THE ZERO PARADOX

ZP-N: The Constructive Snap

the ε_0 snap from below, choice-free on ordinal notations

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The constructive companion to ZP-L. The snap-from-below is rebuilt syntactically on ordinal notations (ONote), where it is choice-free — [propext] only. Beside it: a carrier of notations denoting the ordinals up to ε_0, whose crossing into Mathlib's Ordinal is one named map with a measured price, and a proof that the generality ZP-L borrows implies excluded middle. Proved sorry-free in Lean 4.

ZP-L derived the snap at ε_0, and every ε_0 result in its Lean development carries Classical.choice. ZP-N asks where that choice actually lives and whether the snap needs it. Two answers, and they point in opposite directions. The snap's downward structure — the ω-tower climbs without bound, and no ordinal notation is a fixed point of $x \mapsto \omega^x$ — is genuinely constructive, provable with no Axiom of Choice. But the machinery ZP-L reaches for to state that structure semantically is not: comparing arbitrary well-orders implies excluded middle. So the choice is not a mere artifact of how a proof was written. ZP-L's ε_0 results, as stated, already carry Classical.choice in their statements (the statement controls, printed in ZeroParadox/Ordinal/Epsilon0MinMax.lean), so no proof of those statements can drop it. For the ascent below ε_0 it is the price of a tool stronger than the job requires: the ascent is restated choice-free here, on notations. For ε_0 itself, a choice-free Lean restatement of the fixed-point equation and its uniqueness on E0Note exists, with existence set by definition (E0Note, below); a choice-free derivation in Lean of the fixed point on a well-founded carrier whose statements are choice-free, leastness included, was not located as of 2026-10-08, searched again 2026-10-13 (ZeroParadox/Ordinal/SnapNucleus.md, section Axiom footprint, records the search).

So this layer does two things. It rebuilds the ascent syntactically, where it is choice-free. And it builds a carrier of notations denoting the ordinals up to ε_0 — ordinal notations with one point adjoined on top — where the crossing into Mathlib's semantic ordinals is a single named map whose cost can be read off directly. On the notation side the comparison, the ascent and the carrier's declarations are free of choice, though well-formedness through Mathlib's NF is not (tower_NF, Section III); crossing is not; and the crossing is one function rather than a diffuse dependency.

The move is to work at the level of ordinal notations (ONote, Cantor-normal-form terms), whose comparison ONote.cmp is choice-free (propext-only) and never passes through repr into the semantic Ordinal type. ε_0 is not itself a notation: the notations name exactly the ordinals below ε_0, and ε_0 is their limit: every notation denotes below it (repr_lt_epsilon0, ZeroParadox/Ordinal/PricedInterface.lean), and the tower climbs past every notation (tower_no_upper_bound, ZeroParadox/Ordinal/SnapNucleusConstructive.lean). So the snap threshold is characterised from below: the tower ascends, and the fixed point (ε_0) is precisely what lies beyond every notation. This layer adds no axiom; it isolates where choice actually enters the ordinal snap.

Section I: The Syntactic Substrate

The Cantor normal form gives each ordinal below ε_0 a finite syntactic name. At the notation level, ω^x is simply the term oadd x 1 0 (leading exponent x, coefficient 1, empty remainder) — no general ordinal exponentiation is needed. The ω-tower is built by iterating it. Comparison of notations is the purely syntactic ONote.cmp, which returns lt / eq / gt by structural recursion and never evaluates repr, so it is choice-free.

Definitions (ConstructiveOrdinals.lean)

omegaPow x = oadd x 1 0 — the notation for ω^x; purely syntactic and computable.
tower 0 = 0, tower (n+1) = omegaPow (tower n) — the ω-tower $(\omega^{\cdot})^{[n]}$ 0, constructive.
ONote.cmp — the syntactic three-way comparison on notations (lt / eq / gt); choice-free (it does not touch repr / Ordinal).

