THE ZERO PARADOX

ZP-Q: The Frame-Change

The bottom read as both 0 and ∞, and what that does not settle

Initial: July 2026 | Current: 1.25, November 2026

Synthesis layer, the sequel to ZP-P. The abstract frame-flip schema is proved sorry-free and choice-free in Lean 4 (ZeroParadox/Settheory/ForkFrameChange.lean); the per-domain realizations are witnessed in their home layers (the valuation face — the Riemann sphere — in ZeroParadox/Multihomed/SnapFrameChange.lean / ZeroParadox/Valuation/RiemannSphere.lean; the categorical face in ZeroParadox/Category/SeamFrameChange.lean). The universal cross-category identity is not a theorem — it meets a proven wall (Cantor; ZeroParadox/Category/Lawvere.lean). Adds no axiom; the bottom itself is never positively represented.

The Zero Paradox proves the snap $\bot \to \varepsilon_0$ as a theorem; ZP-Q is about the BOTTOM's two-facedness, which is a different object from the snap. Across every layer the bottom shows the same two features: it is a fixed point — a zero — that carries an infinitude ($v_2(0) = \infty$, unbounded surprisal, the self-referential closure), and every map to it runs one way and reverses only under an inversion that swaps the two poles. These are not two facts. They are two faces of one self-dual object, and the inversion is the frame-change. The frame-change is that pole exchange — $\bot$ being read as both 0 and ∞. Whether the snap is an instance of it is the layer's open conjecture. The snap, T-SNAP's shape, is a join step off $\bot$ of the state semilattice to a different state (t_snap_given); that the step is a cover, $\bot \lessdot a$ (HasFirstStep), is AX-B1's first-step form, a modelling commitment and a further hypothesis (t_snap_given_cover) that MachinePhase supplies and $\mathbb{Q}\geq 0$ does not; the frame-change is an involution and is proved. Merging the two is what makes it look paradoxical that a frame-change reverses while the snap does not, and that reversibility mismatch is exactly what an identification would have to resolve. ZP-Q holds it open rather than settling it either way. Read from one frame the frame-change is a bridge (it joins the poles continuously); read from the other it is a fence (a boundary the map cannot cross). ZP-Q records the abstract frame-flip schema, its per-domain realizations, and the universal form together with the wall that bounds it.

Like ZP-P, this is a synthesis layer, organised in three tiers held to three standards of proof. Tier 1 (Section I) is the abstract schema, proved in Lean over a complete lattice: it extends ZP-P's fork with its second face, the order-duality that swaps the two closures. Tier 2 (Section II) is the face catalogue: each domain's frame-change is a theorem in its home layer, and the valuation face is the framework's originating figure, the p-adic Riemann sphere. Tier 3 (Section III) is the universal form and its walls: the order-theoretic universal holds, the categorical (Lawvere) universal is a proven wall, and the cross-domain identity is a type boundary. The fences in Section III are load-bearing — and they turn out to be built from the same diagonal as the bridges.

A note on what is drawn

ZP-Q characterises the bottom apophatically (by what it is not, never by a positive construction), and the register is itself frame-relative. Seen from outside — from any frame that looks at $\bot$ — there is no interior: nothing sits behind the description waiting to be opened. The boundary is the whole of it. $\bot$ is the seam the frame-change joins, or the tear it leaves, a boundary of measure zero, all edge and no inside (read flat, this is the Quine atom (CC-2) from without, the occupant of the role "steps to itself and to nothing else" under a self-application map supplied by AbstractSelfApp (selfApp_quine_role_iff) — a single node, not a region — and the tear on the sphere — a discontinuity, not a region).

Seen from $\bot$'s own frame — inside the self-reference — the same point is the opposite: infinite depth. $v_2(0) = \infty$; the endless regress $Q \in Q \in Q \ldots$ of the AFA set $Q = \{Q\}$ that plays the role; unbounded surprisal. No-interior and infinite-interior are the $0 \leftrightarrow \infty$ poles of one coin, swapped by the frame-change: what has no thickness from without has nothing but depth from within. So the bottom is not an object behind the boundary — it is the boundary, whose own thickness is the frame-flip.

