THE ZERO PARADOX

ZP-L: Incomputability Convergence

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All theorems §I–§VII proved sorry-free in Lean 4. Axiom footprint: [propext, Classical.choice, Quot.sound] throughout this document's own Lean file (Gentzen.lean). Across the ZP layers the footprints are not uniform — ZP-K Section IV tabulates them.

ZP-L establishes four results connecting the formal axioms of the ZP framework to standard results in ordinal theory and computability. First, the axiom footprints of the layers tabulated in Section I are surveyed, and they are not uniform; ZP-K Section IV tabulates the measured footprints. Second, Rogers' fixed-point theorem for a total computable transformation (Mathlib fixed_point; inter-derivable with Kleene's second recursion theorem, fixed_point$_2$) is formalized as a wrapper, formalizing the computational fixed-point structure. Third, the ordinal ε_0 is fully characterized as the first fixed point of $\alpha \mapsto \omega^\alpha$ and the limit of the tower ω, ω^ω, ω^{ω^ω}, ... Fourth, cnfToZp2 maps ordinals below ε_0 into $\mathbb{Z}_2$ by recursion on their Cantor normal form. From tower stage 1 on, the 2-adic valuation of the image climbs with ordinal height toward ∞ while the 2-adic norm falls toward 0; the images converge to 0, the point the seed stage 0 itself maps to (snap_arc_z2_loop, ZeroParadox/Ordinal/CnfBridge.lean). That 0 is least in $\mathbb{Z}_2$'s norm preorder and greatest in its valuation preorder (z2_two_views, ZeroParadox/Valuation/Scale.lean), the two readings just given.

The central result (§VII) is the canonical snap map: $\phi\,\alpha =$ if $\alpha < \varepsilon_0$ then c_0 else c_1 simultaneously satisfies all five conditions — monotone, tower-aligned, fixed-point-respecting, snapping at ε_0, and ε_0 minimal. All conditions are verified without free hypotheses for this witness. Proved sorry-free, axiom footprint [propext, Classical.choice, Quot.sound] throughout ZeroParadox/Ordinal/Gentzen.lean, measured.

Section I: Axiom Footprint Convergence

Non-constructibility appears across the layers listed below. The axiom footprints are not uniform: ZP-K Section IV tabulates the measured footprints, and the ZPJ/K row below names a witness on each side — bot_self_mem, which measures no axioms, and botCode, which carries them.

Layer	Formal Language	Result cited, or status
ZPB	Topology	C3: no continuous path $\bot \to x \neq \bot$
ZPC	Information Theory	L-INF: surprisal unbounded in depth n, every value finite (l_inf); at Q_2's floor 0 the complexity cxQ2 is $\top$ (infinitude_forces_infinite_complexity, instance q2InfinitudeFloor). That c_0 corresponds to that floor is a reading
ZPJ/K	Set Theory + Computation	bot_self_mem (AFA); botCode (Kleene)

Layer	Formal Language	Result cited, or status
ZPI	Algorithmic IT	K uncomputable. ZP-I § II.B refutes the valuation-complexity bridge (unbounded 2-adic valuation of S_n implies $K(S_n \mid n) / \lvert S_n \rvert \to 1$) with $S_n = 2^n$, the powers of 2 that §IV's tower images take; the valuation half is in Lean, the complexity half is argued outside it

Remark: K is not computed in Lean

Kolmogorov complexity K is not computed in Lean in this framework. The AFA/Kleene route needs no K: da1_paths_unified is a conjunction of two facts, that ⊥ of the lattice is its Quine atom (bot_is_quine_atom) and that the code botCode is a computational quine (the KleeneStructure field botCode_is_quine).

Axiom footprint evidence — the following ZP-K theorems all carry [propext, Classical.choice, Quot.sound]:

t_comp (T-COMP three-way equivalence; the computational face is an assumption, not a clause)

da1_paths_unified

isComputationalQuine_undecidable

infinite_quine_family

Gentzen.lean carries that footprint throughout, measured — and it is not inherited from the theorems above. Mathlib's ordinal theory carries it on its own: Ordinal.nfp and Ordinal.epsilon_zero_eq_nfp each measure [propext, Classical.choice, Quot.sound] with no computability module in scope. ZP-K Section IV tabulates the measured footprints for the computability side.

