THE ZERO PARADOX

ZP Addendum

The Choice-Free Core

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Build the file ZeroParadox/AxiomProfile.lean and read the output of Lean's `#print axioms` command. It reports that the central theorem of this framework — the Binary Snap, T-SNAP, whose SHAPE is derived while its occurrence follows from the occurrence commitment together with DA-1 (closed given DP-2), the transition $\bot \to \varepsilon_0$ — depends on no axioms at all: not the Axiom of Choice, not even propositional extensionality. The lattice algebra (ZP-A) and the Quine-atom self-reference that is the framework's keystone (ZP-J) are likewise choice-free. This is a machine-checked fact, not a claim of the prose: anyone can run `lake build ZeroParadox.AxiomProfile` and see it.

Two boundaries are stated up front, because the claim is narrow and exact. The framework as a whole is not choice-free. Many of its theorems depend on `Classical.choice` (no proportion is measured here). Many of the places it appears are places where the framework builds on Mathlib's classically-built analysis, order, and computability libraries — the layers that realize the snap inside standard analytic structures (p-adic topology, Hilbert space, ordinals), where the dependence is inherited from those libraries. Not all of it is inherited: some is the framework's own bare classical. ZeroParadox/Category/ChoiceCannotBe.lean § IV collects cases, of either provenance, whose principles a reduction shows essential rather than incidental, given a premise stated there (Section III); not every bare classical in the framework is among them. Necessity is a separate axis from provenance: an inherited dependence is shown essential the same way, on the same premise. None of it is used by the core results above. And dependence is not necessity: that those realizations use choice as written does not show the mathematics requires it. Several of their statements carry it as written (Section III), so no proof of those statements can drop it; whether a restatement on a carrier whose statements are choice-free holds is a separate question.

Section I: The Choice-Free Core

Lean's `#print axioms` command reports the complete axiom dependency of any result. Run on the framework's central results, it returns the following. "Does not depend on any axioms" is the strongest possible report — stronger than "choice-free," since it uses not even propositional extensionality.

Verified axiom-free (does not depend on any axioms)
T-SNAP, the Binary Snap and its derivation (ZP-E):
t_snap_machine, t_snap_derived, t_snap_join, t_snap_irreversible,
da1_minimal_path, dp2_execution_distinguishability.

Verified axiom-free (does not depend on any axioms)
The lattice algebra (ZP-A): bot_le, the order laws, cc1.
The Quine-atom self-reference keystone (ZP-J): bot_is_quine_atom, cc1_derived,
t_exec, quine_atom_unique.

A second tier of results is choice-free but uses propositional extensionality and quotient soundness (`[propext, Quot.sound]`), both standard in Lean 4. These include the power set's depth chart (ps_structural_floor, ZeroParadox/Order/PowerSet.lean) and the wheel of fractions (ZeroParadox.instWheel and inf_ne_bot, ZeroParadox/Algebra/WheelFrac.lean). No `Classical.choice`.

Section II: Where Classical.choice Enters

The honest contrast. `Classical.choice` does appear across the framework, in many of its theorems. Each file's `#print axioms` block reports, declaration by declaration, which carry it, and ZeroParadox/AxiomProfile.lean prints a selection (§§ 0–III). Many occurrences are in a layer that realizes the snap floor inside a standard analytic structure, inheriting choice from the Mathlib library that builds that structure classically. Not every occurrence is inherited: the framework writes bare classicals of its own, and ZeroParadox/Category/ChoiceCannotBe.lean § IV collects cases among them whose principles reduce to a taboo, with those reductions — see Section III.

Carries Classical.choice (inherited from Mathlib), e.g.
ZP-B c3_irreversible — p-adic topology (metric / ultrametric library).
ZP-D t4_snap_orthogonal — Hilbert space (inner-product library).
ZP-H fB_functor / fD_functor / fC_functor — TopCat / ModuleCat ℂ /
the Kleisli category of the probability monad.
and the ordinal (ZP-L/M), information (ZP-C), and computability (ZP-K) layers.
Footprint in each case: [propext, Classical.choice, Quot.sound].

The pattern: the core states the result; the analytic layers realize it inside the standard frameworks, and that is where the library's classical foundations enter. For several of those layers the choice is already in the statement as written (Section III), and Section III points at principles that are essential given a premise stated there. Their provenance differs, and that is the point: some are proved with the framework's own bare classical, and some with Mathlib's, among them well-order comparability (in InitialSeg.total) — and each is essential on that premise. Inherited never meant removable.

Section III: Dependence Is Not Necessity

`#print axioms` on a proof shows dependence — that the proof as written uses an axiom. It does not show a principle's necessity — that no choice-free proof exists. Necessity has to be shown a different way, by a reduction: derive a recognised constructive taboo from the principle itself.