Section I: The Frame-Flip Schema (Tier 1)

ZP-P recorded the fork's first face: over a complete lattice a monotone self-map f has a least fixed point (lfp f, the well-founded / inductive closure, μ) and a greatest fixed point (gfp f, the non-well-founded / coinductive closure, ν), and the fork collapses to a single point — the diagonal fixed point $\bot$ — exactly when f has a unique fixed point (fork_collapse_iff). ZP-Q adds the second face. The two closures are the two charts; the frame-change is the order-duality that carries one to the other. It is the abstract form of the concrete frame-changes below: the p-adic inversion $z \mapsto 1/z$, the categorical op-duality, lattice complementation.

Definition: the frame-change (ZeroParadox/Settheory/ForkFrameChange.lean)

For a complete lattice α and a monotone self-map $f : \alpha \to \alpha$:

the two charts are lfp f (μ, the well-founded closure) and gfp f (ν, the non-well-founded closure);

the frame-change is the order-duality $f \mapsto f.dual$ on α^{op} — the abstract analogue of $z \mapsto 1/z$, op, and complementation.

The schema states that the frame-change swaps the two charts, and that they coincide exactly at the diagonal fixed point.

Proposition: lfp_dual_eq_gfp / gfp_dual_eq_lfp

lfp (f.dual) = gfp f and gfp (f.dual) = lfp f.

The frame-change (order-duality) carries the μ-closure to the ν-closure and back: the two charts are swapped, not merged. In Lean both are definitional (rfl) — the swap is built into how gfp is defined (sSup of post-fixed points).

Lean purity: [propext, Quot.sound] — choice-free. ✓

Theorem: fork_is_frameflip — the schema spine

Over any complete lattice, bundling both faces: (i) lfp (f.dual) = gfp f and (ii) gfp (f.dual) = lfp f (the frame-change swaps the charts), together with (iii) lfp f = gfp f ↔ ∃! x, f x = x (the fork collapses at the diagonal fixed point — fork_collapse_iff).

This is the domain-independent form of the POLE BEHAVIOUR: the two charts are swapped by the frame-change, and the bottom is where they meet. Note what (iii) says — least = greatest exactly when the fixed point is unique — and note the hypotheses: it needs a COMPLETE LATTICE and a MONOTONE map. NONE of the framework's min≡max facts satisfies those, so none is an instance of it. epsilon0_min_eq_max is over the ordinals, where the omega-power map has a proper class of fixed points, so nothing collapses; selfApp_bot_is_both_extremal is over a bare join-semilattice whose self-application is not a monotone map; the categorical seam lives in a category, not a lattice. What they share is a SHAPE — one object carrying both extremal characterisations at once — and a shared shape across distinct structures is a type boundary, never a common theorem. State the shape, not an instance-of relation. It does NOT assert that the snap is a change of frame; that identification remains this layer's open conjecture.

Lean purity: [propext, Quot.sound] — choice-free. ✓

The frame-flip schema is choice-free. fork_is_frameflip and both duality lemmas depend only on [propext, Quot.sound] — propositional extensionality and quotient soundness, two of Lean's standard axioms, used throughout Mathlib. None depends on the Axiom of Choice. As in ZP-P, the conceptual skeleton needs no choice; choice enters mostly in the analytic realisations (ZeroParadox/AxiomProfile.lean § III) — not only there, since the framework also carries bare classicals of its own; ZeroParadox/Category/ChoiceCannotBe.lean § IV collects cases among them whose principles reduce to a taboo, with those reductions.

Section II: The Face Catalogue (Tier 2)

The same frame-flip appears across the framework, each domain realising it with its own concrete involution. ZP-Q does not re-prove each face — each is a theorem in its home layer — it records the catalogue and pins the two faces with a full frame-flip (valuation, category) to their Lean witnesses. The two poles of each row are the two charts; the frame-change is the involution that swaps them.