Section II: Rogers Fixed-Point Stability

Rogers' fixed-point theorem (Mathlib fixed_point; inter-derivable with Kleene's second recursion theorem, fixed_point_2) states that any total computable transformation of a code has a behavioral fixed point: a code c such that running f(c) and running c produce the same partial function. For any computable transformation, at least one fixed-point code exists.

Theorem: roger_fixed_point_stability (Gentzen.lean §II)

For any computable f : Code → Code,

∃ c : Code, eval (f c) = eval c

Proof: wrapper around roger_fixed_point_exists.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Note: this is an existential result. The specific code c is not constructively produced. ZP-K Section IV tabulates the measured footprints.

Section III: The Ordinal ε_0

The ordinal ε_0 is the smallest fixed point of the map $\alpha \mapsto \omega^\alpha$. It is the supremum of the tower $0, 1, \omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ — each stage is strictly below ε_0, and ε_0 is not reached by any finite iteration. This section is entirely within Lean scope via Mathlib's ordinal machinery.

Definitions (Gentzen.lean §III)
epsilonZero : Ordinal := Ordinal.epsilon 0 (= nfp (ω^·) 0 = veblen 1 0)
fundamentalSeq : ℕ → Ordinal := fun n ↦ (α ↦ ω^α)^[n] 0
Explicit stages:
fundamentalSeq 0 = 0
fundamentalSeq 1 = 1
fundamentalSeq 2 = ω
fundamentalSeq 3 = ω^ω
fundamentalSeq (n+1) = ω^(fundamentalSeq n)

Theorem: epsilonZero_fixedPoint
$\omega^{\varepsilon_0} = \varepsilon_0$
Proof: Ordinal.omega0_opow_epsilon 0 (Mathlib).
Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: epsilonZero_eq_nfp
$\varepsilon_0 = \text{nfp}\ (\omega^\cdot)\ 0$
Proof: Ordinal.epsilon_zero_eq_nfp (Mathlib).
Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: epsilonZero_eq_iSup
$\varepsilon_0 = \bigsqcup n : \mathbb{N}$, fundamentalSeq n
Proof: iSup_iterate_eq_nfp (Mathlib).
Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: epsilonZero_tower_lt
$\forall\ n : \mathbb{N}$, fundamentalSeq $n < \varepsilon_0$

Theorem: epsilonZero_tower_lt

Every finite stage of the tower is strictly below ε_0.

Proof: Ordinal.iterate_omega0_opow_lt_epsilon_zero (Mathlib).

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: epsilonZero_le_fixedPoint

$\forall\, b : \text{Ordinal},\ \omega^{\wedge} b = b \to \varepsilon_0 \le b$

ε_0 is the least fixed point of $\omega^{\wedge}\cdot$ above 0.

Proof: Ordinal.epsilon_zero_le_of_omega0_opow_le (Mathlib).

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: fundamentalSeq_strictMono

$\forall\, n : \mathbb{N},\ \text{fundamentalSeq}\ n < \text{fundamentalSeq}\ (n + 1)$

The tower is strictly monotone.

Proof: by induction; successor step uses isNormal_opow.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Remark R-L.1: Proof-Theoretic Alignment

Two results pin ε_0 as the proof-theoretic ordinal of Peano Arithmetic — the limit of how far PA's proofs of transfinite induction reach — and they run in opposite directions. Gentzen reports, crediting Hilbert-Bernays, that transfinite induction up to any ordinal strictly below ε_0 is provable in PA. And transfinite induction up to ε_0 itself is not provable in PA, which Gentzen proved directly in 1943. Neither half alone locates the boundary: a ceiling with no witnesses beneath it pins nothing, and witnesses with no ceiling pin nothing either. A third result stands behind them — transfinite induction up to ε_0 suffices to prove PA consistent (1936) — and it is also the indirect route to the unprovability half, via Gödel's theorem together with that 1936 result. This is not claimed or proved here.

ZP-L shares its ε_0 with proof theory: it is Mathlib's Ordinal.epsilon 0 (epsilonZero, ZeroParadox/Ordinal/Gentzen.lean), and the ordinals below it are written in Cantor normal form, the notation ordinal analysis also uses (Gentzen 1943, §1). What is ZP-L's own is the snap: a monotone map sending the tower's stages to c_0 fires nowhere below ε_0 (snap_threshold_is_epsilon_zero), and its firing at ε_0 is the hypothesis $h\varepsilon_0$ (ZeroParadox/Ordinal/Incompleteness.lean § II). That the snap's least possible firing point is the ordinal of Gentzen's consistency proof is offered as a structural observation, not as a link between the two results.

