The Clowder Project

2017–2024

The Clowder Project Authors

Version 01f3714, compiled on Apr 04, 2024.

Contents

Contents

Part I

Sets

Chapter 1

Sets

0000 This chapter (will eventually) contain material on axiomatic set theory, as well as a couple other things.

Contents

0001 1.1 Sets and Functions

0002 1.1.1 Functions

0003 **Definition 1.1.1.1.1.** A **function** is a functional and total relation.

0004 **Notation 1.1.1.1.2.** Throughout this work, we will sometimes denote a function $f: X \to Y$ by

$$f \stackrel{\text{def}}{=} [\![x \mapsto f(x)]\!].$$

1. For example, given a function

$$\Phi: \mathrm{Hom}_{\mathsf{Sets}}(X, Y) \to K$$

taking values on a set of functions such as $\mathrm{Hom}_{\mathsf{Sets}}(X, Y)$, we will sometimes also write

$$\Phi(f) \stackrel{\text{def}}{=} \Phi([\![x \mapsto f(x)]\!]).$$

2. This notational choice is based on the lambda notation

$$f \stackrel{\text{def}}{=} (\lambda x.\ f(x)),$$

but uses a "$\mapsto$" symbol for better spacing and double brackets instead of either:

(a) Square brackets $[x \mapsto f(x)]$;
(b) Parentheses $(x \mapsto f(x))$;

hoping to improve readability when dealing with e.g.:

(a) Equivalence classes, cf.:
   i. $[\![[x] \mapsto f([x])]\!]$
   ii. $[[x] \mapsto f([x])]$
   iii. $(\lambda[x].\ f([x]))$
(b) Function evaluations, cf.:
   i. $\Phi([\![x \mapsto f(x)]\!])$
   ii. $\Phi((x \mapsto f(x)))$
   iii. $\Phi((\lambda x.\ f(x)))$

3. We will also sometimes write $-_1$, $-_2$, etc. for the arguments of a function. Some examples include:

(a) Writing $f(-_1)$ for a function $f\colon A \to B$.
(b) Writing $f(-_1, -_2)$ for a function $f\colon A \times B \to C$.
(c) Given a function $f\colon A \times B \to C$, writing

$$f(a, -)\colon B \to C$$

for the function $[\![b \mapsto f(a, b)]\!]$.

(d) Denoting a composition of the form

$$A \times B \xrightarrow{\phi \times \mathrm{id}_B} A' \times B \xrightarrow{f} C$$

by $f(\phi(-_1), -_2)$.

4. Finally, given a function $f\colon A \to B$, we write

$$\mathrm{ev}_a(f) \stackrel{\text{def}}{=} f(a)$$

for the value of f at some $a \in A$.

For an example of the above notations being used in practice, see the proof of the adjunction

$$(A \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(A, -))\colon\ \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets} \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Sets}}(A,-)"] \end{tikzcd},$$

stated in Item 2 of Proposition 2.1.3.1.2.

1.2 The Enrichment of Sets in Classical Truth
0005 Values

0006 1.2.1 (-2)-Categories

0007 **Definition 1.2.1.1.1.** A (-2)**-category** is the "necessarily true" truth value.[1,2,3]

0008 1.2.2 (-1)-Categories

0009 **Definition 1.2.2.1.1.** A (-1)**-category** is a classical truth value.

000A **Remark 1.2.2.1.2.** [4](-1)-categories should be thought of as being "categories enriched in (-2)-categories", having a collection of objects and, for each pair of objects, a Hom-object $\mathrm{Hom}(x, y)$ that is a (-2)-category (i.e. trivial).
Therefore, a (-1)-category $\mathcal{C}$ is either ([BS10, pp. 33–34]):

1. *Empty*, having no objects;
2. *Contractible*, having a collection of objects $\{a, b, c, \ldots\}$, but with $\mathrm{Hom}_{\mathcal{C}}(a, b)$ being a (-2)-category (i.e. trivial) for all $a, b \in \mathrm{Obj}(\mathcal{C})$, forcing all objects of $\mathcal{C}$ to be uniquely isomorphic to each other.

As such, there are only two (-1)-categories, up to equivalence:

- The (-1)-category false (the empty one);
- The (-1)-category true (the contractible one).

000B **Definition 1.2.2.1.3.** The **poset of truth values**[5] is the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ consisting of

[1] Thus, there is only one (-2)-category.

[2] A $(-n)$-category for $n = 3, 4, \ldots$ is also the "necessarily true" truth value, coinciding with a (-2)-category.

[3] For motivation, see [BS10, p. 13].

[4] For more motivation, see [BS10, p. 13].

[5] *Further Terminology:* Also called the **poset of** (-1)**-categories**.

- *The Underlying Set.* The set $\{\mathsf{true}, \mathsf{false}\}$ whose elements are the truth values true and false.
- *The Partial Order.* The partial order

  $$\preceq\colon \{\mathsf{true}, \mathsf{false}\} \times \{\mathsf{true}, \mathsf{false}\} \to \{\mathsf{true}, \mathsf{false}\}$$

  on $\{\mathsf{true}, \mathsf{false}\}$ defined by[6]

  $$\begin{aligned} \mathsf{false} \preceq \mathsf{false} &\overset{\text{def}}{=} \mathsf{true}, \\ \mathsf{true} \preceq \mathsf{false} &\overset{\text{def}}{=} \mathsf{false}, \\ \mathsf{false} \preceq \mathsf{true} &\overset{\text{def}}{=} \mathsf{true}, \\ \mathsf{true} \preceq \mathsf{true} &\overset{\text{def}}{=} \mathsf{true}. \end{aligned}$$

000C **Notation 1.2.2.1.4.** We also write $\{\mathsf{t}, \mathsf{f}\}$ for the poset $\{\mathsf{true}, \mathsf{false}\}$.

000D **Proposition 1.2.2.1.5.** The poset of truth values $\{\mathsf{t}, \mathsf{f}\}$ is Cartesian closed with product given by[7]

$$\begin{aligned} \mathsf{t} \times \mathsf{t} &= \mathsf{t}, \\ \mathsf{t} \times \mathsf{f} &= \mathsf{f}, \\ \mathsf{f} \times \mathsf{t} &= \mathsf{f}, \\ \mathsf{f} \times \mathsf{f} &= \mathsf{f}, \end{aligned}$$

and internal Hom $\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}$ given by the partial order of $\{\mathsf{t}, \mathsf{f}\}$, i.e. by

$$\begin{aligned} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t}) &= \mathsf{t}, \\ \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f}) &= \mathsf{f}, \\ \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) &= \mathsf{t}, \\ \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) &= \mathsf{t}. \end{aligned}$$

Proof. Existence of Products: We claim that the products $\mathsf{t} \times \mathsf{t}$, $\mathsf{t} \times \mathsf{f}$, $\mathsf{f} \times \mathsf{t}$, and $\mathsf{f} \times \mathsf{f}$ satisfy the universal property of the product in $\{\mathsf{t}, \mathsf{f}\}$. Indeed, consider the diagrams

$$\begin{tikzcd} & P_1 \arrow[dl, "p^1_1"'] \arrow[d, dashed, "\exists!"] \arrow[dr, "p^1_2"] & \\ \mathsf{t} & \underbrace{\mathsf{t} \times \mathsf{t}}_{=\mathsf{t}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{t} \end{tikzcd} \quad \begin{tikzcd} & P_2 \arrow[dl, "p^2_1"'] \arrow[d, dashed, "\exists!"] \arrow[dr, "p^2_2"] & \\ \mathsf{t} & \underbrace{\mathsf{t} \times \mathsf{f}}_{=\mathsf{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{f} \end{tikzcd} \quad \begin{tikzcd} & P_3 \arrow[dl, "p^3_1"'] \arrow[d, dashed, "\exists!"] \arrow[dr, "p^3_2"] & \\ \mathsf{f} & \underbrace{\mathsf{f} \times \mathsf{t}}_{=\mathsf{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{t} \end{tikzcd} \quad \begin{tikzcd} & P_4 \arrow[dl, "p^4_1"'] \arrow[d, dashed, "\exists!"] \arrow[dr, "p^4_2"] & \\ \mathsf{f} & \underbrace{\mathsf{f} \times \mathsf{f}}_{=\mathsf{f}} \arrow[l, two heads, "\mathrm{pr}_1"] \arrow[r, two heads, "\mathrm{pr}_2"'] & \mathsf{f}. \end{tikzcd}$$

Here:

[6]This partial order coincides with logical implication.

[7]Note that $\times$ coincides with the "and" operator, while $\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}$ coincides with

1. If $P_1 = \mathsf{t}$, then $p_1^1 = p_2^1 = \mathrm{id}_{\mathsf{t}}$, and there's indeed a unique morphism from P_1 to t making the diagram commute, namely id_{t};
2. If $P_1 = \mathsf{f}$, then $p_1^1 = p_2^1$ are given by the unique morphism from f to t, and there's indeed a unique morphism from P_1 to t making the diagram commute, namely the unique morphism from f to t;
3. If $P_2 = \mathsf{t}$, then there is no morphism p_2^2.
4. If $P_2 = \mathsf{f}$, then p_1^2 is the unique morphism from f to t while $p_2^2 = \mathrm{id}_{\mathsf{f}}$, and there's indeed a unique morphism from P_2 to f making the diagram commute, namely id_{f};
5. The proof for P_3 is similar to the one for P_2;
6. If $P_4 = \mathsf{t}$, then there is no morphism p_1^4 or p_2^4.
7. If $P_4 = \mathsf{f}$, then $p_1^4 = p_2^4 = \mathrm{id}_{\mathsf{f}}$, and there's indeed a unique morphism from P_4 to f making the diagram commute, namely id_{f}.

Cartesian Closedness: We claim there's a bijection

$$\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(A \times B, C) \cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(A, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(B, C)\Big)$$

natural in $A, B, C \in \{\mathsf{t}, \mathsf{f}\}$. Indeed:

- For $(A, B, C) = (\mathsf{t}, \mathsf{t}, \mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{t}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\\ &= \{\mathrm{id}_{\mathsf{true}}\}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\Big).\end{aligned}$$

- For $(A, B, C) = (\mathsf{t}, \mathsf{t}, \mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{t}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\\ &= \emptyset\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\Big).\end{aligned}$$

- For $(A, B, C) = (\mathsf{t}, \mathsf{f}, \mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{f}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{pt}\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\Big).\end{aligned}$$

- For $(A, B, C) = (\mathsf{t}, \mathsf{f}, \mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t} \times \mathsf{f}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \{\mathrm{id}_{\mathsf{false}}\} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{t}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\Big).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{t}, \mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{t}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{pt} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{t})\Big).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{t}, \mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{t}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \{\mathrm{id}_{\mathsf{false}}\} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{t}, \mathsf{f})\Big).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{f}, \mathsf{t})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{f}, \mathsf{t}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{pt} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{t})\Big).\end{aligned}$$

- For $(A, B, C) = (\mathsf{f}, \mathsf{f}, \mathsf{f})$, we have

$$\begin{aligned}\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f} \times \mathsf{f}, \mathsf{f}) &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &= \{\mathrm{id}_{\mathsf{false}}\} \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f}) \\ &\cong \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(\mathsf{f}, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(\mathsf{f}, \mathsf{f})\Big).\end{aligned}$$

The proof of naturality is omitted. □

the logical implication operator.

000E
1.2.3 0-Categories

000F **Definition 1.2.3.1.1.** A 0**-category** is a poset.[8]

000G **Definition 1.2.3.1.2.** A 0**-groupoid** is a 0-category in which every morphism is invertible.[9]

000H
1.2.4 Tables of Analogies Between Set Theory and Category Theory

Here we record some analogies between notions in set theory and category theory. Note that the analogies relating to presheaves relate equally well to copresheaves, as the opposite X^{op} of a set X is just X again.
Basics:

Set Theory	Category Theory
Enrichment in $\{\mathsf{true}, \mathsf{false}\}$	Enrichment in Sets
Set X	Category $\mathcal{C}$
Element $x \in X$	Object $X \in \mathrm{Obj}(\mathcal{C})$
Function	Functor
Function $X \to \{\mathsf{true}, \mathsf{false}\}$	Functor $\mathcal{C} \to \mathsf{Sets}$
Function $X \to \{\mathsf{true}, \mathsf{false}\}$	Presheaf $\mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$

Powersets and categories of presheaves:

[8] *Motivation:* A 0-category is precisely a category enriched in the poset of (-1)-categories.
[9] That is, a *set*.