Section II: The Choice-Free Snap from Below

Lemma: exp_lt_term (ConstructiveOrdinals.lean)

For every notation e, coefficient n, and remainder a: cmp e (oadd e n a) = lt.
An exponent is strictly below its own term. Proof: structural induction on the exponent, pure syntax — no repr, no Ordinal.
Lean purity: [propext] only — choice-free, and free even of Quot.sound. ✓

Theorem: omegaPow_no_fixedpoint (ConstructiveOrdinals.lean)

For every notation x: cmp x (omegaPow x) = lt.
No ordinal notation is a fixed point of $x \mapsto \omega^x$: every notation is strictly below its own ω-power, in the choice-free syntactic comparison (holds for all notations, well-formed or not). This is the constructive shadow of "ε_0 is the least fixed point of $x \mapsto \omega^x$, lying beyond every notation."
Lean purity: [propext] only — choice-free. ✓

Theorem: tower_strictMono (ConstructiveOrdinals.lean)

For every n: cmp (tower n) (tower (n+1)) = lt.
The ω-tower is strictly increasing: the snap stages climb without bound below ε_0. Immediate from omegaPow_no_fixedpoint applied at tower n.
Lean purity: [propext] only — choice-free. ✓

The snap from below is choice-free. All three results depend on [propext] alone — not Classical.choice, and not even Quot.sound. The ascent of the ω-tower and the absence of any notation fixed point are proved by pure structural recursion on the syntax. This is the sharpest form of the framework's choice-free-core pattern (see ZeroParadox/AxiomProfile.lean): the snap's downward structure needs nothing beyond propositional extensionality.

Section III: The Finding and Its Fences

The point of ZP-N is the contrast with ZP-L. There, every ε_0 result carries Classical.choice; here, the same downward structure is proved with [propext] only. It is tempting to conclude that the choice is therefore an artifact of representation. It is not, and the reason is worth stating precisely, because the natural diagnosis is wrong twice over.

First, the choice is not in the type. #print axioms Ordinal reports [propext, Quot.sound]. It is in the order instance and the operations built on it — Ordinal.partialOrder, instLinearOrder, typein, omega0, nfp, deriv, epsilon. Second, and consequently, an axiom footprint on any ε_0 result measures that ambient instance rather than the proof: $a \leq a$ carries Classical.choice while $a = a$ does not, for the same element, differing only in whether the statement mentions the order (order_footprint_le, order_footprint_eq). The instrument cannot answer the question it was being asked.

What it can answer is a different question: is that instance's classical content real? It is. Merely knowing, of any two well-orders, that one of them embeds in the other — without being handed which — already yields excluded middle (em_of_wellOrder_comparable, itself choice-free: the classical content is the hypothesis, not the proof). The hypothesis is a bare disjunction, not a procedure that computes the answer, which makes the implication sharper than it first sounds; it is also the form Mathlib's le_total actually takes. This is a known taboo in constructive mathematics, not a result of this framework: it is Theorem 38(d) of Kraus, Nordvall Forsberg and Xu, stated there in the data form, and their witnesses are the ones used here. So a choice-free proof of the generality Mathlib's order provides — comparing any two well-orders — would be a choice-free proof of excluded middle, which this corpus takes to be impossible in Lean but does not prove (ZeroParadox/Category/ChoiceCannotBe.lean § IV). The ascent below ε_0 never needed that generality: its counterpart on notations (Section II) uses propext alone.

Remark: even well-formedness inherits choice (through NF's definition)

tower_NF — the statement that each tower stage is in normal form — DOES carry Classical.choice ([propext, Classical.choice, Quot.sound]), because Mathlib's NF predicate is defined through repr into Ordinal. The snap facts (Section II) do not depend on NF, so they stay choice-free; but even "this notation is well-formed" inherits choice, through NF's definition. That is one site where choice enters, not the only one: ZP-L's least-fixed-point results (epsilon0_min_eq_max and epsilon0_eq_nfp_bot, ZeroParadox/Ordinal/Epsilon0MinMax.lean) are not routed through tower_NF (their files do not import ZeroParadox/Ordinal/ConstructiveOrdinals.lean, where tower_NF is declared), and their choice is carried by their own statements, through Ordinal's order instance and operations (the statement controls epsilon0MinMaxStatement and epsilon0EqNfpBotStatement, printed in that file; where the choice enters: ZeroParadox/Ordinal/SnapNucleus.md § Axiom footprint).