Remark R-L.2: Surreal Numbers

Every ordinal is a surreal number (Conway, 1976), so ε_0 lives in the surreal number field No. No satisfies the axioms of a real-closed field, and by Tarski's completeness theorem for the theory of real-closed fields, every first-order sentence in the language of ordered fields that holds in $\mathbb{R}$ also holds in No, and vice versa.

Remark R-L.2: Surreal Numbers

ZP-F (The Counterexamples) proves that in any linearly ordered field 0 has no first step above it, AX-B1's form (f_snap_impossible, which, despite its name, states that no linearly ordered field has a least positive element). Its ε_0 is a bound variable over the positive elements of the field, each a candidate first step above the field's 0, and does not name the ordinal ε_0 of this section; in No that ordinal is one of them, and $\varepsilon_0/2$ lies below it. This result applies directly to No considered as a linearly ordered field. Note the scope: T-SNAP's SHAPE does not need a first step (t_snap_given holds at a step taken in a densely ordered join-semilattice, $\mathbb{Q}_{\geq 0}$ under max, whose $\bot$ is 0; ZeroParadox/Order/Snap.lean); what an ordered field lacks is AX-B1's first step.

The surreals therefore contain ε_0 as an ordinal while simultaneously satisfying the density condition that rules out a first step above 0 in their ordered field structure. Both structures coexist in No: ε_0 is present as an ordinal, and the field density that rules out a first step is present in the field structure. The two results apply to different structural aspects of No.

Remark R-L.3: Hyperreals and Łoś's Theorem

The hyperreals $^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/U$ (ultrafilter U on $\mathbb{N}$) satisfy a transfer result: by Łoś's theorem, every first-order sentence true in $\mathbb{R}$ holds in $^*\mathbb{R}$. Łoś's theorem applies to the full first-order theory, so any first-order property of $\mathbb{R}$ — including field density — transfers.

Field density transfers: between any two hyperreals there is another. The density condition that rules out a first step above 0 in $\mathbb{R}$ (proved in ZP-F) therefore holds in $^*\mathbb{R}$ as well.

The non-standard naturals $^*\mathbb{N}$ contain infinite elements, but every infinite $H \in {}^*\mathbb{N}$ has a predecessor $H-1$ in $^*\mathbb{N}$, so none of them is limit-like. ε_0, by contrast, is a limit ordinal: it has no predecessor and is the supremum of the tower below it.

The monad of 0 in $^*\mathbb{R}$ — the external set $\{\varepsilon : |\varepsilon| < 1/n \text{ for all standard } n\}$ — is external to the ultrapower. Between any two distinct elements of the monad there is another (their arithmetic mean in $^*\mathbb{R}$), so no discrete jump at 0 is possible in $^*\mathbb{R}$.

Field density is an internal property of the ordered field structure of $^*\mathbb{R}$ and transfers by Łoś's theorem. Limit ordinal structure is set-theoretic and is not preserved by the ultrapower construction: $^*\mathbb{N}$ contains only successor-like infinite elements, not limit-like ones. The two results — transfer of density, failure of limit-ordinal structure — come from different theorems.

Section IV: Cantor Normal Form Bridge

Every ordinal below ε_0 has a unique Cantor normal form — a finite sum $\omega^{e_1}\cdot a_1 + \omega^{e_2}\cdot a_2 + \ldots$ with strictly decreasing exponents $e_1 > e_2 > \ldots$ and each coefficient a nonzero natural number (a nonzero ordinal strictly below ω). In Lean: NONote (Mathlib.SetTheory.Ordinal.Notation). The map cnfToZp2 sends each such ordinal to $\mathbb{Z}_2$ by structural recursion on the Cantor normal form. From tower stage 1 on, the 2-adic valuation of the image tracks ordinal height (tower_orders_agree, ZeroParadox/Ordinal/CnfBridge.lean).