Set Theory	Category Theory
Powerset $\mathcal{P}(X)$	Presheaf category $\mathsf{PSh}(\mathcal{C})$
Characteristic function $\chi_{\{x\}}$	Representable presheaf h_X
Characteristic embedding $\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$	Yoneda embedding $\text{よ} \colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$
Characteristic relation $\chi_X(-_1, -_2)$	Hom profunctor $\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)$
The Yoneda lemma for sets $\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$	The Yoneda lemma for categories $\mathrm{Nat}(h_X, \mathcal{F}) \cong \mathcal{F}(X)$
The characteristic embedding is fully faithful, $\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y)$	The Yoneda embedding is fully faithful, $\mathrm{Nat}(h_X, h_Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$
Subsets are unions of their elements $U = \bigcup_{x \in U} \{x\}$ or $\chi_U = \operatorname*{colim}_{\chi_x \in \mathsf{Sets}(U, \{\mathsf{t}, \mathsf{f}\})} (\chi_x)$	Presheaves are colimits of representables, $\mathcal{F} \cong \operatorname*{colim}_{h_X \in \int_{\mathcal{C}} \mathcal{F}} (h_X)$

Categories of elements:

Set Theory	Category Theory
Assignment $U \mapsto \chi_U$	Assignment $\mathcal{F} \mapsto \int_{\mathcal{C}} \mathcal{F}$ (the category of elements)
Assignment $U \mapsto \chi_U$ giving an isomorphism $\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$	Assignment $\mathcal{F} \mapsto \int_{\mathcal{C}} \mathcal{F}$ giving an equivalence $\mathsf{PSh}(\mathcal{C}) \overset{\text{eq.}}{\cong} \mathsf{DFib}(\mathcal{C})$

Functions between powersets and functors between presheaf categories:

Set Theory	Category Theory
Direct image function $f_* \colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Inverse image functor $f^{-1} \colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$
Inverse image function $f^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$	Direct image functor $f_* \colon \mathsf{PSh}(\mathcal{D}) \to \mathsf{PSh}(\mathcal{C})$
Direct image with compact support function $f_! \colon \mathcal{P}(X) \to \mathcal{P}(Y)$	Direct image with compact support functor $f_! \colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$

Relations and profunctors:

Set Theory	Category Theory
Relation $R\colon X\times Y\to\{\mathsf{t},\mathsf{f}\}$	Profunctor $\mathfrak{p}\colon\mathcal{D}^{\mathsf{op}}\times\mathcal{C}\to\mathsf{Sets}$
Relation $R\colon X\to\mathcal{P}(Y)$	Profunctor $\mathfrak{p}\colon\mathcal{C}\to\mathsf{PSh}(\mathcal{D})$
Relation as a cocontinuous morphism of posets $R\colon(\mathcal{P}(X),\subset)\to(\mathcal{P}(Y),\subset)$	Profunctor as a colimit-preserving functor $\mathfrak{p}\colon\mathsf{PSh}(\mathcal{C})\to\mathsf{PSh}(\mathcal{D})$

Appendices

1.A Other Chapters

Chapter 2

Constructions With Sets

000J This chapter develops some material relating to constructions with sets with an eye towards its categorical and higher-categorical counterparts to be introduced later in this work. In particular, it contains:

1. Explicit descriptions of the major types of co/limits in Sets, including in particular explicit descriptions of pushouts and coequalisers (see Definitions 2.2.4.1.1 and 2.2.5.1.1 and Remarks 2.2.4.1.2 and 2.2.5.1.2).

2. A discussion of powersets as decategorifications of categories of presheaves (Remarks 2.4.1.1.2 and 2.4.3.1.2), including a (-1)-categorical analogue of un/straightening, described in Items 1 and 2 of Proposition 2.4.3.1.6 and Remark 2.4.3.1.7.

3. A lengthy discussion of the adjoint triple

$$f_* \dashv f^{-1} \dashv f_! \colon \mathcal{P}(A) \overset{\rightleftarrows}{\to} \mathcal{P}(B)$$

   of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a map of sets $f \colon A \to B$, along with a discussion of the properties of f_*, f^{-1}, and $f_!$.

   In line with the categorical viewpoint developed here, this adjoint triple may be described in terms of Kan extensions, and, as it turns out, it also shows up in some definitions and results in point-set topology, such as in e.g. notions of continuity for functions (**??**).

Contents

000K 2.1 Limits of Sets

000L 2.1.1 The Terminal Set

000M **Definition 2.1.1.1.1.** The **terminal set** is the pair $\left(\mathrm{pt}, \{!_A\}_{A \in \mathrm{Obj}(\mathsf{Sets})}\right)$ consisting of:

- *The Limit.* The punctual set $\mathrm{pt} \stackrel{\text{def}}{=} \{\star\}$.

- *The Cone.* The collection of maps

$$\{!_A \colon A \to \mathrm{pt}\}_{A \in \mathrm{Obj}(\mathsf{Sets})}$$

defined by

$$!_A(a) \overset{\mathrm{def}}{=} \star$$

for each $a \in A$ and each $A \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that pt is the terminal object of Sets. Indeed, suppose we have a diagram of the form

$$A \qquad \mathrm{pt}$$

in Sets. Then there exists a unique map $\phi \colon A \to \mathrm{pt}$ making the diagram

$$A \overset{\phi}{\underset{\exists!}{\dashrightarrow}} \mathrm{pt}$$

commute, namely $!_A$. □

000N 2.1.2 Products of Families of Sets

Let $\{A_i\}_{i \in I}$ be a family of sets.

000P **Definition 2.1.2.1.1.** The **product**[1] **of** $\{A_i\}_{i \in I}$ is the pair $(\prod_{i \in I} A_i, \{\mathrm{pr}_i\}_{i \in I})$ consisting of:

- *The Limit.* The set $\prod_{i \in I} A_i$ defined by[2]

$$\prod_{i \in I} A_i \overset{\mathrm{def}}{=} \left\{ f \in \mathsf{Sets}\left(I, \bigcup_{i \in I} A_i\right) \;\middle|\; \begin{array}{l}\text{for each } i \in I,\\ \text{we have } f(i) \in \\ A_i\end{array} \right\}.$$

- *The Cone.* The collection

$$\left\{ \mathrm{pr}_i \colon \prod_{i \in I} A_i \to A_i \right\}_{i \in I}$$

of maps given by

$$\mathrm{pr}_i(f) \overset{\mathrm{def}}{=} f(i)$$

for each $f \in \prod_{i \in I} A_i$ and each $i \in I$.

[1] *Further Terminology:* Also called the **Cartesian product of** $\{A_i\}_{i \in I}$.

[2] Less formally, $\prod_{i \in I} A_i$ is the set whose elements are I-indexed collections $(a_i)_{i \in I}$

Proof. We claim that $\prod_{i\in I} A_i$ is the categorical product of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} P \arrow[dr, "p_i"] & \\ \prod\limits_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon P \to \prod_{i\in I} A_i$ making the diagram

$$\begin{tikzcd} P \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\ \prod\limits_{i\in I} A_i \arrow[r, "\mathrm{pr}_i"'] & A_i \end{tikzcd}$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. $\square$

000Q **Proposition 2.1.2.1.2.** Let $\{A_i\}_{i\in I}$ be a family of sets.

000R 1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \prod_{i\in I} A_i$ defines a functor

$$\prod_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\prod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\text{def}}{=} \prod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$,

---

with $a_i \in A_i$ for each $i \in I$. The projection maps

$$\left\{\mathrm{pr}_i\colon \prod_{i\in I} A_i \to A_i\right\}_{i\in I}$$

are then given by

$$\mathrm{pr}_i\big((a_j)_{j\in I}\big) \stackrel{\text{def}}{=} a_i$$

for each $(a_j)_{j\in I} \in \prod_{i\in I} A_i$ and each $i \in I$.

the action on Hom-sets

$$\left(\prod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}} : \mathrm{Nat}\big((A_i)_{i\in I},(B_i)_{i\in I}\big) \to \mathsf{Sets}\left(\prod_{i\in I} A_i, \prod_{i\in I} B_i\right)$$

of $\prod_{i\in I}$ at $\big((A_i)_{i\in I},(B_i)_{i\in I}\big)$ is defined by sending a map

$$\{f_i \colon A_i \to B_i\}_{i\in I}$$

in $\mathrm{Nat}\big((A_i)_{i\in I},(B_i)_{i\in I}\big)$ to the map of sets

$$\prod_{i\in I} f_i \colon \prod_{i\in I} A_i \to \prod_{i\in I} B_i$$

defined by

$$\left[\prod_{i\in I} f_i\right]\big((a_i)_{i\in I}\big) \stackrel{\text{def}}{=} (f_i(a_i))_{i\in I}$$

for each $(a_i)_{i\in I} \in \prod_{i\in I} A_i$.

Proof. *Item 1*, *Functoriality*: This follows from **??** of **??**. □

000S 2.1.3 Binary Products of Sets

Let A and B be sets.

000T **Definition 2.1.3.1.1.** The **product**[3] **of** A **and** B is the pair $(A \times B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

- *The Limit.* The set $A \times B$ defined by[4]

$$\begin{aligned} A \times B &\stackrel{\text{def}}{=} \prod_{z\in\{A,B\}} z \\ &\stackrel{\text{def}}{=} \{f \in \mathsf{Sets}(\{0,1\}, A \cup B) \mid \text{we have } f(0) \in A \text{ and } f(1) \in B\} \\ &\cong \{\{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{we have } a \in A \text{ and } b \in B\}. \end{aligned}$$

- *The Cone.* The maps

$$\begin{aligned} \mathrm{pr}_1 &\colon A \times B \to A, \\ \mathrm{pr}_2 &\colon A \times B \to B \end{aligned}$$

[3] *Further Terminology:* Also called the **Cartesian product of** A **and** B or the **binary Cartesian product of** A **and** B, for emphasis.
This can also be thought of as the $(\mathbb{E}_{-1}, \mathbb{E}_{-1})$**-tensor product of** A **and** B.
[4] In other words, $A \times B$ is the set whose elements are ordered pairs (a, b) with $a \in A$

defined by

$$\begin{aligned}\mathrm{pr}_1(a,b) &\overset{\text{def}}{=} a,\\ \mathrm{pr}_2(a,b) &\overset{\text{def}}{=} b\end{aligned}$$

for each $(a,b) \in A \times B$.

Proof. We claim that $A \times B$ is the categorical product of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ A & A \times B \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon P \to A \times B$ making the diagram

$$\begin{tikzcd} & P \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists!"] & \\ A & A \times B \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & B \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\mathrm{pr}_1 \circ \phi &= p_1,\\ \mathrm{pr}_2 \circ \phi &= p_2\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. □

000U **Proposition 2.1.3.1.2.** Let A, B, C, and X be sets.

000V 1. *Functoriality.* The assignments $A, B, (A,B) \mapsto A \times B$ define functors

$$\begin{aligned} A \times -\colon \mathsf{Sets} &\to \mathsf{Sets},\\ - \times B\colon \mathsf{Sets} &\to \mathsf{Sets},\\ -_1 \times -_2\colon \mathsf{Sets} \times \mathsf{Sets} &\to \mathsf{Sets},\end{aligned}$$

where $-_1 \times -_2$ is the functor where

- *Action on Objects.* For each $(A,B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have

$$[-_1 \times -_2](A,B) \overset{\text{def}}{=} A \times B.$$

- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\times_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \times B, X \times Y)$$

  of $\times$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function

$$f \times g \colon A \times B \to X \times Y$$

  defined by

$$[f \times g](a, b) \stackrel{\text{def}}{=} (f(a), g(b))$$

  for each $(a, b) \in A \times B$.

and where $A \times -$ and $- \times B$ are the partial functors of $-_1 \times -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

000W 2. *Adjointness.* We have adjunctions

$$\begin{tikzcd}[ampersand replacement=\&] (A \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(A, -)) \colon \mathsf{Sets} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] \& \mathsf{Sets}, \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Sets}}(A,-)"] \end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&] (- \times B \dashv \mathrm{Hom}_{\mathsf{Sets}}(B, -)) \colon \mathsf{Sets} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] \& \mathsf{Sets}, \arrow[l, bend left, "\mathrm{Hom}_{\mathsf{Sets}}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) &\cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)), \\ \mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) &\cong \mathrm{Hom}_{\mathsf{Sets}}(B, \mathrm{Hom}_{\mathsf{Sets}}(A, C)), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

000X 3. *Associativity.* We have an isomorphism of sets

$$(A \times B) \times C \cong A \times (B \times C),$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

000Y 4. *Unitality.* We have isomorphisms of sets

$$\begin{aligned} \mathrm{pt} \times A &\cong A, \\ A \times \mathrm{pt} &\cong A, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

000Z 5. *Commutativity.* We have an isomorphism of sets

$$A \times B \cong B \times A,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

0010 6. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{aligned} A \times \emptyset &\cong \emptyset, \\ \emptyset \times A &\cong \emptyset, \end{aligned}$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

0011 7. *Distributivity Over Unions.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \cup C) &= (A \times B) \cup (A \times C), \\ (A \cup B) \times C &= (A \times C) \cup (B \times C). \end{aligned}$$

0012 8. *Distributivity Over Intersections.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \cap C) &= (A \times B) \cap (A \times C), \\ (A \cap B) \times C &= (A \times C) \cap (B \times C). \end{aligned}$$

0013 9. *Middle-Four Exchange with Respect to Intersections.* We have an isomorphism of sets

$$(A \times B) \cap (C \times D) \cong (A \cap B) \times (C \cap D).$$

0014 10. *Distributivity Over Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \setminus C) &= (A \times B) \setminus (A \times C), \\ (A \setminus B) \times C &= (A \times C) \setminus (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

0015 11. *Distributivity Over Symmetric Differences.* We have isomorphisms of sets

$$\begin{aligned} A \times (B \mathbin{\triangle} C) &= (A \times B) \mathbin{\triangle} (A \times C), \\ (A \mathbin{\triangle} B) \times C &= (A \times C) \mathbin{\triangle} (B \times C), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

0016 12. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times, \mathrm{pt})$ is a symmetric monoidal category.