Such reductions do exist here, and they bound what this document claims. ZeroParadox/Category/ChoiceCannotBe.lean § IV collects such reductions, with the premise they rest on, and that file indexes others besides. For example, `em_of_wellOrder_comparable` derives excluded middle from comparability of well-orders (prior art: Kraus, Nordvall Forsberg and Xu, arXiv:2104.02549, Thm 38(d)), and `wem_of_fixedPointFree` derives weak excluded middle from the general fixed-point-free principle (related prior art, cited not claimed, as recorded in ZeroParadox/Category/LawvereTaboo.lean: Bauer, "On fixed-point theorems in synthetic computability", Tbilisi Math. J. 10(3) (2017), Cor 4.3; de Jong–Escardó, "Predicative Aspects of Order Theory in Univalent Foundations", arXiv:2102.08812, Cor 39(iii)-(iv); Booij–Escardó–Lumsdaine–Shulman, "Parametricity, automorphisms of the universe, and excluded middle", arXiv:1701.05617, Thm 3, and Thm 5 (Simpson), whose proof technique the reduction uses) — the latter sitting on the keystone rather than on an imported order instance; § IV's cases 3 to 5 are reductions for principles proved with bare classicals in ZeroParadox/Die.lean. Each is a statement about the principle: re-proving it constructively would prove a taboo constructively, so no choice-free re-proof exists given that the taboo is not derivable in Lean without Classical.choice — a premise this corpus does not prove (§ IV). None is an independence result, and none is established by a footprint measurement — which is precisely why the accidental side needs a measurement and the essential side needs a reduction.

So the remaining analytic-layer dependence may be removable, and that is a different question from the reductions above. It is also not settled by measuring a proof: where an axiom sits in a type rather than a proof, no proof of any statement mentioning that type can be clean, and removability there means changing the statement, not cleaning the argument. That case is measured: the statements of c3_irreversible, t4_snap_orthogonal, fD_functor and fC_functor, each only assumed, already report Classical.choice (the statement-carried row of ZeroParadox/AxiomProfile.md), so they are statement-carried, and the open question for them is a restatement on a carrier whose statements are choice-free. fB_functor's statement is choice-free; its dependence stays open.

One layer has been classified directly. In the "choice-probe" experiment, the `Classical.choice` in the 2-adic tree construction (PadicTree) decomposed into three sources: incidental tactic artifacts (removed, leaving those results choice-free); Mathlib's classically-proved connectivity API (routable by a path-uniqueness reformulation); and `sInf` on a complete lattice (routable by a redefinition). The verdict for that layer was "mostly not structurally required." Whether this generalizes — and whether the snap geometry forces choice anywhere — is an open question. On ordinal notations, ZP-N restates the ordinal ascent choice-free; the state of the restatement question is in ZeroParadox/Ordinal/SnapNucleus.md, section Axiom footprint.

Remark — Why this matters

The Zero Paradox argues that the foundational axioms are not freely chosen but forced by the structure of the bottom element. It would be a tension if the framework's own central results leaned on the Axiom of Choice — the canonical free, non-constructive selection. They do not. T-SNAP is axiom-free; the keystone is choice-free. The "forced, not chosen" thesis is internally consistent at the level of what the framework actually asserts. Where choice appears it is often the supporting library's classical foundation showing through the realizations rather than an assumption of the argument. The reductions Section III points at are the exception on a different axis: they are about necessity, not provenance. Some of their principles are proved with the framework's own bare classical and some with Mathlib's, among them well-order comparability, and each is essential on the premise stated in Section III — so "inherited" is not a synonym for "removable", and none of these cases is absorbed into the generalization.

The Artifact

Checkable evidence

ZeroParadox/AxiomProfile.lean — a file of `#print axioms` commands. Its § 0 prints footprints for some proofs written with the framework's own classical, for reductions to taboos of the kind ZeroParadox/Category/ChoiceCannotBe.lean § IV collects, and for related declarations; § I prints the choice-free core and § II further choice-free results; § III prints the analytic-realization results that carry choice, as an honest contrast. Build with `lake build ZeroParadox.AxiomProfile` and read the `#print axioms` report. The per-theorem map is in the project repository.

Choice-probe (the one-layer classification): branch `choice-probe`; the verdict and the three decomposed sources are recorded in the project notes.

Scope of the claim

The claim is exactly: the framework's central results — T-SNAP, the lattice, the Quine-atom self-reference — are choice-free, and T-SNAP is axiom-free. NOT claimed: that the whole framework is choice-free (it is not), nor that the analytic-layer choice is removable or necessary. Where a statement carries it as written (Section III), no proof of that statement can drop it and the open question is a restatement on a carrier whose statements are choice-free (for the ε_0 results, its two readings and the search record are in ZeroParadox/Ordinal/SnapNucleus.md, section Axiom footprint); elsewhere necessity is open. The fact surfaced here is the verified one.

Endnote: This is a framework-wide note; the footprints it cites are copies, and the live record is the #print axioms output of the files it names. The central theorem T-SNAP depends on no axioms; the conceptual core is free of the Axiom of Choice; `Classical.choice` appears in many places where the framework builds on Mathlib's classical analysis, order, and computability libraries, where several statements carry it as written (for those, the open question is a restatement) and elsewhere whether it is necessary remains open. It is not only there: the framework carries bare classicals of its own, and the reductions Section III points at (ZeroParadox/Category/ChoiceCannotBe.lean § IV) settle necessity for the principles they cover, given the premise stated there. All of this is checkable in ZeroParadox/AxiomProfile.lean.