The diagnosis above, that choice enters through Ordinal's order instance and the operations on it, never through the notations' own comparison, suggests its own remedy: build a carrier of notations denoting the ordinals up to ε_0. E0Note is ordinal notations with a single point adjoined on top, that point standing for ε_0. Below the top, comparison is the existing syntactic comparator and stays decidable; the map e0Repr sends the carrier into Mathlib's Ordinal. Measured, the boundary is priced: the carrier side's declarations carry no Classical.choice, and the map carries it at every declaration. On the carrier, staying constructive is free; crossing costs the classical assumption; and the crossing is one named function.

Scope fences on the carrier — what it is and is not

The fixed point at the top is stipulated, not discovered. Defining the tower operator to fix the adjoined point makes the closure exist by fiat at the added point; only its uniqueness is a theorem. This does not weaken no_snap_closure, which says no such closure exists on the notations alone.

E0Note's points denote the ordinals up to ε_0: those below ε_0, and ε_0 itself at the adjoined top. Its own order is not well-founded, so it is not a notation system for $\varepsilon_0 + 1$ in the well-ordered sense (the example stating not WellFoundedLT E0Note, in ZeroParadox/Ordinal/PricedInterface.lean). The standard alternative is a Veblen system where ε_0 is an ordinary term; the top's honest advantage is that it is minimal and leaves the comparator untouched.

e0Repr is not a member of Castéran's ON_correct class, and the first reason comes before any field: ON_correct is stated over an ON, whose order must be well-founded (ON_wf), and E0Note's order is not (the bullet above). Read one by one, its three fields go as follows. Two hold: every point denotes at or below ε_0 (e0Repr_le_epsilon0, from repr_lt_epsilon0), and e0Repr is onto the ordinals at or below ε_0 (from repr_surj_below_epsilon0). The comparator field fails: on raw notations the representation map is not injective (e0Repr_not_injective). See ZeroParadox/Ordinal/PricedInterface.md, "What is NOT proved here". Restricting to normal forms is the standard repair and is not done here, because that predicate is itself defined through the representation map.

The construction is not new. The carrier has the shape of Castéran's generic sum of notation systems (hydra-battles, ON_plus — that file carries his name alone, while the library as a whole is the work of the hydra-battles authors) with a one-point right summand, and is not an instance of it: ON_plus sums well-founded notation systems, and this carrier's order is not well-founded (the second bullet above). The map is likewise not an ON_correct member (the bullet above). What is contributed is the measurement.

Scope fence: the snap from below, not minimality

ZP-N proves the snap from below: ε_0 is the fixed point unreachable from within the notation system (no notation is a fixed point of $x \mapsto \omega^x$; the tower climbs without bound). The matching minimality — that ε_0 is the LEAST fixed point of $x \mapsto \omega^x$ — is a separate target: it quantifies over the limit, which no notation names (tower_no_upper_bound; across the crossing, repr_lt_epsilon0), so it cannot be stated purely syntactically on ONote. Leastness is proved on Ordinal (epsilon0_min_eq_max, with choice); a choice-free derivation of it in Lean on a well-founded carrier whose statements are choice-free was not located as of 2026-10-08, searched again 2026-10-13 (ZeroParadox/Ordinal/SnapNucleus.md, section Axiom footprint, records the search and the defined reading on E0Note). What is proved here is the from-below half, choice-free.

Prior art for leastness, classical: hydra-battles (Coq) proves epsilon0_lfp, that ε_0 is the least fixed point of phi0, its name for $x \mapsto \omega^x$ (theories/ordinals/Schutte/AP.v, commit ed8e604), a file carrying Castéran's name alone, in the library's Schütte module, which axiomatises the countable ordinals and works in classical logic.