0017 13. *Symmetric Bimonoidality.* The quintuple $(\mathsf{Sets}, \coprod, \emptyset, \times, \mathrm{pt})$ is a symmetric bimonoidal category.

Proof. *Item 1, Functoriality*: This follows from **??** of **??**.
Item 2, Adjointness: We prove only that there's an adjunction $- \times B \dashv \mathrm{Hom}_{\mathsf{Sets}}(B, -)$, witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \cong \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)),$$

natural in $B, C \in \mathrm{Obj}(\mathsf{Sets})$, as the proof of the existence of the adjunction $A \times - \dashv \mathrm{Hom}_{\mathsf{Sets}}(A, -)$ follows almost exactly in the same way.

- *Map I.* We define a map

  $$\Phi_{B,C} \colon \mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \to \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)),$$

  by sending a function

  $$\xi \colon A \times B \to C$$

  to the function

  $$\begin{aligned} \xi^{\dagger} \colon A &\to \mathrm{Hom}_{\mathsf{Sets}}(B, C), \\ a &\longmapsto \left(\xi^{\dagger}_{a} \colon B \to C\right), \end{aligned}$$

  where we define

  $$\xi^{\dagger}_{a}(b) \stackrel{\text{def}}{=} \xi(a, b)$$

  for each $b \in B$. In terms of the $[\![a \mapsto f(a)]\!]$ notation of Notation 1.1.1.1.2, we have

  $$\xi^{\dagger} \stackrel{\text{def}}{=} [\![a \mapsto [\![b \mapsto \xi(a, b)]\!]]\!].$$

- *Map II.* We define a map

  $$\Psi_{B,C} \colon \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C)), \to \mathrm{Hom}_{\mathsf{Sets}}(A \times B, C)$$

  given by sending a function

  $$\begin{aligned} \xi \colon A &\to \mathrm{Hom}_{\mathsf{Sets}}(B, C), \\ a &\longmapsto (\xi_{a} \colon B \to C), \end{aligned}$$

  to the function

  $$\xi^{\dagger} \colon A \times B \to C$$

defined by

$$\begin{aligned}\xi^{\dagger}(a,b) &\overset{\text{def}}{=} \mathrm{ev}_b(\mathrm{ev}_a(\xi))\\ &\overset{\text{def}}{=} \mathrm{ev}_b(\xi_a)\\ &\overset{\text{def}}{=} \xi_a(b)\end{aligned}$$

for each $(a,b) \in A \times B$.

- *Invertibility I.* We claim that

$$\Psi_{A,B} \circ \Phi_{A,B} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(A\times B,C)}.$$

Indeed, given a function $\xi \colon A \times B \to C$, we have

$$\begin{aligned}[\Psi_{A,B} \circ \Phi_{A,B}](\xi) &= \Psi_{A,B}(\Phi_{A,B}(\xi))\\ &= \Psi_{A,B}(\Phi_{A,B}([\![(a,b) \mapsto \xi(a,b)]\!]))\\ &= \Psi_{A,B}([\![a \mapsto [\![b \mapsto \xi(a,b)]\!]]\!])\\ &= \Psi_{A,B}([\![a' \mapsto [\![b' \mapsto \xi(a',b')]\!]]\!])\\ &= [\![(a,b) \mapsto \mathrm{ev}_b(\mathrm{ev}_a([\![a' \mapsto [\![b' \mapsto \xi(a',b')]\!]]\!]))]\!]\\ &= [\![(a,b) \mapsto \mathrm{ev}_b([\![b' \mapsto \xi(a,b')]\!])]\!]\\ &= [\![(a,b) \mapsto \xi(a,b)]\!]\\ &= \xi.\end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_{A,B} \circ \Psi_{A,B} = \mathrm{id}_{\mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,C))}.$$

Indeed, given a function

$$\begin{aligned}\xi \colon A &\longrightarrow \mathrm{Hom}_{\mathsf{Sets}}(B,C),\\ a &\longmapsto (\xi_a \colon B \to C),\end{aligned}$$

we have

$$\begin{aligned}[\Phi_{A,B} \circ \Psi_{A,B}](\xi) &\overset{\text{def}}{=} \Phi_{A,B}(\Psi_{A,B}(\xi))\\ &\overset{\text{def}}{=} \Phi_{A,B}([\![(a,b) \mapsto \xi_a(b)]\!])\\ &\overset{\text{def}}{=} \Phi_{A,B}([\![(a',b') \mapsto \xi_{a'}(b')]\!])\\ &\overset{\text{def}}{=} [\![a \mapsto [\![b \mapsto \mathrm{ev}_{(a,b)}([\![(a',b') \mapsto \xi_{a'}(b')]\!])]\!]]\!]\\ &\overset{\text{def}}{=} [\![a \mapsto [\![b \mapsto \xi_a(b)]\!]]\!]\\ &\overset{\text{def}}{=} [\![a \mapsto \xi_a]\!]\\ &\overset{\text{def}}{=} \xi.\end{aligned}$$

- *Naturality for Φ, Part I.* We need to show that, given a function $g\colon B \to B'$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sets}}(A \times B', C) \arrow[r, "\Phi_{B',C}"] \arrow[d, "\mathrm{id}_A \times g^*"'] & \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B', C)), \arrow[d, "(g^*)_*"] \\
\mathrm{Hom}_{\mathsf{Sets}}(A \times B, C) \arrow[r, "\Phi_{B,C}"'] & \mathrm{Hom}_{\mathsf{Sets}}(A, \mathrm{Hom}_{\mathsf{Sets}}(B, C))
\end{tikzcd}$$

commutes. Indeed, given a function

$$\xi\colon A \times B' \to C,$$

we have

$$\begin{aligned}
[\Phi_{B,C} \circ (\mathrm{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\mathrm{id}_A \times g^*](\xi)) \\
&= \Phi_{B,C}(\xi(-_1, g(-_2))) \\
&= [\xi(-_1, g(-_2))]^\dagger \\
&= \xi^\dagger_{-_1}(g(-_2)) \\
&= (g^*)_*\Big(\xi^\dagger\Big) \\
&= (g^*)_*(\Phi_{B',C}(\xi)) \\
&= \big[(g^*)_* \circ \Phi_{B',C}\big](\xi).
\end{aligned}$$

Alternatively, using the $[\![a \mapsto f(a)]\!]$ notation of Notation 1.1.1.1.2, we have

$$\begin{aligned}
[\Phi_{B,C} \circ (\mathrm{id}_A \times g^*)](\xi) &= \Phi_{B,C}([\mathrm{id}_A \times g^*](\xi)) \\
&= \Phi_{B,C}([\mathrm{id}_A \times g^*]([\![(a,b') \mapsto \xi(a,b')]\!])) \\
&= \Phi_{B,C}([\![(a,b) \mapsto \xi(a,g(b))]\!]) \\
&= [\![a \mapsto [\![b \mapsto \xi(a,g(b))]\!]]\!] \\
&= [\![a \mapsto g^*([\![b' \mapsto \xi(a,b')]\!])]\!] \\
&= (g^*)_*([\![a \mapsto [\![b' \mapsto \xi(a,b')]\!]]\!]) \\
&= (g^*)_*(\Phi_{B',C}([\![(a,b') \mapsto \xi(a,b')]\!])) \\
&= (g^*)_*(\Phi_{B',C}(\xi)) \\
&= \big[(g^*)_* \circ \Phi_{B',C}\big](\xi).
\end{aligned}$$

- *Naturality for Φ, Part II.* We need to show that, given a function

and $b \in B$ as in Definition 2.3.4.1.1

$h\colon C\to C'$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Sets}}(A\times B,C) \arrow[r, "\Phi_{B,C}"] \arrow[d, "h_*"'] & \mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,C)), \arrow[d, "(h_*)_*"] \\
\mathrm{Hom}_{\mathsf{Sets}}(A\times B,C') \arrow[r, "\Phi_{B,C'}"'] & \mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,C'))
\end{tikzcd}$$

commutes. Indeed, given a function

$$\xi\colon A\times B\to C,$$

we have

$$\begin{aligned}
[\Phi_{B,C}\circ h_*](\xi) &= \Phi_{B,C}(h_*(\xi))\\
&= \Phi_{B,C}(h_*(\llbracket (a,b)\mapsto \xi(a,b)\rrbracket))\\
&= \Phi_{B,C}(\llbracket (a,b)\mapsto h(\xi(a,b))\rrbracket)\\
&= \llbracket a\mapsto \llbracket b\mapsto h(\xi(a,b))\rrbracket\rrbracket\\
&= \llbracket a\mapsto h_*(\llbracket b\mapsto \xi(a,b)\rrbracket\rrbracket)\\
&= (h_*)_*(\llbracket a\mapsto \llbracket b\mapsto \xi(a,b)\rrbracket\rrbracket)\\
&= (h_*)_*(\Phi_{B,C}(\llbracket (a,b)\mapsto \xi(a,b)\rrbracket))\\
&= (h_*)_*(\Phi_{B,C}(\xi))\\
&= [(h_*)_*\circ\Phi_{B,C}](\xi).
\end{aligned}$$

- *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Item 2 of Proposition 8.8.6.1.2 that Ψ is also natural in each argument.

Item 3, Associativity: See [Pro24a].
Item 4, Unitality: Clear.
Item 5, Commutativity: See [Pro24b].
Item 6, Annihilation With the Empty Set: See [Pro24f].
Item 7, Distributivity Over Unions: See [Pro24e].
Item 8, Distributivity Over Intersections: See [Pro24g, Corollary 1].
Item 9, Middle-Four Exchange With Respect to Intersections: See [Pro24g, Corollary 1].
Item 10, Distributivity Over Differences: See [Pro24c].
Item 11, Distributivity Over Symmetric Differences: See [Pro24d].
Item 12, Symmetric Monoidality: See [`MO 382264`].
Item 13, Symmetric Bimonoidality: Omitted. □

0018 ### 2.1.4 Pullbacks

Let A, B, and C be sets and let $f\colon A \to C$ and $g\colon B \to C$ be functions.

0019 **Definition 2.1.4.1.1.** The **pullback of** A **and** B **over** C **along** f **and** g[5] is the pair[6] $(A \times_C B, \{\mathrm{pr}_1, \mathrm{pr}_2\})$ consisting of:

- *The Limit.* The set $A \times_C B$ defined by
$$A \times_C B \overset{\text{def}}{=} \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$
- *The Cone.* The maps
$$\begin{aligned}\mathrm{pr}_1\colon A \times_C B &\to A,\\ \mathrm{pr}_2\colon A \times_C B &\to B\end{aligned}$$
defined by
$$\begin{aligned}\mathrm{pr}_1(a, b) &\overset{\text{def}}{=} a,\\ \mathrm{pr}_2(a, b) &\overset{\text{def}}{=} b\end{aligned}$$
for each $(a, b) \in A \times_C B$.

Proof. We claim that $A \times_C B$ is the categorical pullback of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} A \times_C B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C. \end{tikzcd}$$

Indeed, given $(a, b) \in A \times_C B$, we have

$$\begin{aligned}[f \circ \mathrm{pr}_1](a, b) &= f(\mathrm{pr}_1(a, b))\\ &= f(a)\\ &= g(b)\\ &= g(\mathrm{pr}_2(a, b))\\ &= [g \circ \mathrm{pr}_2](a, b),\end{aligned}$$

where $f(a) = g(b)$ since $(a, b) \in A \times_C B$. Next, we prove that $A \times_C B$

[5] *Further Terminology:* Also called the **fibre product of** A **and** B **over** C **along** f **and** g.

[6] *Further Notation:* Also written $A \times_{f,C,g} B$.

satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd}
P \arrow[rrd, bend left, "p_2"] \arrow[rdd, bend right, "p_1"'] & & \\
& A \times_C B \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon P \to A \times_C B$ making the diagram

$$\begin{tikzcd}
P \arrow[rrd, bend left, "p_2"] \arrow[rdd, bend right, "p_1"'] \arrow[rd, dashed, "\phi", "\exists!"'] & & \\
& A \times_C B \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\mathrm{pr}_1 \circ \phi &= p_1, \\
\mathrm{pr}_2 \circ \phi &= p_2
\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in A \times B$ indeed lies in $A \times_C B$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in A \times_C B$. □

001A **Example 2.1.4.1.2.** Here are some examples of pullbacks of sets.

001B 1. *Unions via Intersections.* Let $A, B \subset X$. We have a bijection of sets

$$A \cap B \cong A \times_{A \cup B} B, \qquad \begin{tikzcd} A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\iota_B"] \\ A \arrow[r, "\iota_A"'] & A \cup B. \end{tikzcd}$$

Proof. *Item 1, Unions via Intersections*: Indeed, we have

$$\begin{aligned} A \times_{A \cup B} B &\cong \{(x, y) \in A \times B \mid x = y\} \\ &\cong A \cap B. \end{aligned}$$

This finishes the proof. □

001C **Proposition 2.1.4.1.3.** Let A, B, C, and X be sets.