Definition: cnfToZp2 (Gentzen.lean §IV)

cnfToZp2 : NONote → $\mathbb{Z}_2$

Definition: cnfToZp2 (Gentzen.lean §IV)

Base: cnfToZp2(0) = 0

Recursive: cnfToZp2(ω^e · n + a) = 2^(v_2(cnfToZp2(e)) + 1) · n + cnfToZp2(a) [n : ℕ⁺]

where v_2 denotes Lean's PadicInt.valuation, which sets $v_2(0) = 0$; the recursion uses that value.

Valuation of the n-th tower stage:

cnfToZp2(towerNONote 0) = 0 (Lean: PadicInt.valuation 0 = 0; the standard 2-adic valuation of 0 is +∞)

cnfToZp2(towerNONote 1) = 2 (valuation 1)

cnfToZp2(towerNONote 2) = 4 (valuation 2)

cnfToZp2(towerNONote n) = 2^n (valuation n; n ≥ 1)

Theorem: cnfToZp2_tower_valuation

∀ n : ℕ, (cnfToZp2 (towerNONote n)).valuation = n

The 2-adic valuation of the image of the n-th tower stage equals n (at n = 0, by Lean's convention PadicInt.valuation 0 = 0).

Proof: by induction; successor step uses PadicInt.valuation_pow.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: cnfToZp2_valuation_unbounded

∀ k : ℕ, ∃ α : NONote, k ≤ (cnfToZp2 α).valuation

The 2-adic valuation is unbounded across ordinals below ε_0.

Proof: towerNONote k witnesses the bound.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: tower_converges_to_zero

Filter.Tendsto (fun n ↦ cnfToZp2 (towerNONote n)) Filter.atTop (nhds 0)

The images of the tower stages converge to 0 (nhds 0) in $\mathbb{Z}_2$; the theorem names no order.

Proof: each stage's image cnfToZp2(towerNONote (n+1)) = 2^(n+1) in $\mathbb{Z}_2$, so its 2-adic norm is ‖2‖^(n+1) = (1/2)^(n+1) → 0. Uses Metric.tendsto_atTop and exists_pow_lt_of_lt_one.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Section V: Ordinal Tower Limit and Snap Threshold

This section connects the ordinal tower to the ZPE machine-phase snap. The cofinality theorem shows that the fundamental sequence approaches ε_0 from below. The lower-bound theorem shows that any monotone tower-aligned map must assign c_0 to all pre-ε_0 ordinals. A canonical witness snapping exactly at ε_0 is exhibited.

What this does NOT claim (§V)
That ε_0 is the proof-theoretic ordinal of PA (not claimed — see Remark R-L.1).
Any statement about formal provability in PA.
That ε_0 is the UNIQUE minimal snap boundary: snap_threshold_is_epsilon_zero shows no ordinal below ε_0 works (for maps satisfying the stated hypotheses), but does not rule out maps that snap at some ordinal strictly above ε_0.
That the snap threshold result applies to all maps Ordinal → MachinePhase regardless of the monotonicity and tower-alignment hypotheses.

Theorem: fundamentalSeq_cofinal
$\forall\ \alpha$: Ordinal, $\alpha < \varepsilon_0 \to \exists\ n : \mathbb{N}, \alpha <$ fundamentalSeq n
The fundamental sequence is cofinal in ε_0: for any ordinal below ε_0, some tower stage exceeds it.
Proof: $\varepsilon_0 =$ nfp ($\omega^{\wedge}\cdot$) 0 (epsilonZero_eq_nfp), and lt_nfp_iff gives a < nfp f b ↔ ∃ n, a < f^[n] b.
Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: snap_threshold_is_epsilon_zero
For any ϕ : Ordinal → MachinePhase satisfying:
(a) hmono: $\forall\ \alpha \le \beta$, join ($\phi\ \alpha$) ($\phi\ \beta$) = $\phi\ \beta$ (order-non-decreasing in the ZPSemilattice sense: $\phi\ \alpha \le \phi\ \beta$)
(b) h0: $\forall\ n : \mathbb{N}$, ϕ (fundamentalSeq n) = c_0
we have: $\forall\ \alpha < \varepsilon_0$, $\phi\ \alpha = c_0$
This is a lower bound: no ordinal below ε_0 is a snap point for maps satisfying (a) and (b). "Minimal" not "unique."
Proof: cofinality gives a stage above α; monotonicity chains $\phi\ \alpha \le \phi$(stage) = c_0; $c_0 = \bot$ forces $\phi\ \alpha = c_0$.
Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: c1_epsilon_zero_identification
$\exists\ \phi$: Ordinal → MachinePhase,
($\forall\ n : \mathbb{N}$, ϕ (fundamentalSeq n) = c_0) ∧ $\phi\ \varepsilon_0 = c_1$
Witness: $\phi\ \alpha =$ if $\alpha < \varepsilon_0$ then c_0 else c_1.

