_mul_inf_eq_bot and zpw_top_val_iff_inv_is_inf each carry Classical.choice; zpw_inf_ne_bot is [propext].

Remark R-J.W — Relationship to Carlström's Theorem

The name and the original construction are Setzer's, not Carlström's. Carlström records the lineage on his own p. 3: Edalat and Potts adjoined $\infty = 1/0$ and $\bot = 0/0$ to the reals; Martin-Löf proposed building them into the construction of the rationals from the integers; "such structures were called 'wheels' (the term inspired by the topological picture $\odot$ of the projective line together with an extra point 0/0) by Setzer [Set97], who showed how to modify the construction of fields of fractions from integral domains so that wheels are obtained instead of fields." ([Set97] is Carlström's key for A. Setzer, Wheels (draft), 1997.)

Carlström is the generalizer: "in this paper, we generalize Setzer's construction, so that it applies not only to integral domains, but to any commutative semiring." That is Wheels — On Division by Zero (Research Reports in Mathematics No. 11, Department of Mathematics, Stockholm University, 2001; a Licentiate thesis), where Definition 1.1 (p. 5) gives the eight wheel axioms and the wheel of fractions is constructed (§1.2, §4.2). The work was later published as Wheels — on division by zero, Mathematical Structures in Computer Science 14(1):143–184, 2004. His theorem covers commutative semirings; the Lean encoding in ZeroParadox/Algebra/WheelFrac.lean is stated for commutative rings, so it is the narrower case of his result, not the same generality. The result here is that theorem, encoded in Lean 4 against a typeclass that reproduces his Definition 1.1 field for field, and discharged without the axiom of choice. The encoding is what is new: a third party can read the 14 fields against Carlström's eight axioms and confirm the correspondence, and can read the axiom footprint to confirm the choice-free claim. The condition $\mathrm{val}(0) = \top$ stays an assumed class field, motivated and not forced (ZeroParadox/Algebra/Wheel.lean §VII–VIII); given multiplicativity it holds exactly when the valuation is not the zero map (the examples closing §VII-b there).

Endnote: This document is an addendum to ZP-J Self-Reference and reads after it. ZP-J established the valuation condition $\mathrm{val}(\bot) = \infty$ and $\bot$ as the lattice-level occupant of the Quine-atom role (CC-2); this document gives the algebraic form of the $0/\infty$ exchange, the wheel of fractions, and the machine-verified proof that the construction is a wheel and a non-trivial one. All results sorry-free in Lean 4 as of September 2026. Footprints are per result, and [propext, Quot.sound] is the headline results'; the Axiom Footprint box above gives the scope and the measured exceptions.