001D 1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \times_{f,C,g} B$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd} A \times_C B \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[dr, "\psi"] \arrow[dd, "g" near start] & \\ & A' \times_{C'} B' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & B' \arrow[dd, "g'"] \\ A \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\ & A' \arrow[rr, "f'"'] & & C' \end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \times_C B \xrightarrow{\exists!} A' \times_{C'} B'$ given by

$$\xi(a, b) \stackrel{\text{def}}{=} (\phi(a), \psi(b))$$

for each $(a,b) \in A \times_C B$, which is the unique map making the diagram

$$\begin{tikzcd}
A \times_C B \arrow[rr] \arrow[dr, dashed] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & B \arrow[dr, "\psi"] \arrow[dd, "g"] & \\
& A' \times_{C'} B' \arrow[rr] \arrow[dd] \arrow[dr, phantom, very near start, "\lrcorner"] & & B' \arrow[dd, "g'"] \\
A \arrow[rr, "f"] \arrow[dr, "\phi"'] & & C \arrow[dr, "\chi"] & \\
& A' \arrow[rr, "f'"'] & & C'
\end{tikzcd}$$

commute.

001E 2. *Associativity.* Given a diagram

$$\begin{tikzcd}
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}$$

in Sets, we have isomorphisms of sets

$$(A \times_X B) \times_Y C \cong (A \times_X B) \times_B (B \times_Y C) \cong A \times_X (B \times_Y C),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}
& & (A \times_X B) \times_Y C \arrow[dl] \arrow[ddrr] \arrow[d, phantom, very near start, "\lrcorner"] & & \\
& A \times_X B \arrow[dl] \arrow[dr] \arrow[d, phantom, very near start, "\lrcorner"] & {} & & \\
A \arrow[dr, "f"'] & {} & B \arrow[dl, "g"] \arrow[dr, "h"'] & & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & (A \times_X B) \times_B (B \times_Y C) \arrow[dl] \arrow[dr] \arrow[d, phantom, very near start, "\lrcorner"] & & \\
& A \times_X B \arrow[dl] \arrow[dr] \arrow[d, phantom, very near start, "\lrcorner"] & {} & B \times_Y C \arrow[dl] \arrow[dr] \arrow[d, phantom, very near start, "\lrcorner"] & \\
A \arrow[dr, "f"'] & {} & B \arrow[dl, "g"] \arrow[dr, "h"'] & {} & C, \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & A \times_X (B \times_Y C) \arrow[ddll] \arrow[dr] \arrow[d, phantom, very near start, "\lrcorner"] & & \\
& & {} & B \times_Y C \arrow[dl] \arrow[dr] \arrow[d, phantom, very near start, "\lrcorner"] & \\
A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \arrow[dr, "h"'] & {} & C. \arrow[dl, "k"] \\
& X & & Y &
\end{tikzcd}$$

001F 3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
X \times_X A &\cong A, \\
A \times_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}$$

001G 4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd} A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & X, \end{tikzcd} \qquad A \times_X B \cong B \times_X A \qquad \begin{tikzcd} B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & X. \end{tikzcd}$$

001H 5. *Annihilation With the Empty Set.* We have isomorphisms of sets

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \emptyset \arrow[d] \\ A \arrow[r, "f"'] & X, \end{tikzcd} \qquad \begin{gathered} A \times_X \emptyset \cong \emptyset, \\ \emptyset \times_X A \cong \emptyset, \end{gathered} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ \emptyset \arrow[r] & X. \end{tikzcd}$$

001J 6. *Interaction With Products.* We have an isomorphism of sets

$$A \times_{\mathrm{pt}} B \cong A \times B, \qquad \begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "!_B"] \\ A \arrow[r, "!_A"'] & \mathrm{pt}. \end{tikzcd}$$

001K 7. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \times_X, X)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.
Item 2, Associativity: Indeed, we have

$$\begin{aligned} (A \times_X B) \times_Y C &\cong \{((a,b),c) \in (A \times_X B) \times C \mid h(b) = k(c)\} \\ &\cong \{((a,b),c) \in (A \times B) \times C \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong \{(a,(b,c)) \in A \times (B \times_Y C) \mid f(a) = g(b)\} \\ &\cong A \times_X (B \times_Y C) \end{aligned}$$

and

$$\begin{aligned}(A \times_X B) \times_B (B \times_Y C) &\cong \left\{ \left((a,b),\left(b',c\right)\right) \in (A \times_X B) \times (B \times_Y C) \;\middle|\; b = b' \right\} \\ &\cong \left\{ \left((a,b),\left(b',c\right)\right) \in (A \times B) \times (B \times C) \;\middle|\; \begin{array}{l} f(a) = g(b),\ b = b', \\ \text{and } h\left(b'\right) = k(c) \end{array} \right\} \\ &\cong \left\{ \left(a,\left(b,\left(b',c\right)\right)\right) \in A \times (B \times (B \times C)) \;\middle|\; \begin{array}{l} f(a) = g(b),\ b = b', \\ \text{and } h\left(b'\right) = k(c) \end{array} \right\} \\ &\cong \left\{ \left(a,\left(\left(b,b'\right),c\right)\right) \in A \times ((B \times B) \times C) \;\middle|\; \begin{array}{l} f(a) = g(b),\ b = b', \\ \text{and } h\left(b'\right) = k(c) \end{array} \right\} \\ &\cong \left\{ \left(a,\left(\left(b,b'\right),c\right)\right) \in A \times ((B \times_B B) \times C) \;\middle|\; \begin{array}{l} f(a) = g(b) \text{ and} \\ h\left(b'\right) = k(c) \end{array} \right\} \\ &\cong \{(a,(b,c)) \in A \times (B \times C) \mid f(a) = g(b) \text{ and } h(b) = k(c)\} \\ &\cong A \times_X (B \times_Y C),\end{aligned}$$

where we have used Item 3 for the isomorphism $B \times_B B \cong B$.
Item 3, Unitality: Indeed, we have

$$\begin{aligned} X \times_X A &\cong \{(x,a) \in X \times A \mid f(a) = x\}, \\ A \times_X X &\cong \{(a,x) \in X \times A \mid f(a) = x\}, \end{aligned}$$

which are isomorphic to A via the maps $(x,a) \mapsto a$ and $(a,x) \mapsto a$.
Item 4, Commutativity: Clear.
Item 5, Annihilation With the Empty Set: Clear.
Item 6, Interaction With Products: Clear.
Item 7, Symmetric Monoidality: Omitted. □

001L 2.1.5 Equalisers

Let A and B be sets and let $f, g \colon A \rightrightarrows B$ be functions.

001M **Definition 2.1.5.1.1.** The **equaliser of f and g** is the pair $(\mathrm{Eq}(f,g), \mathrm{eq}(f,g))$ consisting of:

- *The Limit.* The set $\mathrm{Eq}(f,g)$ defined by
$$\mathrm{Eq}(f,g) \overset{\text{def}}{=} \{a \in A \mid f(a) = g(a)\}.$$
- *The Cone.* The inclusion map
$$\mathrm{eq}(f,g) \colon \mathrm{Eq}(f,g) \hookrightarrow A.$$

Proof. We claim that $\mathrm{Eq}(f,g)$ is the categorical equaliser of f and g in Sets. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f,g) = g \circ \mathrm{eq}(f,g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f,g)$. Next, we prove that $\mathrm{Eq}(f,g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd} \mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ E \arrow[ur, "e"'] & & \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon E \to \mathrm{Eq}(f,g)$ making the diagram

$$\begin{tikzcd} \mathrm{Eq}(f,g) \arrow[r, hook, "\mathrm{eq}(f,g)"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ E \arrow[u, dashed, "\phi", "\exists!"'] \arrow[ur, "e"'] & & \end{tikzcd}$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g)$. □

001N **Proposition 2.1.5.1.2.** Let A, B, and C be sets.

001P 1. *Associativity.* We have isomorphisms of sets[7]

$$\underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f\circ\mathrm{eq}(g,h),h\circ\mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g\circ\mathrm{eq}(f,g),h\circ\mathrm{eq}(f,g))},$$

[7]That is, the following three ways of forming "the" equaliser of (f,g,h) agree:

1. Take the equaliser of (f,g,h), i.e. the limit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

where $\mathrm{Eq}(f, g, h)$ is the limit of the diagram

$$A \overset{f}{\underset{h}{\substack{\longrightarrow \\ -g\rightarrow \\ \longrightarrow}}} B$$

in Sets, being explicitly given by

$$\mathrm{Eq}(f, g, h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

001Q 4. *Unitality.* We have an isomorphism of sets

$$\mathrm{Eq}(f, f) \cong A.$$

001R 5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

001S 6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

2. First take the equaliser of f and g, forming a diagram

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(f, g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{h}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g)) = \mathrm{Eq}(g \circ \mathrm{eq}(f, g), h \circ \mathrm{eq}(f, g))$$

of $\mathrm{Eq}(f, g)$.

3. First take the equaliser of g and h, forming a diagram

$$\mathrm{Eq}(g, h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{h}{\overset{g}{\rightrightarrows}} B$$

and then take the equaliser of the composition

$$\mathrm{Eq}(g, h) \overset{\mathrm{eq}(g,h)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

obtaining a subset

$$\mathrm{Eq}(f \circ \mathrm{eq}(g, h), g \circ \mathrm{eq}(g, h)) = \mathrm{Eq}(f \circ \mathrm{eq}(g, h), h \circ \mathrm{eq}(g, h))$$

of $\mathrm{Eq}(g, h)$.

be functions. We have an inclusion of sets

$$\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g)) \subset \mathrm{Eq}(h \circ f, k \circ g),$$

where $\mathrm{Eq}(h \circ f \circ \mathrm{eq}(f,g), k \circ g \circ \mathrm{eq}(f,g))$ is the equaliser of the composition

$$\mathrm{Eq}(f,g) \overset{\mathrm{eq}(f,g)}{\hookrightarrow} A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C.$$

Proof. *Item 1, Associativity*: We first prove that $\mathrm{Eq}(f,g,h)$ is indeed given by

$$\mathrm{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

Indeed, suppose we have a diagram of the form

$$\begin{tikzcd}
\mathrm{Eq}(f,g,h) \arrow[r, hook, "\mathrm{eq}(f,g,h)"] & A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \\
E \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon E \to \mathrm{Eq}(f,g,h)$, uniquely determined by the condition

$$\mathrm{eq}(f,g) \circ \phi = e$$

being necessarily given by

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g,h)$ by the condition

$$f \circ e = g \circ e = h \circ e,$$

which gives

$$f(e(x)) = g(e(x)) = h(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g,h)$.

We now check the equalities

$$\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) \cong \mathrm{Eq}(f,g,h) \cong \mathrm{Eq}(f \circ \mathrm{eq}(f,g), h \circ \mathrm{eq}(f,g)).$$

Indeed, we have

$$\begin{aligned}
\mathrm{Eq}(f \circ \mathrm{eq}(g,h), g \circ \mathrm{eq}(g,h)) &\cong \{x \in \mathrm{Eq}(g,h) \mid [f \circ \mathrm{eq}(g,h)](a) = [g \circ \mathrm{eq}(g,h)](a)\} \\
&\cong \{x \in \mathrm{Eq}(g,h) \mid f(a) = g(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) \text{ and } g(a) = h(a)\} \\
&\cong \{x \in A \mid f(a) = g(a) = h(a)\} \\
&\cong \mathrm{Eq}(f,g,h).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\mathrm{Eq}(f\circ \mathrm{eq}(f,g), h\circ \mathrm{eq}(f,g)) &\cong \{x\in \mathrm{Eq}(f,g) \mid [f\circ \mathrm{eq}(f,g)](a) = [h\circ \mathrm{eq}(f,g)](a)\}\\
&\cong \{x\in \mathrm{Eq}(f,g) \mid f(a) = h(a)\}\\
&\cong \{x\in A \mid f(a) = h(a) \text{ and } f(a) = g(a)\}\\
&\cong \{x\in A \mid f(a) = g(a) = h(a)\}\\
&\cong \mathrm{Eq}(f,g,h).
\end{aligned}$$

Item 4, Unitality: Clear.
Item 5, Commutativity: Clear.
Item 6, Interaction With Composition: Indeed, we have

$$\begin{aligned}
\mathrm{Eq}(h\circ f\circ \mathrm{eq}(f,g), k\circ g\circ \mathrm{eq}(f,g)) &\cong \{a\in \mathrm{Eq}(f,g) \mid h(f(a)) = k(g(a))\}\\
&\cong \{a\in A \mid f(a) = g(a) \text{ and } h(f(a)) = k(g(a))\}.
\end{aligned}$$

and

$$\mathrm{Eq}(h\circ f, k\circ g) \cong \{a\in A \mid h(f(a)) = k(g(a))\},$$

and thus there's an inclusion from $\mathrm{Eq}(h\circ f\circ \mathrm{eq}(f,g), k\circ g\circ \mathrm{eq}(f,g))$ to $\mathrm{Eq}(h\circ f, k\circ g)$. □

001T 2.2 Colimits of Sets

001U 2.2.1 The Initial Set

001V **Definition 2.2.1.1.1.** The **initial set** is the pair $\left(\emptyset, \{\iota_A\}_{A\in \mathrm{Obj}(\mathsf{Sets})}\right)$ consisting of:

- *The Limit.* The empty set $\emptyset$ of Definition 2.3.1.1.1.
- *The Cone.* The collection of maps

  $$\{\iota_A \colon \emptyset \to A\}_{A\in \mathrm{Obj}(\mathsf{Sets})}$$

  given by the inclusion maps from $\emptyset$ to A.