Domain	The two poles (charts)	Frame-change — witness
Valuation	$0 = \bot$ of Q_2 (2-adic norm) ↔ ∞	z ↦ 1/z (rInv) — snap_is_frameflip
Category	initial (μ) ↔ terminal (ν)	op-duality — catseam_is_frameflip
Set theory	Foundation (μ) ↔ AFA (ν)	the fork — fork_collapse_iff
Computability	code ↔ self-application	the diagonal — AbstractSelfApp.unique_fp
State / Hilbert	X ↔ X ⊞ X	the biproduct diagonal — biprod_diagonal_only_zero
Information	measure 0 ↔ surprisal ∞	−log — info_bottom_diverges

Domain	The two poles (charts)	Frame-change — witness
Reals	$0 \leftrightarrow \infty$ (no first step above 0)	$x \mapsto 1/x$ on the one-point compactification — onePointInv_swaps_real; holds, and is not a discriminator (the remark below)

The valuation realization is the originating figure of the framework: the p-adic Riemann sphere. Under the tower-rank encoding (ZeroParadox/Ordinal/P8.lean) the ω-tower climbing to ε_0 has stage-encodings that converge to $\mathbb{Z}_{[2]}$'s 0 (cnf_encode_tower_tendsto_zero), least under the 2-adic norm and greatest under the 2-adic valuation — so in that chart, read by the norm, the ascent toward ε_0 resolves back onto the floor (the encodings converge to 0; ε_0 has no encoding, and $\varepsilon_0 \neq \bot$ of Ordinal, epsilon0_ne_bot). Viewed through the inversion rInv (the one-point compactification of $\mathbb{Q}_2$, ZeroParadox/Valuation/RiemannSphere.lean), the same encodings diverge to ∞. rInv is the homeomorphism that swaps the poles $0 \leftrightarrow \infty$.

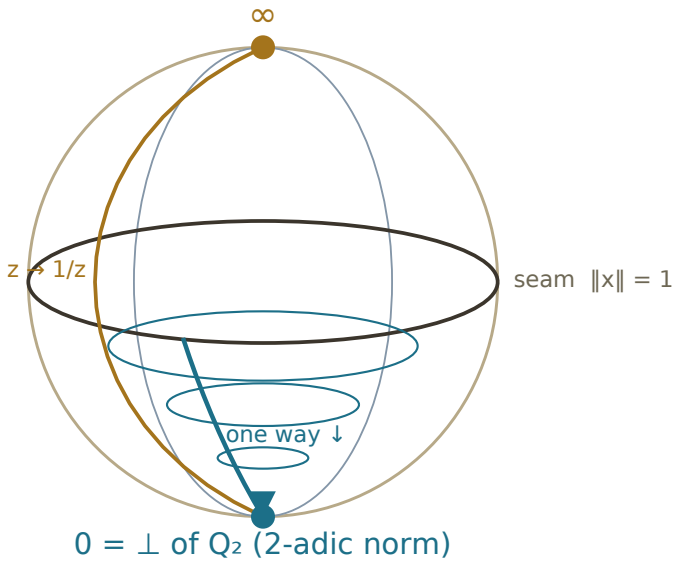


Figure 1. The p-adic Riemann sphere. The poles $0 = \bot$ (the floor, least under the 2-adic norm) and ∞ (the ceiling) are antipodal; the equator $\|x\| = 1$ is the seam. Inside $\mathbb{Z}_p$ (the lower hemisphere) the ω-tower falls one way to 0; the inversion $z \to 1/z$ (rInv) swaps the poles $0 \leftrightarrow \infty$ — continuous on the sphere, torn at 0 in the field.