Theorem: c1_epsilon_zero_identification

One specific map sends all tower stages to c_0 and ε_0 to c_1.

The witness map is order-non-decreasing in ZP-A's join order (snap_map_mono), and no map Ordinal → MachinePhase is compatible with the CNF → $\mathbb{Z}_2$ map along the tower: for every g : MachinePhase → $\mathbb{Z}_2$ the square fails, because MachinePhase has two values and the images of stages 1, 2, 3 have three distinct 2-adic valuations (the example after c1_epsilon_zero_identification in ZeroParadox/Ordinal/Gentzen.lean); through snapEmbed it fails at every stage n ≥ 1, because snapEmbed takes only the values 1 and 0, and stage n ≥ 1 maps to 2^n, which is neither (ZeroParadox/Ordinal/Incompleteness.lean § II).

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Section VI: Kleene-Ordinal Fixed-Point Bridge

The ordinal fixed-point structure ($\varepsilon_0 = \text{nfp}\ (\omega^{\wedge}\cdot)\ 0$, $\omega^{\wedge}\varepsilon_0 = \varepsilon_0$) and the computational fixed-point structure (Kleene's recursion theorem, roger_fixed_point_stability) both carry Classical.choice in the proofs recorded here — parallel structure, not a proved isomorphism. The hypothesis hfp encodes that ordinal fixed points of $\omega^{\wedge}\cdot$ map to the snap state c_1. Under this hypothesis plus monotonicity and tower alignment, ε_0 is the minimal snap threshold.

Theorem: snap_exactly_at_epsilon_zero

For any ϕ : Ordinal → MachinePhase satisfying:

(a) hmono: order-non-decreasing (join ($\phi\ \alpha$) ($\phi\ \beta$) = $\phi\ \beta$ for $\alpha \le \beta$)

(b) h0: every tower stage maps to c_0

(c) hfp: every fixed point of $\omega^{\wedge}\cdot$ maps to c_1

we have: $\phi\ \varepsilon_0 = c_1$ AND $\forall\ \alpha$, $\phi\ \alpha = c_1 \to \varepsilon_0 \le \alpha$

ε_0 is the minimal snap threshold: ϕ assigns c_1 first at ε_0.

Note: "minimal" not "unique." A map satisfying these hypotheses could also assign c_1 to ordinals above ε_0; what is ruled out is any snap strictly before ε_0.

Proof: upper bound from hfp + epsilonZero_fixedPoint; lower bound from snap_threshold_is_epsilon_zero.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: kleene_ordinal_snap_bridge

$\exists\ \phi$: Ordinal → MachinePhase,

($\forall$ n : $\mathbb{N}$, ϕ (fundamentalSeq n) = c_0) ∧

($\forall\ \alpha$, $\omega^{\wedge}\alpha = \alpha \to \phi\ \alpha = c_1$) ∧

Theorem: kleene_ordinal_snap_bridge

$\phi\ \varepsilon_0 = c_1 \wedge$

$\forall\ \alpha,\ \phi\ \alpha = c_1 \to \varepsilon_0 \leq \alpha$

Witness: $\phi\ \alpha =$ if $\alpha < \varepsilon_0$ then c_0 else c_1.

Note: the theorem name refers to the informal §VI conceptual parallel between Kleene recursion fixed points and ordinal fixed points of ω^·. The theorem itself is purely ordinal — no Code or eval appears.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Section VII: Canonical Snap Map — Full Closure

The canonical threshold map $\phi\ \alpha =$ if $\alpha < \varepsilon_0$ then c_0 else c_1 satisfies all three conditions of snap_exactly_at_epsilon_zero (hmono, h0, hfp). For this specific map, the snap identification is unconditional: all five conditions are simultaneously verified without free hypotheses.