Proof. We claim that $\emptyset$ is the initial object of Sets. Indeed, suppose we have a diagram of the form

$$\emptyset \qquad A$$

in Sets. Then there exists a unique map $\phi\colon \emptyset \to A$ making the diagram

$$\begin{tikzcd} \emptyset \arrow[r, dashed, "\phi", "\exists!"'] & A \end{tikzcd}$$

commute, namely the inclusion map ι_A. □

001W 2.2.2 Coproducts of Families of Sets

Let $\{A_i\}_{i\in I}$ be a family of sets.

001X **Definition 2.2.2.1.1.** The **disjoint union of the family** $\{A_i\}_{i\in I}$ is the pair $(\coprod_{i\in I} A_i, \{\mathrm{inj}_i\}_{i\in I})$ consisting of:

- *The Colimit.* The set $\coprod_{i\in I} A_i$ defined by
$$\coprod_{i\in I} A_i \overset{\mathrm{def}}{=} \left\{(i,x) \in I \times \left(\bigcup_{i\in I} A_i\right) \;\middle|\; x \in A_i\right\}.$$
- *The Cocone.* The collection
$$\left\{\mathrm{inj}_i \colon A_i \to \coprod_{i\in I} A_i\right\}_{i\in I}$$
of maps given by
$$\mathrm{inj}_i(x) \overset{\mathrm{def}}{=} (i,x)$$
for each $x \in A_i$ and each $i \in I$.

Proof. We claim that $\coprod_{i\in I} A_i$ is the categorical coproduct of $\{A_i\}_{i\in I}$ in Sets. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod_{i\in I} A_i \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi\colon \coprod_{i\in I} A_i \to C$ making the diagram

$$\begin{tikzcd} & C \\ A_i \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \coprod_{i\in I} A_i \arrow[u, dashed, "\phi", "\exists!"'] \end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via
$$\phi((i,x)) = \iota_i(x)$$
for each $(i,x) \in \coprod_{i\in I} A_i$. $\square$

001Y **Proposition 2.2.2.1.2.** Let $\{A_i\}_{i\in I}$ be a family of sets.

001Z 1. *Functoriality.* The assignment $\{A_i\}_{i\in I} \mapsto \coprod_{i\in I} A_i$ defines a functor

$$\coprod_{i\in I} \colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}) \to \mathsf{Sets}$$

where

- *Action on Objects.* For each $(A_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, we have

$$\left[\coprod_{i\in I}\right]((A_i)_{i\in I}) \stackrel{\text{def}}{=} \coprod_{i\in I} A_i$$

- *Action on Morphisms.* For each $(A_i)_{i\in I}, (B_i)_{i\in I} \in \mathrm{Obj}(\mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}))$, the action on Hom-sets

$$\left(\coprod_{i\in I}\right)_{(A_i)_{i\in I},(B_i)_{i\in I}} \colon \mathrm{Nat}\big((A_i)_{i\in I}, (B_i)_{i\in I}\big) \to \mathsf{Sets}\left(\coprod_{i\in I} A_i, \coprod_{i\in I} B_i\right)$$

  of $\coprod_{i\in I}$ at $\big((A_i)_{i\in I}, (B_i)_{i\in I}\big)$ is defined by sending a map

$$\{f_i \colon A_i \to B_i\}_{i\in I}$$

  in $\mathrm{Nat}\big((A_i)_{i\in I}, (B_i)_{i\in I}\big)$ to the map of sets

$$\coprod_{i\in I} f_i \colon \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$$

  defined by

$$\left[\coprod_{i\in I} f_i\right](i,a) \stackrel{\text{def}}{=} f_i(a)$$

  for each $(i,a) \in \coprod_{i\in I} A_i$.

Proof. *Item 1*, *Functoriality*: This follows from **??** of **??**. □

0020 2.2.3 Binary Coproducts

Let A and B be sets.

0021 **Definition 2.2.3.1.1.** The **coproduct**[8] **of** A **and** B is the pair $(A \coprod B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

- *The Colimit.* The set $A \coprod B$ defined by

$$\begin{aligned} A \coprod B &\stackrel{\text{def}}{=} \coprod_{z\in\{A,B\}} z \\ &\cong \{(0,a) \mid a \in A\} \cup \{(1,b) \mid b \in B\}. \end{aligned}$$

[8] *Further Terminology:* Also called the **disjoint union of** A **and** B, or the **binary**

- *The Cocone.* The maps

$$\begin{aligned}\mathrm{inj}_1 \colon A &\to A \coprod B,\\ \mathrm{inj}_2 \colon B &\to A \coprod B,\end{aligned}$$

  given by

$$\begin{aligned}\mathrm{inj}_1(a) &\overset{\mathrm{def}}{=} (0, a),\\ \mathrm{inj}_2(b) &\overset{\mathrm{def}}{=} (1, b),\end{aligned}$$

  for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \coprod B$ is the categorical coproduct of A and B in Sets. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & C & \\ A \arrow[ur, bend left, "\iota_A"] \arrow[r, "\mathrm{inj}_A"'] & A \coprod B & B \arrow[ul, bend right, "\iota_B"'] \arrow[l, "\mathrm{inj}_B"] \end{tikzcd}$$

in Sets. Then there exists a unique map $\phi \colon A \coprod B \to C$ making the diagram

$$\begin{tikzcd} & C & \\ A \arrow[ur, bend left, "\iota_A"] \arrow[r, "\mathrm{inj}_A"'] & A \coprod B \arrow[u, dashed, "\phi", "\exists!"'] & B \arrow[ul, bend right, "\iota_B"'] \arrow[l, "\mathrm{inj}_B"] \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}\phi \circ \mathrm{inj}_A &= \iota_A,\\ \phi \circ \mathrm{inj}_B &= \iota_B\end{aligned}$$

via

$$\phi(x) = \begin{cases} \iota_A(a) & \text{if } x = (0, a),\\ \iota_B(b) & \text{if } x = (1, b)\end{cases}$$

for each $x \in A \coprod B$. □

0022 **Proposition 2.2.3.1.2.** Let A, B, C, and X be sets.

0023 1. *Functoriality.* The assignment $A, B, (A, B) \mapsto A \coprod B$ defines functors

$$\begin{aligned}A \coprod - \colon \mathsf{Sets} &\to \mathsf{Sets},\\ - \coprod B \colon \mathsf{Sets} &\to \mathsf{Sets},\\ -_1 \coprod -_2 \colon \mathsf{Sets} \times \mathsf{Sets} &\to \mathsf{Sets},\end{aligned}$$

   where $-_1 \coprod -_2$ is the functor where

- *Action on Objects.* For each $(A, B) \in \mathrm{Obj}(\mathsf{Sets} \times \mathsf{Sets})$, we have
$$[-_1 \coprod -_2](A, B) \stackrel{\text{def}}{=} A \coprod B.$$
- *Action on Morphisms.* For each $(A, B), (X, Y) \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\coprod_{(A,B),(X,Y)} \colon \mathsf{Sets}(A, X) \times \mathsf{Sets}(B, Y) \to \mathsf{Sets}(A \coprod B, X \coprod Y)$$

of $\coprod$ at $((A, B), (X, Y))$ is defined by sending (f, g) to the function
$$f \coprod g \colon A \coprod B \to X \coprod Y$$
defined by
$$[f \coprod g](x) \stackrel{\text{def}}{=} \begin{cases} (0, f(a)) & \text{if } x = (0, a), \\ (1, g(b)) & \text{if } x = (1, b), \end{cases}$$
for each $x \in A \coprod B$.

and where $A \coprod -$ and $- \coprod B$ are the partial functors of $-_1 \coprod -_2$ at $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

0024 2. *Associativity.* We have an isomorphism of sets
$$(A \coprod B) \coprod C \cong A \coprod (B \coprod C),$$
natural in $A, B, C \in \mathrm{Obj}(\mathsf{Sets})$.

0025 3. *Unitality.* We have isomorphisms of sets
$$A \coprod \emptyset \cong A,$$
$$\emptyset \coprod A \cong A,$$
natural in $A \in \mathrm{Obj}(\mathsf{Sets})$.

0026 4. *Commutativity.* We have an isomorphism of sets
$$A \coprod B \cong B \coprod A,$$
natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

0027 5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod, \emptyset)$ is a symmetric monoidal category.

Proof. *Item 1*, *Functoriality*: This follows from **??** of **??**.
Item 2, *Associativity*: Clear.
Item 3, *Unitality*: Clear.
Item 4, *Commutativity*: Clear.
Item 5, *Symmetric Monoidality*: Omitted. □

0028 2.2.4 Pushouts

Let A, B, and C be sets and let $f\colon C \to A$ and $g\colon C \to B$ be functions.

0029 **Definition 2.2.4.1.1.** The **pushout of A and B over C along f and g**[9] is the pair[10] $(A \coprod_C B, \{\mathrm{inj}_1, \mathrm{inj}_2\})$ consisting of:

- *The Colimit.* The set $A \coprod_C B$ defined by

$$A \coprod_C B \overset{\text{def}}{=} A \coprod B/{\sim_C},$$

where $\sim_C$ is the equivalence relation on $A \coprod B$ generated by $(0, f(c)) \sim_C (1, g(c))$.

- *The Cocone.* The maps

$$\begin{aligned}\mathrm{inj}_1 &\colon A \to A \textstyle\coprod_C B,\\ \mathrm{inj}_2 &\colon B \to A \textstyle\coprod_C B\end{aligned}$$

given by

$$\begin{aligned}\mathrm{inj}_1(a) &\overset{\text{def}}{=} [(0, a)]\\ \mathrm{inj}_2(b) &\overset{\text{def}}{=} [(1, b)]\end{aligned}$$

for each $a \in A$ and each $b \in B$.

Proof. We claim that $A \coprod_C B$ is the categorical pushout of A and B over C with respect to (f, g) in Sets. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} A \coprod_C B & B \arrow[l, "\mathrm{inj}_2"'] \\ A \arrow[u, "\mathrm{inj}_1"] & C. \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $c \in C$, we have

$$\begin{aligned}[\mathrm{inj}_1 \circ f](c) &= \mathrm{inj}_1(f(c))\\ &= [(0, f(c))]\\ &= [(1, g(c))]\\ &= \mathrm{inj}_2(g(c))\\ &= [\mathrm{inj}_2 \circ g](c),\end{aligned}$$

disjoint union of A and B, for emphasis.

[9] *Further Terminology:* Also called the **fibre coproduct of A and B over C along f and g**.

[10] *Further Notation:* Also written $A \coprod_{f,C,g} B$.

where $[(0, f(c))] = [(1, g(c))]$ by the definition of the relation $\sim$ on $A \coprod B$. Next, we prove that $A \coprod_C B$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$
\begin{tikzcd}
P & & \\
& A \coprod_C B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& A \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
$$

in Sets. Then there exists a unique map $\phi \colon A \coprod_C B \to P$ making the diagram

$$
\begin{tikzcd}
P & & \\
& A \coprod_C B \arrow[ul, dashed, "\exists!", "\phi"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& A \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & C \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
$$

commute, being uniquely determined by the conditions

$$
\begin{aligned}
\phi \circ \mathrm{inj}_1 &= \iota_1, \\
\phi \circ \mathrm{inj}_2 &= \iota_2
\end{aligned}
$$

via

$$
\phi(x) = \begin{cases} \iota_1(a) & \text{if } x = [(0, a)], \\ \iota_2(b) & \text{if } x = [(1, b)] \end{cases}
$$

for each $x \in A \coprod_C B$, where the well-definedness of ϕ is guaranteed by the equality $\iota_1 \circ f = \iota_2 \circ g$ and the definition of the relation $\sim$ on $A \coprod B$ as follows:

1. *Case 1:* Suppose we have $x = [(0, a)] = [(0, a')]$ for some $a, a' \in A$. Then, by Remark 2.2.4.1.2, we have a sequence

$$
(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a').
$$

2. *Case 2:* Suppose we have $x = [(1, b)] = [(1, b')]$ for some $b, b' \in B$. Then, by Remark 2.2.4.1.2, we have a sequence

$$
(1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b').
$$

3. *Case 3:* Suppose we have $x = [(0, a)] = [(1, b)]$ for some $a \in A$ and $b \in B$. Then, by Remark 2.2.4.1.2, we have a sequence

$$(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b).$$

In all these cases, we declare $x \sim' y$ iff there exists some $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$ or $x = (1, g(c))$ and $y = (0, f(c))$. Then, the equality $\iota_1 \circ f = \iota_2 \circ g$ gives

$$\begin{aligned}
\phi([x]) &= \phi([(0, f(c))]) \\
&\overset{\text{def}}{=} \iota_1(f(c)) \\
&= \iota_2(g(c)) \\
&\overset{\text{def}}{=} \phi([(1, g(c))]) \\
&= \phi([y]),
\end{aligned}$$

with the case where $x = (1, g(c))$ and $y = (0, f(c))$ similarly giving $\phi([x]) = \phi([y])$. Thus, if $x \sim' y$, then $\phi([x]) = \phi([y])$. Applying this equality pairwise to the sequences

$$\begin{aligned}
(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (0, a'), \\
(1, b) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b'), \\
(0, a) \sim' x_1 \sim' \cdots \sim' x_n \sim' (1, b)
\end{aligned}$$

gives

$$\begin{aligned}
\phi([(0, a)]) = \phi([(0, a')]), \\
\phi([(1, b)]) = \phi([(1, b')]), \\
\phi([(0, a)]) = \phi([(1, b)]),
\end{aligned}$$

showing ϕ to be well-defined. □

002A **Remark 2.2.4.1.2.** In detail, by Construction 7.4.2.1.2, the relation $\sim$ of Definition 2.2.4.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a, b \in A$ and $a = b$;
- We have $a, b \in B$ and $a = b$;
- There exist $x_1, \ldots, x_n \in A \coprod B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  1. There exists $c \in C$ such that $x = (0, f(c))$ and $y = (1, g(c))$.
  2. There exists $c \in C$ such that $x = (1, g(c))$ and $y = (0, f(c))$.