Theorem: snap_is_frameflip (ZeroParadox/Multihomed/SnapFrameChange.lean)

The one ω-tower's encodings have two limits, one per chart: they fall to the floor 0 of $\mathbb{Z}_{[2]}$ in the encoding chart, read by the 2-adic norm (cnf_encode_tower_tendsto_zero; cnf_encode takes the values 2^k and never starts at 0, so the return to the SAME 0 the arc started from is a fact about the other encoding, cnfToZp2: tower_image_loops_to_seed, snap_arc_z2_loop), and rise to the antipode ∞ in the rInv chart (snap_frameflip_tower_tendsto_infty), with rInv the frame-change swapping the poles (rInv_swaps: $0 \leftrightarrow \infty$). The encodings converge to 0; $\varepsilon_0 \neq \bot$ of Ordinal (epsilon0_ne_bot).

Theorem: snap_is_frameflip (ZeroParadox/Multihomed/SnapFrameChange.lean)

The descent to 0 and the ascent to ∞ are the same tower-encodings under the frame-change — the valuation-frame realisation of the BOTTOM's two chart-readings. Note the scope: no snap transition appears in this statement (no $c_0 \to c_1$, no $\bot \lessdot a$). The declaration handle snap_is_frameflip is retained for cross-reference but overclaims; the statement is the authority.

Lean purity: [propext, Classical.choice, Quot.sound] — choice-carrying (Classical.choice from the Riemann-sphere continuity machinery). ✓

Remark: continuity is frame-relative — the bridge and the fence

RiemannSphere.lean proves the content that makes the sphere the right figure. In the affine field $\mathbb{Q}_2$ the inversion $z \mapsto 1/z$ is discontinuous at 0 (1/0 is undefined): the map tears. Adjoining ∞ and sending $0 \mapsto \infty$ repairs the tear — on the sphere the inversion is a continuous homeomorphism (rInvHomeo). So the same point 0 is a discontinuity (a fence) from the field frame and a continuous join to ∞ (a bridge) from the sphere frame. Continuity itself is frame-relative; one chart tears where the other heals. The tear at 0 is the apophatic bottom — the place the map fails, never positively drawn.

Theorem: catseam_is_frameflip (ZeroParadox/Category/SeamFrameChange.lean)

The zero-object seam Z = fD_functor.obj 0 is the μ-bottom (initial) in one chart and the ν-top (terminal) in the other; the op-duality frame-change swaps the two — carrying Z's initial-ness to a terminal-ness of Z^{op} and its terminal-ness to an initial-ness of Z^{op} (terminalOpOfInitial / initialOpOfTerminal). The seam is the op-self-dual zero object.

Off the zero object the coincidence fails (ZeroParadox/Category/SeamLimColim.lean: generic_object_empty_lim_ne_colim) — as with 0 on the sphere, the frame-change fixes only the bottom.

Lean purity: [propext, Classical.choice, Quot.sound] — choice-carrying (inherited from the ModuleCat / category-theory library). ✓

The remaining realizations are referenced, not re-proved here. Set theory: the Quine atom (CC-2) is the collapse of the Foundation / AFA fork (ZP-J, ZP-P; fork_collapse_iff, selfMem_eq_singleton_bot), stated as a role, a point that steps to itself and to nothing else, whose occupancy is settled chart by chart, with no equation across charts: occupied by ⊥ in the lattice chart, under a self-application map supplied by AbstractSelfApp (selfApp_quine_role_iff) and in the QuineAtomHost an AFAStructure on L builds (quineAtomHost_quine_role_iff with afaStructure_isQuineAtomHost), by no set in Mathlib's ZFSet (zfset_quine_role_unoccupied). Computability: ZP-K's class KleeneStructure requires a code meeting a periodicity condition (botCode_is_quine) and equates it with nothing. The reading that this code is the bottom's computational witness splits: its identity half is retired (no single code occupies the Quine-atom role under the eval step, evalStepMotion_not_selfOnlyAt, while a class of codes up to equal eval does, for a computable map respecting equal eval, codes_vs_classes), and its running half, that the bottom executes, is the occurrence commitment. That the bottom is the unique fixed point of self-application is the AbstractSelfApp class field unique_fp (ZP-J), a requirement rather than a theorem. State / Hilbert: ⊥ is the unique finite-dimensional fixed point of the biproduct-diagonal $X \mapsto X \boxplus X$ (ZP-H; biprod_diagonal_only_zero). Information: measure runs to 0 at the floor exactly as surprisal runs to ∞ (ZP-C; info_bottom_diverges, addVal_bot), the two joined by −log.