Theorem: snap_map_mono

$\forall\ \alpha\ \beta$: Ordinal, $\alpha \leq \beta \to$

join (if $\alpha < \varepsilon_0$ then c_0 else c_1)

(if $\beta < \varepsilon_0$ then c_0 else c_1) =

if $\beta < \varepsilon_0$ then c_0 else c_1

The canonical map is order-non-decreasing.

Proof: four cases (α/β vs ε_0). The case $\alpha \geq \varepsilon_0$ with $\beta < \varepsilon_0$ contradicts $\alpha \leq \beta$. Each live case closes by rfl from the MachinePhase join definition.

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: epsilon_zero_snap_canonical

$\exists\ \phi$: Ordinal → MachinePhase,

$(\forall\ \alpha\ \beta, \alpha \leq \beta \to \text{join}\ (\phi\ \alpha)\ (\phi\ \beta) = \phi\ \beta) \wedge$ [hmono]

$(\forall\ n : \mathbb{N}, \phi\ (\text{fundamentalSeq}\ n) = c_0) \wedge$ [h0]

$(\forall\ \alpha, \omega^\alpha = \alpha \to \phi\ \alpha = c_1) \wedge$ [hfp]

$\phi\ \varepsilon_0 = c_1 \wedge$ [snap]

$\forall\ \alpha, \phi\ \alpha = c_1 \to \varepsilon_0 \leq \alpha$ [minimality]

All five conditions verified for the explicit witness $\phi\ \alpha =$ if $\alpha < \varepsilon_0$ then c_0 else c_1, with no free hypotheses.

Theorem: epsilon_zero_snap_canonical

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

Theorem: snap_zp2_correspondence

Four independent facts about the same tower sequence:

(i) $\forall\, n : \mathbb{N}$, fundamentalSeq n $< \varepsilon_0$

(ii) $\forall\, n : \mathbb{N}$, ϕ (fundamentalSeq n) $= c_0$ where ϕ is the canonical map

(iii) Filter.Tendsto (fun n ↦ cnfToZp2 (towerNONote n)) Filter.atTop (nhds 0)

(iv) $\phi\ \varepsilon_0 = c_1$

The same tower sequence witnesses both the ordinal approach to ε_0 and the 2-adic approach to 0. This is a co-witness, not an identity (cnf_bridge_type_boundary, ZeroParadox/Ordinal/CnfBridge.lean): ε_0 and the $\mathbb{Z}_2$ zero live in different types with no coercion between them, so an identity between them fails to elaborate.

Proof: ⟨epsilonZero_tower_lt, fun n ↦ if_pos …, tower_converges_to_zero, if_neg (lt_irrefl ε_0)⟩

Lean purity: [propext, Classical.choice, Quot.sound]. ✓

The Bridge: What Is Formal and What Is Assumed

What the Bridge Supplies

c_1 is a MachinePhase state and ε_0 an ordinal, so an identity between them is not a well-formed statement; the formal link runs through maps: a canonical map Ordinal → MachinePhase and snapEmbed : MachinePhase → $\mathbb{Z}_2$.

What is proved: a canonical map Ordinal → MachinePhase assigns c_1 exactly at ε_0 and nowhere earlier.

Two of the three pieces a type bridge from the ordinals through MachinePhase to $\mathbb{Z}_2$ was expected to need are built, in ZPM (ZeroParadox/Ordinal/Incompleteness.lean §I–§II), and the third is not supplied by them:

(1) snapEmbed : MachinePhase → $\mathbb{Z}_2$ exists, sending $c_0 \mapsto 1$ and $c_1 \mapsto 0$ — the snap state to 2-adic zero, the pre-snap state to a unit (snapEmbed_c0, snapEmbed_c1, both by rfl)

(2) snapEmbed_mul_morphism proves snapEmbed (join a b) = snapEmbed a × snapEmbed b, carrying join to multiplication because c_1 absorbs joins as 0 absorbs products. That is an absorbing-element morphism, and it does not send bottom to bottom in the Scale.lean chart ($\bot = 0$): there the bottom role is filled by 0 (a ZPSemilattice on $\mathbb{Z}_2$ with $\bot = 0$ is built in ZeroParadox/Valuation/Scale.lean § V), and snapEmbed sends c_0 to 1 (snapEmbed_c0); in the multiplicative reading snapEmbed $c_0 = 1$ is the identity ($1 \cdot x = x$), as c_0 is the identity of join, and that reading is not a ZPSemilattice, since multiplication on $\mathbb{Z}_2$ is not idempotent ($2 \cdot 2 \neq 2$)