That is: we require the following condition to be satisfied:

($\star$) There exist $x_1, \ldots, x_n \in A \coprod B$ satisfying the following conditions:

1. There exists $c_0 \in C$ satisfying one of the following conditions:
   (a) We have $a = f(c_0)$ and $x_1 = g(c_0)$.
   (b) We have $a = g(c_0)$ and $x_1 = f(c_0)$.
2. For each $1 \leq i \leq n-1$, there exists $c_i \in C$ satisfying one of the following conditions:
   (a) We have $x_i = f(c_i)$ and $x_{i+1} = g(c_i)$.
   (b) We have $x_i = g(c_i)$ and $x_{i+1} = f(c_i)$.
3. There exists $c_n \in C$ satisfying one of the following conditions:
   (a) We have $x_n = f(c_n)$ and $b = g(c_n)$.
   (b) We have $x_n = g(c_n)$ and $b = f(c_n)$.

002B **Example 2.2.4.1.3.** Here are some examples of pushouts of sets.

002C 1. *Wedge Sums of Pointed Sets.* The wedge sum of two pointed sets of Definition 3.3.3.1.1 is an example of a pushout of sets.

002D 2. *Intersections via Unions.* Let $A, B \subset X$. We have a bijection of sets

$$A \cup B \cong A \coprod_{A \cap B} B, \qquad \begin{tikzcd} A \cup B \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[l] \\ A \arrow[u] & A \cap B. \arrow[l, hook] \arrow[u, hook] \end{tikzcd}$$

Proof. *Item 1, Wedge Sums of Pointed Sets*: Follows by definition.
Item 2, Intersections via Unions: Indeed, $A \coprod_{A \cap B} B$ is the quotient of $A \coprod B$ by the equivalence relation obtained by declaring $(0, a) \sim (1, b)$ iff $a = b \in A \cap B$, which is in bijection with $A \cup B$ via the map with $[(0, a)] \mapsto a$ and $[(1, b)] \mapsto b$. $\square$

002E **Proposition 2.2.4.1.4.** Let A, B, C, and X be sets.

002F 1. *Functoriality.* The assignment $(A, B, C, f, g) \mapsto A \coprod_{f,C,g} B$ defines a functor

$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets},$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \\ \bullet & \bullet. \arrow[l] \arrow[u] \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$
\begin{tikzcd}
A \coprod_C B \arrow[dr, phantom, very near start, "\ulcorner"] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, very near start, "\ulcorner"] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, crossing over, "f"'] \arrow[uu, crossing over, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu, crossing over] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}
$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets})$ to the map $\xi \colon A \coprod_C B \xrightarrow{\exists!} A' \coprod_{C'} B'$ given by

$$
\xi(x) \stackrel{\text{def}}{=} \begin{cases} \phi(a) & \text{if } x = [(0, a)], \\ \psi(b) & \text{if } x = [(1, b)] \end{cases}
$$

for each $x \in A \coprod_C B$, which is the unique map making the diagram

$$
\begin{tikzcd}
A \coprod_C B \arrow[dr, phantom, very near start, "\ulcorner"] \arrow[dr, dashed] & & B \arrow[ll] \arrow[dr, "\psi"] & \\
& A' \coprod_{C'} B' \arrow[dr, phantom, very near start, "\ulcorner"] & & B' \arrow[ll] \\
A \arrow[uu] \arrow[dr, "\phi"'] & & C \arrow[ll, crossing over, "f"'] \arrow[uu, crossing over, "g"] \arrow[dr, "\chi"] & \\
& A' \arrow[uu, crossing over] & & C' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}
$$

commute.

002G 2. *Associativity.* Given a diagram

$$
\begin{tikzcd}
A & & B & & C \\
& X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
$$

in Sets, we have isomorphisms of sets

$$
(A \textstyle\coprod_X B) \coprod_Y C \cong (A \coprod_X B) \coprod_B (B \coprod_Y C) \cong A \coprod_X (B \coprod_Y C),
$$

where these pullbacks are built as in the diagrams

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & (A \coprod_X B) \coprod_Y C & & \\
 & A \coprod_X B \arrow[ur] & & & \\
A \arrow[ur] & & B \arrow[ul] & & C, \arrow[uull] \\
 & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
 & & (A \coprod_X B) \coprod_B (B \coprod_Y C) & & \\
 & A \coprod_X B \arrow[ur] & & B \coprod_Y C \arrow[ul] & \\
A \arrow[ur] & & B \arrow[ul] \arrow[ur] & & C, \arrow[ul] \\
 & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
 & & A \coprod_X (B \coprod_Y C) & & \\
 & & & B \coprod_Y C \arrow[ul] & \\
A \arrow[uurr] & & B \arrow[ur] & & C. \arrow[ul] \\
 & X \arrow[ul, "f"] \arrow[ur, "g"'] & & Y \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
$$

002H 3. *Unitality.* We have isomorphisms of sets

$$
\begin{tikzcd}
A \arrow[r, equal] & A \\
X \arrow[u, "f"] \arrow[r, equal] \arrow[ur, phantom, very near start, "\ulcorner"] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned}
X \textstyle\coprod_X A &\cong A, \\
A \textstyle\coprod_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A & X \arrow[l, "f"'] \\
X \arrow[u, equal] \arrow[ur, phantom, very near start, "\ulcorner"] & X. \arrow[l, "f"] \arrow[u, equal]
\end{tikzcd}
$$

002J 4. *Commutativity.* We have an isomorphism of sets

$$
\begin{tikzcd}
A \coprod_X B & B \arrow[l] \\
A \arrow[u] \arrow[ur, phantom, very near start, "\ulcorner"] & X, \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
\qquad
A \textstyle\coprod_X B \cong B \textstyle\coprod_X A
\qquad
\begin{tikzcd}
B \coprod_X A & A \arrow[l] \\
B \arrow[u] \arrow[ur, phantom, very near start, "\ulcorner"] & X. \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}
$$

002K 5. *Interaction With Coproducts.* We have

$$
A \textstyle\coprod_\emptyset B \cong A \coprod B,
\qquad
\begin{tikzcd}
A \coprod B & B \arrow[l] \\
A \arrow[u] \arrow[ur, phantom, very near start, "\ulcorner"] & \emptyset. \arrow[l, hook', "\iota_A"] \arrow[u, hook, "\iota_B"']
\end{tikzcd}
$$

002L 6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}, \coprod_X, X)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.

Item 2, Associativity: Omitted.

Item 3, Unitality: Omitted.

Item 4, Commutativity: Clear.

Item 5, Interaction With Coproducts: Clear.

Item 6, Symmetric Monoidality: Omitted. □

002M 2.2.5 Coequalisers

Let A and B be sets and let $f, g\colon A \rightrightarrows B$ be functions.

002N **Definition 2.2.5.1.1.** The **coequaliser of** f **and** g is the pair $(\mathrm{CoEq}(f,g), \mathrm{coeq}(f,g))$ consisting of:

- *The Colimit.* The set $\mathrm{CoEq}(f,g)$ defined by

  $$\mathrm{CoEq}(f,g) \overset{\text{def}}{=} B/{\sim},$$

  where $\sim$ is the equivalence relation on B generated by $f(a) \sim g(a)$.

- *The Cocone.* The map

  $$\mathrm{coeq}(f,g)\colon B \to \mathrm{CoEq}(f,g)$$

  given by the quotient map $\pi\colon B \twoheadrightarrow B/{\sim}$ with respect to the equivalence relation generated by $f(a) \sim g(a)$.

Proof. We claim that $\mathrm{CoEq}(f,g)$ is the categorical coequaliser of f and g in Sets. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f,g) \circ f = \mathrm{coeq}(f,g) \circ g.$$

Indeed, we have

$$\begin{aligned}
[\mathrm{coeq}(f,g) \circ f](a) &\overset{\text{def}}{=} [\mathrm{coeq}(f,g)](f(a)) \\
&\overset{\text{def}}{=} [f(a)] \\
&= [g(a)] \\
&\overset{\text{def}}{=} [\mathrm{coeq}(f,g)](g(a)) \\
&\overset{\text{def}}{=} [\mathrm{coeq}(f,g) \circ g](a)
\end{aligned}$$

for each $a \in A$. Next, we prove that $\mathrm{CoEq}(f,g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & \mathrm{CoEq}(f,g) \\
& & C
\end{tikzcd}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Items 4 and 5 of Proposition 7.5.2.1.3 that there exists a unique map

$\mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[rd, "c"'] & \mathrm{CoEq}(f,g) \arrow[d, dashed, "\exists!"] \\ & & C \end{tikzcd}$$

commute. □

002P **Remark 2.2.5.1.2.** In detail, by Construction 7.4.2.1.2, the relation $\sim$ of Definition 2.2.5.1.1 is given by declaring $a \sim b$ iff one of the following conditions is satisfied:

- We have $a = b$;
- There exist $x_1, \ldots, x_n \in B$ such that $a \sim' x_1 \sim' \cdots \sim' x_n \sim' b$, where we declare $x \sim' y$ if one of the following conditions is satisfied:
  1. There exists $z \in A$ such that $x = f(z)$ and $y = g(z)$.
  2. There exists $z \in A$ such that $x = g(z)$ and $y = f(z)$.

  That is: we require the following condition to be satisfied:

  ($\star$) There exist $x_1, \ldots, x_n \in B$ satisfying the following conditions:
  1. There exists $z_0 \in A$ satisfying one of the following conditions:
     (a) We have $a = f(z_0)$ and $x_1 = g(z_0)$.
     (b) We have $a = g(z_0)$ and $x_1 = f(z_0)$.
  2. For each $1 \leq i \leq n-1$, there exists $z_i \in A$ satisfying one of the following conditions:
     (a) We have $x_i = f(z_i)$ and $x_{i+1} = g(z_i)$.
     (b) We have $x_i = g(z_i)$ and $x_{i+1} = f(z_i)$.
  3. There exists $z_n \in A$ satisfying one of the following conditions:
     (a) We have $x_n = f(z_n)$ and $b = g(z_n)$.
     (b) We have $x_n = g(z_n)$ and $b = f(z_n)$.

002Q **Example 2.2.5.1.3.** Here are some examples of coequalisers of sets.

002R 1. *Quotients by Equivalence Relations.* Let R be an equivalence relation on a set X. We have a bijection of sets

$$X/\sim_R \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right).$$

Proof. *Item 1*, *Quotients by Equivalence Relations*: See [Pro24ad]. □

002S **Proposition 2.2.5.1.4.** Let A, B, and C be sets.

002T 1. *Associativity.* We have isomorphisms of sets[11]

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

[11] That is, the following three ways of forming "the" coequaliser of (f,g,h) agree:

1. Take the coequaliser of (f,g,h), i.e. the colimit of the diagram

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & B \end{tikzcd}$$

in Sets.

2. First take the coequaliser of f and g, forming a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g)$$

and then take the coequaliser of the composition

$$A \underset{h}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(f,g)}{\twoheadrightarrow} \mathrm{CoEq}(f,g),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f,\mathrm{coeq}(f,g)\circ h) = \mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g,\mathrm{coeq}(f,g)\circ h)$$

of $\mathrm{CoEq}(f,g)$

3. First take the coequaliser of g and h, forming a diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h)$$

and then take the coequaliser of the composition

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{\mathrm{coeq}(g,h)}{\twoheadrightarrow} \mathrm{CoEq}(g,h),$$

obtaining a quotient

$$\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ g) = \mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f,\mathrm{coeq}(g,h)\circ h)$$

of $\mathrm{CoEq}(g,h)$.

002U 4. *Unitality.* We have an isomorphism of sets

$$\mathrm{CoEq}(f, f) \cong B.$$

002V 5. *Commutativity.* We have an isomorphism of sets

$$\mathrm{CoEq}(f, g) \cong \mathrm{CoEq}(g, f).$$

002W 6. *Interaction With Composition.* Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{k}{\overset{h}{\rightrightarrows}} C$$

be functions. We have a surjection

$$\mathrm{CoEq}(h \circ f, k \circ g) \twoheadrightarrow \mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$$

exhibiting $\mathrm{CoEq}(\mathrm{coeq}(h, k) \circ h \circ f, \mathrm{coeq}(h, k) \circ k \circ g)$ as a quotient of $\mathrm{CoEq}(h \circ f, k \circ g)$ by the relation generated by declaring $h(y) \sim k(y)$ for each $y \in B$.