Remark: the reals beside $\mathbb{Q}_2$ (what separates them)

Inversion does not separate $\mathbb{R}$ from $\mathbb{Q}_2$. Inside each field it is discontinuous at 0. On the one-point compactification of any proper normed field it is continuous and swaps 0 and ∞, a standard fact for the Riemann sphere (Forster, Lectures on Riemann Surfaces, GTM 81, §§ 1–2, as cited by TauCeti; not retrieved), which TauCeti formalizes for proper nontrivially normed fields: OnePoint.invChart (a chart) and OnePoint.instContinuousConstSMul (the global action). It is proved here with Mathlib alone (continuous_onePointInv, onePointInv_swaps, ZeroParadox/Valuation/RiemannSphere.lean), so for $\mathbb{R}$ as well as $\mathbb{Q}_2$ (continuous_onePointInv_real, onePointInv_swaps_real; for $\mathbb{Q}_2$, the sphere above, continuous_rInv and rInv_swaps). What separates the two is topology: $\mathbb{R}$ is connected (place_topological_signature), so not totally disconnected (real_no_snap), while $\mathbb{Q}_2$ is totally disconnected (padic_snaps). Neither has a first step above 0, AX-B1's form: in $\mathbb{R}$ because no linearly ordered field has a least positive element (ZP-F; f_snap_impossible, which, despite its name, states exactly that), and in $\mathbb{Q}_2$, which carries no order of its own, under the norm preorder (ZeroParadox/BottomCannotBe.lean § ROLE). T-SNAP's shape does not need a first step: t_snap_given holds at a step taken in a densely ordered join-semilattice ($\mathbb{Q}\geq 0$ under max, ZeroParadox/Order/Snap.lean).

Section III: The Universal Form and Its Walls (Tier 3)

Is there a single universal frame-flip — one theorem all the faces are cases of? No, and the question resolves into two walls of different kinds. At the order-theoretic level a single choice-free theorem does exist over any complete lattice — fork_is_frameflip (Section I) — but as Section I records, none of the framework's min≡max faces satisfies its hypotheses of a complete lattice and a monotone map, so none of them is an instance of it. It is universal over its own domain, not over the faces. At the categorical level the universal is a proven wall (below). What the faces share is a SHAPE realized concretely in each domain. The face table's last row, the reals, carries the same pole exchange on its one-point compactification (onePointInv_swaps_real) and has no first step above 0 (f_snap_impossible), so the frame-flip does not separate $\mathbb{R}$ from $\mathbb{Q}_2$; total disconnectedness does (padic_snaps, real_no_snap). The two walls are not hedging: they mark where the universal form stops, and each per-domain frame-flip remains a full theorem in its own domain.

The wall: the categorical universal is category-relative (ZeroParadox/Category/Lawvere.lean)

Lawvere's fixed-point theorem derives a fixed point from a point-surjection $\alpha \to (\alpha \to \beta)$. ZP asks whether each bottom is such a Lawvere-produced fixed point. The answer is category-relative, and it is proved. In Set (all endofunctions) no nontrivial total type carries a Lawvere witness — Cantor forbids it (nontrivial_lattice_no_witness, q2_no_witness) — so the lattice and 2-adic bottoms share the diagonal shape but are provably NOT literal Lawvere instances. Only in the effective category is there a genuine one, where the fixed-point-free diagonal construction is not admissible — which is not the claim that no fixed-point-free computable endomap exists, since fun c => Code.pair c c is one (code_pair_diag_fixedPointFree); removing the obstruction does not by itself supply the witness, and the fixed point is cited rather than derived (computability_face_fixedPoint). So the keystone unifies a shape, not a single mechanism.