(3) not supplied — for a map ϕ sending the tower's stages to c_0 (h0), continuity of snapEmbed ∘ ϕ along the tower would force $\phi\ \varepsilon_0 = c_0$, the opposite (the examples after snap_unconditional, ZeroParadox/Ordinal/Incompleteness.lean § II), while the cnfToZp2 images of the stages do tend to snapEmbed $c_1 = 0$; the placement enters as $h\varepsilon_0$.

What the Bridge Supplies

So hfp is derived rather than assumed: hfp_from_epsilon_zero obtains it from monotonicity together with $h\varepsilon_0 : \phi\ \varepsilon_0 = c_1$, and snap_unconditional uses it in place of the free hypothesis of snap_exactly_at_epsilon_zero. $h\varepsilon_0$ ($\phi\ \varepsilon_0 = c_1$; in the $\mathbb{Z}_2$ chart, snapEmbed ($\phi\ \varepsilon_0$) = 0) is the snap's occurrence at ε_0, taken as a hypothesis: of the infinitely many admissible firing points that monotonicity and tower alignment (every tower stage sent to c_0) leave open, $h\varepsilon_0$ selects the least, and so fixes ϕ uniquely (ZeroParadox/Ordinal/Incompleteness.lean § II). Whether Classical.choice is forced by the metric collapse is a separate open question (ZeroParadox/Ordinal/SyntacticCollapse.md). The canonical witness (epsilon_zero_snap_canonical) satisfies all five conditions without any such type bridge.

Theorem Summary

Theorem	Section	Status
roger_fixed_point_stability	§II	Proved ✓
epsilonZero_fixedPoint	§III	Proved ✓
epsilonZero_eq_nfp	§III	Proved ✓
epsilonZero_eq_iSup	§III	Proved ✓
epsilonZero_tower_lt	§III	Proved ✓
epsilonZero_le_fixedPoint	§III	Proved ✓
fundamentalSeq_strictMono	§III	Proved ✓
tower_stage_zero / _one / _two	§III	Proved ✓
towerNONote_repr	§IV	Proved ✓
cnfToZp2_tower_valuation	§IV	Proved ✓
cnfToZp2_valuation_unbounded	§IV	Proved ✓
fundamentalSeq_zp2_converges	§IV	Proved ✓
tower_converges_to_zero	§IV	Proved ✓
zpe_snap_ordinal_correspondence	§V	Proved ✓
epsilonZero_tower_bound	§V	Proved ✓
c1_epsilon_zero_identification	§V	Proved ✓
fundamentalSeq_cofinal	§V	Proved ✓
snap_threshold_is_epsilon_zero	§V	Proved ✓
snap_exactly_at_epsilon_zero	§VI	Proved ✓
kleene_ordinal_snap_bridge	§VI	Proved ✓

Theorem	Section	Status
snap_map_mono	§VII	Proved ✓
epsilon_zero_snap_canonical	§VII	Proved ✓
snap_zp2_correspondence	§VII	Proved ✓

Axiom Purity

Every theorem in the Theorem Summary above carries axiom footprint: [propext, Classical.choice, Quot.sound]. Section I's table and ZP-K Section IV record where the ZP layers differ — Section I's own ZPJ/K row names bot_self_mem, which measures no axioms.

These are axioms of Lean's core library, reached here through Mathlib's ordinal theory, p-adic analysis and computability. They are not ZP-L commitments.

The footprints across the ZP layers are not uniform; ZP-K Section IV tabulates the measurements. In the computability layer, Rogers' fixed-point theorem as Mathlib states it carries Classical.choice in its statement, through Mathlib's numbering of program codes (the Denumerable Code instance; ZeroParadox/Computability/Kleene.md § VIII). ZP-K's machinePhaseKleene also picks botCode with Classical.choose, which is what makes that instance noncomputable.

The footprints listed here are copies; to re-measure one, run #print axioms on the named declaration. ZeroParadox/Ordinal/Gentzen.lean, this document's own Lean file, prints those of the results it holds.