Proof. *Item 1*, *Associativity*: Omitted.
Item 4, *Unitality*: Clear.
Item 5, *Commutativity*: Clear.
Item 6, *Interaction With Composition*: Omitted. □

002X 2.3 Operations With Sets

002Y 2.3.1 The Empty Set

002Z **Definition 2.3.1.1.1.** The **empty set** is the set $\emptyset$ defined by

$$\emptyset \overset{\text{def}}{=} \{x \in X \mid x \neq x\},$$

where A is the set in the set existence axiom, **??** of **??**.

0030 2.3.2 Singleton Sets

Let X be a set.

0031 **Definition 2.3.2.1.1.** The **singleton set containing** X is the set $\{X\}$ defined by

$$\{X\} \overset{\text{def}}{=} \{X, X\},$$

where $\{X, X\}$ is the pairing of X with itself (Definition 2.3.3.1.1).

0032 2.3.3 Pairings of Sets

Let X and Y be sets.

0033 **Definition 2.3.3.1.1.** The **pairing of** X **and** Y is the set $\{X, Y\}$ defined by

$$\{X, Y\} \overset{\text{def}}{=} \{x \in A \mid x = X \text{ or } x = Y\},$$

where A is the set in the axiom of pairing, **??** of **??**.

0034 2.3.4 Ordered Pairs

Let A and B be sets.

0035 **Definition 2.3.4.1.1.** The **ordered pair associated to** A **and** B is the set (A, B) defined by

$$(A, B) \overset{\text{def}}{=} \{\{A\}, \{A, B\}\}.$$

0036 **Proposition 2.3.4.1.2.** Let A and B be sets.

0037 1. *Uniqueness.* Let A, B, C, and D be sets. The following conditions are equivalent:

0038 (a) We have $(A, B) = (C, D)$.

0039 (b) We have $A = C$ and $B = D$.

Proof. *Item 1, Uniqueness*: See [Cie97, Theorem 1.2.3]. □

003A 2.3.5 Sets of Maps

Let A and B be sets.

003B **Definition 2.3.5.1.1.** The **set of maps from** A **to** B[12] is the set $\mathrm{Hom}_{\mathsf{Sets}}(A, B)$[13] whose elements are the functions from A to B.

003C **Proposition 2.3.5.1.2.** Let A and B be sets.

003D 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto \mathrm{Hom}_{\mathsf{Sets}}(X, Y)$ define functors

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sets}}(X, -) &\colon \mathsf{Sets} \to \mathsf{Sets}, \\ \mathrm{Hom}_{\mathsf{Sets}}(-, Y) &\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets}, \\ \mathrm{Hom}_{\mathsf{Sets}}(-_1, -_2) &\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets} \to \mathsf{Sets}. \end{aligned}$$

Proof. *Item 1, Functoriality*: This follows from Items 2 and 5 of Proposition 8.1.6.1.2. □

[12] *Further Terminology:* Also called the **Hom set from** A **to** B.
[13] *Further Notation:* Also written $\mathsf{Sets}(A, B)$.

003E 2.3.6 Unions of Families

Let $\{A_i\}_{i\in I}$ be a family of sets.

003F **Definition 2.3.6.1.1.** The **union of the family** $\{A_i\}_{i\in I}$ is the set $\bigcup_{i\in I} A_i$ defined by

$$\bigcup_{i\in I} A_i \stackrel{\text{def}}{=} \{x \in F \mid \text{there exists some } i \in I \text{ such that } x \in A_i\},$$

where F is the set in the axiom of union, **??** of **??**.

003G 2.3.7 Binary Unions

Let A and B be sets.

003H **Definition 2.3.7.1.1.** The **union**[14] **of** A **and** B is the set $A \cup B$ defined by

$$A \cup B \stackrel{\text{def}}{=} \bigcup_{z\in\{A,B\}} z.$$

003J **Proposition 2.3.7.1.2.** Let X be a set.

003K 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cup V$ define functors

$$U \cup -\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset),$$
$$- \cup V\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset),$$
$$-_1 \cup -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) \to (\mathcal{P}(X), \subset),$$

where $-_1 \cup -_2$ is the functor where

- *Action on Objects.* For each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \cup -_2](U, V) \stackrel{\text{def}}{=} U \cup V.$$

- *Action on Morphisms.* For each pair of morphisms

$$\iota_U\colon U \hookrightarrow U',$$
$$\iota_V\colon V \hookrightarrow V'$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \cup \iota_V\colon U \cup V \hookrightarrow U' \cup V'$$

of (ι_U, ι_V) by $\cup$ is the inclusion

$$U \cup V \subset U' \cup V'$$

i.e. where we have

[14] *Further Terminology:* Also called the **binary union of** A **and** B, for emphasis.

($\star$) If $U \subset U'$ and $V \subset V'$, then $U \cup V \subset U' \cup V'$.

and where $U \cup -$ and $- \cup V$ are the partial functors of $-_1 \cup -_2$ at $U, V \in \mathcal{P}(X)$.

003L 2. *Via Intersections and Symmetric Differences.* We have an equality of sets

$$U \cup V = (U \bigtriangleup V) \bigtriangleup (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

003M 3. *Associativity.* We have an equality of sets

$$(U \cup V) \cup W = U \cup (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

003N 4. *Unitality.* We have equalities of sets

$$U \cup \emptyset = U,$$
$$\emptyset \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

003P 5. *Commutativity.* We have an equality of sets

$$U \cup V = V \cup U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

003Q 6. *Idempotency.* We have an equality of sets

$$U \cup U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

003R 7. *Distributivity Over Intersections.* We have equalities of sets

$$U \cup (V \cap W) = (U \cup V) \cap (U \cup W),$$
$$(U \cap V) \cup W = (U \cup W) \cap (V \cup W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

003S 8. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

003T 9. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cup V} = \chi_U + \chi_V - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

003U 10. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1, Functoriality*: See [Pro24ar].
Item 2, Via Intersections and Symmetric Differences: See [Pro24bc].
Item 3, Associativity: See [Pro24be].
Item 4, Unitality: This follows from [Pro24bh] and Item 5.
Item 5, Commutativity: See [Pro24bf].
Item 6, Idempotency: See [Pro24aq].
Item 7, Distributivity Over Intersections: See [Pro24bd].
Item 8, Interaction With Characteristic Functions I: See [Pro24k].
Item 9, Interaction With Characteristic Functions II: See [Pro24k].
Item 10, Interaction With Powersets and Semirings: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 2.3.9.1.2. □

003V 2.3.8 Intersections of Families

Let $\mathcal{F}$ be a family of sets.

003W **Definition 2.3.8.1.1.** The **intersection of a family $\mathcal{F}$ of sets** is the set $\bigcap_{X \in \mathcal{F}} X$ defined by

$$\bigcap_{X \in \mathcal{F}} X \stackrel{\text{def}}{=} \left\{ z \in \bigcup_{X \in \mathcal{F}} X \;\middle|\; \text{for each } X \in \mathcal{F}, \text{ we have } z \in X \right\}.$$

003X 2.3.9 Binary Intersections

Let X and Y be sets.

003Y **Definition 2.3.9.1.1.** The **intersection**[15] **of X and Y** is the set $X \cap Y$ defined by

$$X \cap Y \stackrel{\text{def}}{=} \bigcap_{z \in \{X, Y\}} z.$$

003Z **Proposition 2.3.9.1.2.** Let X be a set.

0040 1. *Functoriality.* The assignments $U, V, (U, V) \mapsto U \cap V$ define

[15] *Further Terminology:* Also called the **binary intersection of X and Y**, for emphasis.

functors

$$\begin{aligned} U\cap -&\colon (\mathcal{P}(X),\subset)\to(\mathcal{P}(X),\subset),\\ -\cap V&\colon (\mathcal{P}(X),\subset)\to(\mathcal{P}(X),\subset),\\ -_1\cap -_2&\colon (\mathcal{P}(X)\times\mathcal{P}(X),\subset\times\subset)\to(\mathcal{P}(X),\subset), \end{aligned}$$

where $-_1\cap -_2$ is the functor where

- *Action on Objects.* For each $(U,V)\in\mathcal{P}(X)\times\mathcal{P}(X)$, we have

$$[-_1\cap -_2](U,V)\stackrel{\text{def}}{=}U\cap V.$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_U&\colon U\hookrightarrow U',\\ \iota_V&\colon V\hookrightarrow V' \end{aligned}$$

  of $\mathcal{P}(X)\times\mathcal{P}(X)$, the image

$$\iota_U\cap\iota_V\colon U\cap V\hookrightarrow U'\cap V'$$

  of (ι_U,ι_V) by $\cap$ is the inclusion

$$U\cap V\subset U'\cap V'$$

  i.e. where we have

  ($\star$) If $U\subset U'$ and $V\subset V'$, then $U\cap V\subset U'\cap V'$.

and where $U\cap -$ and $-\cap V$ are the partial functors of $-_1\cap -_2$ at $U,V\in\mathcal{P}(X)$.

0041 2. *Adjointness.* We have adjunctions

$$\Big(U\cap - \dashv \mathbf{Hom}_{\mathcal{P}(X)}(U,-)\Big)\colon\quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "U\cap -"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(U,-)"] \end{tikzcd}$$

$$\Big(-\cap V \dashv \mathbf{Hom}_{\mathcal{P}(X)}(V,-)\Big)\colon\quad \begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "-\cap V"] \arrow[r, phantom, "\perp"] & \mathcal{P}(X), \arrow[l, bend left, "\mathbf{Hom}_{\mathcal{P}(X)}(V,-)"] \end{tikzcd}$$

where

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1,-_2)\colon\mathcal{P}(X)^{\mathsf{op}}\times\mathcal{P}(X)\to\mathcal{P}(X)$$

is the bifunctor defined by[16]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U,V) \overset{\text{def}}{=} (X \setminus U) \cup V$$

witnessed by bijections

$$\begin{aligned}\mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\Big(U, \mathbf{Hom}_{\mathcal{P}(X)}(V,W)\Big),\\ \mathrm{Hom}_{\mathcal{P}(X)}(U \cap V, W) &\cong \mathrm{Hom}_{\mathcal{P}(X)}\Big(V, \mathbf{Hom}_{\mathcal{P}(X)}(U,W)\Big),\end{aligned}$$

natural in $U, V, W \in \mathcal{P}(X)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $U \cap V \subset W$.
   ii. We have $U \subset \mathbf{Hom}_{\mathcal{P}(X)}(V,W)$.
   iii. We have $U \subset (X \setminus V) \cup W$.

(b) The following conditions are equivalent:
   i. We have $V \cap U \subset W$.
   ii. We have $V \subset \mathbf{Hom}_{\mathcal{P}(X)}(U,W)$.
   iii. We have $V \subset (X \setminus U) \cup W$.

0042 3. *Associativity.* We have an equality of sets

$$(U \cap V) \cap W = U \cap (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

0043 4. *Unitality.* Let X be a set and let $U \in \mathcal{P}(X)$. We have equalities of sets

$$\begin{aligned}X \cap U &= U,\\ U \cap X &= U\end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

0044 5. *Commutativity.* We have an equality of sets

$$U \cap V = V \cap U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

0045 6. *Idempotency.* We have an equality of sets

$$U \cap U = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

[16]For intuition regarding the expression defining $\mathbf{Hom}_{\mathcal{P}(X)}(U,V)$, see

0046 7. *Distributivity Over Unions.* We have equalities of sets

$$U \cap (V \cup W) = (U \cap V) \cup (U \cap W),$$
$$(U \cup V) \cap W = (U \cap W) \cup (V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

0047 8. *Annihilation With the Empty Set.* We have an equality of sets

$$\emptyset \cap X = \emptyset,$$
$$X \cap \emptyset = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

0048 9. *Interaction With Characteristic Functions I.* We have

$$\chi_{U \cap V} = \chi_U \chi_V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

0049 10. *Interaction With Characteristic Functions II.* We have

$$\chi_{U \cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

004A 11. *Interaction With Powersets and Monoids With Zero.* The quadruple $((\mathcal{P}(X), \emptyset), \cap, X)$ is a commutative monoid with zero.

004B 12. *Interaction With Powersets and Semirings.* The quintuple $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is an idempotent commutative semiring.

Proof. *Item 1, Functoriality*: See [Pro24ap].
Item 2, Adjointness: See [MSE 267469].
Item 3, Associativity: See [Pro24v].
Item 4, Unitality: This follows from [Pro24z] and Item 5.
Item 5, Commutativity: See [Pro24w].
Item 6, Idempotency: See [Pro24ao].
Item 7, Distributivity Over Unions: See [Pro24an].
Item 8, Annihilation With the Empty Set: This follows from [Pro24x] and Item 5.
Item 9, Interaction With Characteristic Functions I: See [Pro24h].
Item 10, Interaction With Characteristic Functions II: See [Pro24h].
Item 11, Interaction With Powersets and Monoids With Zero: This follows from Items 3 to 5 and 8.
Item 12, Interaction With Powersets and Semirings: This follows from Items 3 to 6 and Items 3 to 5, 7 and 8 of Proposition 2.3.9.1.2. □

004C **Remark 2.3.9.1.3.** Since intersections are the products in $\mathcal{P}(X)$ (Item 1 of Proposition 2.4.3.1.3), the left adjoint $\mathbf{Hom}_{\mathcal{P}(X)}(U,V)$ may be thought of as a function type $[U,V]$.
Then, under the Curry–Howard correspondence, the function type $[U,V]$ corresponds to implication $U \implies V$, which is logically equivalent to the statement $\neg U \vee V$. This in turn corresponds to the set $U^{\mathsf{c}} \vee V = (X \setminus U) \cup V$.