Hard fence (permanent): cross-domain identity is a type boundary

The claim that the contact points of all the faces — the p-adic 0, the Quine atom, the Kleene quine, the categorical zero object — are numerically one object is not a theorem and cannot become one. Identity requires a shared type; these are terms of different types in different categories, so "x = y" across them is not false but not well-formed. This is proved at the measure level too: BottomMeasure.lean shows "+∞ at every floor" is a per-domain bundle in incompatible types, not one invariant. The interconnect is therefore point-of-view-specific. What is claimed formally is only that each domain forks, and flips, at its own contact point.

The fence is the bridge. The frame-change that joins the two poles (the bridge) and the diagonal that forbids collapsing them into one object (the fence) are the same diagonal. The bridge only exists between poles that stay two: were they merged there would be nothing to cross. So the boundary is not the price of the connection — it is its condition. The keystone is bounded by the exact mechanism it is built from: the diagonal argument makes the fixed point and, in the same gesture, forbids its universal identification. Per-face frame-flips are theorems and are not hedged; the fences scope only the universal cross-category claim.

Theorem Summary

Result	Lean source (full path)	Axioms
lfp_dual_eq_gfp	ZeroParadox/Settheory/ForkFrameChange.lean	choice-free
gfp_dual_eq_lfp	ZeroParadox/Settheory/ForkFrameChange.lean	choice-free
fork_is_frameflip	ZeroParadox/Settheory/ForkFrameChange.lean	choice-free
fork_collapse_iff	ZeroParadox/Settheory/FixedPointFork.lean	choice-free
snap_is_frameflip	ZeroParadox/Multihomed/SnapFrameChange.lean	Classical.choice
rInv_swaps	ZeroParadox/Valuation/RiemannSphere.lean	Classical.choice
onePointInv_swaps	ZeroParadox/Valuation/RiemannSphere.lean	Classical.choice
onePointInv_swaps_real	ZeroParadox/Valuation/RiemannSphere.lean	Classical.choice
catseam_is_frameflip	ZeroParadox/Category/SeamFrameChange.lean	Classical.choice
nontrivial_lattice_no_witness	ZeroParadox/Category/Lawvere.lean	Classical.choice
computability_face_fixedPoint	ZeroParadox/Category/Lawvere.lean	Classical.choice

Axiom Purity

Frame-flip schema (ZeroParadox/Settheory/ForkFrameChange.lean): [propext, Quot.sound] — choice-free. The universal needs no Axiom of Choice; the duality-swap is definitional.

Valuation face (ZeroParadox/Multihomed/SnapFrameChange.lean / ZeroParadox/Valuation/RiemannSphere.lean): [propext, Classical.choice, Quot.sound] — Classical.choice inherited from the p-adic / one-point-compactification analysis.

Axiom Purity
Categorical face (ZeroParadox/Category/SeamFrameChange.lean): [propext, Classical.choice, Quot.sound] — inherited from the ModuleCat / category-theory library.
The wall (ZeroParadox/Category/Lawvere.lean): the concrete Cantor no-witness results (nontrivial_lattice_no_witness, q2_no_witness) and computability_face_fixedPoint (Kleene's recursion theorem) carry Classical.choice; the abstract witness machinery (fixedPoint_of_witness, no_witness_of_fixedPointFree) is axiom-free.
Core choice-free, realisation choice-carrying — the framework's standing pattern.
The footprints listed here are copies; the live record is the #print axioms output of each named .lean file.

End of ZP-Q | The Frame-Change | fork_is_frameflip (choice-free order-theoretic universal) | the Riemann-sphere valuation face (snap_is_frameflip) and the op-duality categorical face (catseam_is_frameflip), neither an instance of it | the categorical universal is a proven wall (Cantor; ZeroParadox/Category/Lawvere.lean) | the fence is the bridge | the bottom is never positively represented.