Theorem Summary

Result	Lean source (full path)	Axioms
exp_lt_term	ZeroParadox/Ordinal/ConstructiveOrdinals.lean	propext only
omegaPow_no_fixedpoint	ZeroParadox/Ordinal/ConstructiveOrdinals.lean	propext only

Result	Lean source (full path)	Axioms
tower_strictMono	ZeroParadox/Ordinal/ConstructiveOrdinals.lean	propext only
tower_NF	ZeroParadox/Ordinal/ConstructiveOrdinals.lean	Classical.choice
tower_cofinal	ZeroParadox/Ordinal/SnapNucleusConstructive.lean	propext only
no_snap_closure	ZeroParadox/Ordinal/SnapNucleusConstructive.lean	propext only
em_of_wellOrder_comparable	ZeroParadox/Ordinal/OrdinalChoiceEssential.lean	propext, Quot.sound
order_footprint_eq	ZeroParadox/Ordinal/OrdinalChoiceEssential.lean	propext, Quot.sound
order_footprint_le	ZeroParadox/Ordinal/OrdinalChoiceEssential.lean	Classical.choice
E0Note (carrier)	ZeroParadox/Ordinal/PricedInterface.lean	no axioms
e0Repr (the crossing)	ZeroParadox/Ordinal/PricedInterface.lean	Classical.choice
e0Repr_not_injective	ZeroParadox/Ordinal/PricedInterface.lean	Classical.choice

Axiom Purity

The snap from below (exp_lt_term, omegaPow_no_fixedpoint, tower_strictMono): [propext] only — choice-free, and free even of Quot.sound. Proved by structural recursion on ONote; repr / Ordinal are never touched.

Well-formedness (tower_NF): [propext, Classical.choice, Quot.sound] — Classical.choice inherited through Mathlib's NF, defined via repr into Ordinal, whose order instance and operations carry it; the type Ordinal itself reports [propext, Quot.sound] (Section III).

The taboo (em_of_wellOrder_comparable): [propext, Quot.sound] — choice-free by design, because the classical content is the hypothesis rather than the proof. Its converse (comparable_of_classical) carries Classical.choice, as it must.

The carrier (E0Note, e0Coe): no axioms at all; its decidable-order instances [propext, Quot.sound]. The crossing (e0Repr and every lemma about it): [propext, Classical.choice, Quot.sound]. No declaration on the carrier side carries choice.

Together these locate the Classical.choice in ZP-L's ε_0 results: it is not in the Ordinal type, and not in the ascent (which is proved choice-free here), but in the order instance the semantic statement passes through, whose comparability implies excluded middle. Locating it is not eliminating it: those ε_0 results, as stated, are statement-carried (their statements carry choice); for ε_0 itself, a choice-free Lean restatement of the fixed-point equation and its uniqueness on E0Note sets existence by definition, and a choice-free derivation in Lean of the fixed point on a well-founded carrier whose statements are choice-free, leastness included, was not located as of 2026-10-08, searched again 2026-10-13 (ZeroParadox/Ordinal/SnapNucleus.md, section Axiom footprint).

The footprints listed here are copies; the live record is the #print axioms output of each named .lean file.

End of ZP-N | The Constructive Snap | the snap from below on ordinal notations, choice-free ([propext] only): exp_lt_term, omegaPow_no_fixedpoint, tower_strictMono | the classical dependency is in Mathlib's order instance, where comparing arbitrary well-orders implies excluded middle (taboo cited to Kraus, Nordvall Forsberg and Xu) | a carrier of notations denoting the ordinals up to ε_0, with the crossing priced at one named map | leastness of ε_0 is proved

on Ordinal (epsilon0_min_eq_max, with choice); a choice-free derivation of it in Lean on a well-founded carrier whose statements are choice-free was not located as of 2026-10-08, searched again 2026-10-13 (ZeroParadox/Ordinal/SnapNucleus.md, section Axiom footprint).