004D 2.3.10 Differences

Let X and Y be sets.

004E **Definition 2.3.10.1.1.** The **difference of X and Y** is the set $X \setminus Y$ defined by

$$X \setminus Y \overset{\text{def}}{=} \{a \in X \mid a \notin Y\}.$$

004F **Proposition 2.3.10.1.2.** Let X be a set.

004G 1. *Functoriality.* The assignments $U, V, (U,V) \mapsto U \cap V$ define functors

$$\begin{aligned} U \setminus -&\colon (\mathcal{P}(X), \supset) \to (\mathcal{P}(X), \subset), \\ - \setminus V&\colon (\mathcal{P}(X), \subset) \to (\mathcal{P}(X), \subset), \\ -_1 \setminus -_2&\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \supset) \to (\mathcal{P}(X), \subset), \end{aligned}$$

where $-_1 \setminus -_2$ is the functor where

- *Action on Objects.* For each $(U,V) \in \mathcal{P}(X) \times \mathcal{P}(X)$, we have

$$[-_1 \setminus -_2](U,V) \overset{\text{def}}{=} U \setminus V.$$

- *Action on Morphisms.* For each pair of morphisms

$$\begin{aligned} \iota_A &\colon A \hookrightarrow B, \\ \iota_U &\colon U \hookrightarrow V \end{aligned}$$

of $\mathcal{P}(X) \times \mathcal{P}(X)$, the image

$$\iota_U \setminus \iota_V \colon A \setminus V \hookrightarrow B \setminus U$$

of (ι_U, ι_V) by $\setminus$ is the inclusion

$$A \setminus V \subset B \setminus U$$

i.e. where we have

Remark 2.3.9.1.3.

(⋆) If $A \subset B$ and $U \subset V$, then $A \setminus V \subset B \setminus U$.

and where $U \setminus -$ and $- \setminus V$ are the partial functors of $-_1 \setminus -_2$ at $U, V \in \mathcal{P}(X)$.

004H 2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} X \setminus (U \cup V) &= (X \setminus U) \cap (X \setminus V), \\ X \setminus (U \cap V) &= (X \setminus U) \cup (X \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

004J 3. *Interaction With Unions I.* We have equalities of sets

$$U \setminus (V \cup W) = (U \setminus V) \cap (U \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

004K 4. *Interaction With Unions II.* We have equalities of sets

$$(U \setminus V) \cup W = (U \cup W) \setminus (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

004L 5. *Interaction With Unions III.* We have equalities of sets

$$\begin{aligned} U \setminus (V \cup W) &= (U \cup W) \setminus (V \cup W) \\ &= (U \setminus V) \setminus W \\ &= (U \setminus W) \setminus V \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

004M 6. *Interaction With Unions IV.* We have equalities of sets

$$(U \cup V) \setminus W = (U \setminus W) \cup (V \setminus W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

004N 7. *Interaction With Intersections.* We have equalities of sets

$$\begin{aligned} (U \setminus V) \cap W &= (U \cap W) \setminus V \\ &= U \cap (W \setminus V) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

004P 8. *Interaction With Complements.* We have an equality of sets

$$U \setminus V = U \cap V^{\mathsf{c}}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

004Q 9. *Interaction With Symmetric Differences.* We have an equality of sets

$$U \setminus V = U \bigtriangleup (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

004R 10. *Triple Differences.* We have

$$U \setminus (V \setminus W) = (U \cap W) \cup (U \setminus V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

004S 11. *Left Annihilation.* We have

$$\emptyset \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

004T 12. *Right Unitality.* We have

$$U \setminus \emptyset = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

004U 13. *Invertibility.* We have

$$U \setminus U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

004V 14. *Interaction With Containment.* The following conditions are equivalent:

004W (a) We have $V \setminus U \subset W$.

004X (b) We have $V \setminus W \subset U$.

004Y 15. *Interaction With Characteristic Functions.* We have

$$\chi_{U \setminus V} = \chi_U - \chi_{U \cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

Proof. *Item 1*, *Functoriality*: See [Pro24ah] and [Pro24al].
Item 2, *De Morgan's Laws*: See [Pro24p].
Item 3, *Interaction With Unions I*: See [Pro24q].
Item 4, *Interaction With Unions II*: Omitted.
Item 5, *Interaction With Unions III*: See [Pro24am].
Item 6, *Interaction With Unions IV*: See [Pro24ag].
Item 7, *Interaction With Intersections*: See [Pro24y].

Item 8, Interaction With Complements: See [Pro24ae].
Item 9, Interaction With Symmetric Differences: See [Pro24af].
Item 10, Triple Differences: See [Pro24ak].
Item 11, Left Annihilation: Clear.
Item 12, Right Unitality: See [Pro24ai].
Item 13, Invertibility: See [Pro24aj].
Item 14, Interaction With Containment: Omitted.
Item 15, Interaction With Characteristic Functions: See [Pro24i]. □

004Z 2.3.11 Complements

Let X be a set and let $U \in \mathcal{P}(X)$.

0050 **Definition 2.3.11.1.1.** The **complement of** U is the set U^{c} defined by

$$\begin{aligned} U^{\mathsf{c}} &\overset{\text{def}}{=} X \setminus U \\ &\overset{\text{def}}{=} \{a \in X \mid a \notin U\}. \end{aligned}$$

0051 **Proposition 2.3.11.1.2.** Let X be a set.

0052 1. *Functoriality.* The assignment $U \mapsto U^{\mathsf{c}}$ defines a functor

$$(-)^{\mathsf{c}} \colon \mathcal{P}(X)^{\mathsf{op}} \to \mathcal{P}(X),$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(X)$, we have

$$[(-)^{\mathsf{c}}](U) \overset{\text{def}}{=} U^{\mathsf{c}}.$$

- *Action on Morphisms.* For each morphism $\iota_U \colon U \hookrightarrow V$ of $\mathcal{P}(X)$, the image

$$\iota_U^{\mathsf{c}} \colon V^{\mathsf{c}} \hookrightarrow U^{\mathsf{c}}$$

of ι_U by $(-)^{\mathsf{c}}$ is the inclusion

$$V^{\mathsf{c}} \subset U^{\mathsf{c}}$$

i.e. where we have

($\star$) If $U \subset V$, then $V^{\mathsf{c}} \subset U^{\mathsf{c}}$.

0053 2. *De Morgan's Laws.* We have equalities of sets

$$\begin{aligned} (U \cup V)^{\mathsf{c}} &= U^{\mathsf{c}} \cap V^{\mathsf{c}}, \\ (U \cap V)^{\mathsf{c}} &= U^{\mathsf{c}} \cup V^{\mathsf{c}} \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

0054 3. *Involutority.* We have

$$(U^{\mathsf{c}})^{\mathsf{c}} = U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

0055 4. *Interaction With Characteristic Functions.* We have

$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

Proof. *Item 1, Functoriality*: This follows from Item 1 of Proposition 2.3.10.1.2.
Item 2, De Morgan's Laws: See [Pro24p].
Item 3, Involutority: See [Pro24l].
Item 4, Interaction With Characteristic Functions: Clear. □

0056 2.3.12 Symmetric Differences

Let A and B be sets.

0057 **Definition 2.3.12.1.1.** The **symmetric difference of A and B** is the set $A \mathbin{\triangle} B$ defined by

$$A \mathbin{\triangle} B \overset{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

0058 **Proposition 2.3.12.1.2.** Let X be a set.

0059 1. *Lack of Functoriality.* The assignment $(U, V) \mapsto U \mathbin{\triangle} V$ ***need not*** define functors

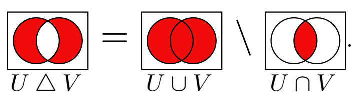
$$\begin{aligned} U \mathbin{\triangle} -\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ - \mathbin{\triangle} V\colon (\mathcal{P}(X), \subset) &\to (\mathcal{P}(X), \subset), \\ -_1 \mathbin{\triangle} -_2\colon (\mathcal{P}(X) \times \mathcal{P}(X), \subset \times \subset) &\to (\mathcal{P}(X), \subset). \end{aligned}$$

005A 2. *Via Unions and Intersections.* We have[17]

$$U \mathbin{\triangle} V = (U \cup V) \setminus (U \cap V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

[17] *Illustration:*



005B 3. *Associativity.* We have[18]

$$(U \triangle V) \triangle W = U \triangle (V \triangle W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

005C 4. *Commutativity.* We have

$$U \triangle V = V \triangle U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

005D 5. *Unitality.* We have

$$\begin{aligned} U \triangle \emptyset &= U, \\ \emptyset \triangle U &= U \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

005E 6. *Invertibility.* We have

$$U \triangle U = \emptyset$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

005F 7. *Interaction With Unions.* We have

$$(U \triangle V) \cup (V \triangle T) = (U \cup V \cup W) \setminus (U \cap V \cap W)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

005G 8. *Interaction With Complements I.* We have

$$U \triangle U^{\mathsf{c}} = X$$

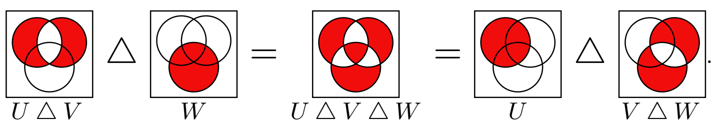
for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

005H 9. *Interaction With Complements II.* We have

$$\begin{aligned} U \triangle X &= U^{\mathsf{c}}, \\ X \triangle U &= U^{\mathsf{c}} \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

[18] *Illustration:*



005J 10. *Interaction With Complements III.* We have

$$U^{\mathsf{c}} \triangle V^{\mathsf{c}} = U \triangle V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

005K 11. *"Transitivity".* We have

$$(U \triangle V) \triangle (V \triangle W) = U \triangle W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

005L 12. *The Triangle Inequality for Symmetric Differences.* We have

$$U \triangle W \subset U \triangle V \cup V \triangle W$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

005M 13. *Distributivity Over Intersections.* We have

$$\begin{aligned} U \cap (V \triangle W) &= (U \cap V) \triangle (U \cap W), \\ (U \triangle V) \cap W &= (U \cap W) \triangle (V \cap W) \end{aligned}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V, W \in \mathcal{P}(X)$.

005N 14. *Interaction With Characteristic Functions.* We have

$$\chi_{U \triangle V} = \chi_U + \chi_V - 2\chi_{U \cap V}$$

and thus, in particular, we have

$$\chi_{U \triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

005P 15. *Bijectivity.* Given $A, B \subset \mathcal{P}(X)$, the maps

$$\begin{aligned} A \triangle - &\colon \mathcal{P}(X) \to \mathcal{P}(X), \\ - \triangle B &\colon \mathcal{P}(X) \to \mathcal{P}(X) \end{aligned}$$

are bijections with inverses given by

$$\begin{aligned} (A \triangle -)^{-1} &= - \cup (A \cap -), \\ (- \triangle B)^{-1} &= - \cup (B \cap -). \end{aligned}$$

Moreover, the map

$$C \mapsto C \triangle (A \triangle B)$$

is a bijection of $\mathcal{P}(X)$ onto itself sending A to B and B to A.

005Q 16. *Interaction With Powersets and Groups.* Let X be a set.

005R (a) The quadruple $\left(\mathcal{P}(X), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(X)}\right)$ is an abelian group. [19]

(b) Every element of $\mathcal{P}(X)$ has order 2 with respect to $\triangle$, and
005S thus $\mathcal{P}(X)$ is a *Boolean group* (i.e. an abelian 2-group).

005T 4. *Interaction With Powersets and Vector Spaces I.* The pair $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ consisting of

- The group $\mathcal{P}(X)$ of **??**;
- The map $\alpha_{\mathcal{P}(X)} \colon \mathbb{F}_2 \times \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$\begin{aligned} 0 \cdot U &\overset{\text{def}}{=} \emptyset, \\ 1 \cdot U &\overset{\text{def}}{=} U; \end{aligned}$$

is an $\mathbb{F}_2$-vector space.

005U 5. *Interaction With Powersets and Vector Spaces II.* If X is finite, then:

(a) The set of singletons sets on the elements of X forms a basis for the $\mathbb{F}_2$-vector space $\left(\mathcal{P}(X), \alpha_{\mathcal{P}(X)}\right)$ of Item 4.

(b) We have
$$\dim(\mathcal{P}(X)) = \#\mathcal{P}(X).$$

005V 6. *Interaction With Powersets and Rings.* The quintuple $(\mathcal{P}(X), \triangle, \cap, \emptyset, X)$ is a commutative ring.[20]

Proof. Item 1, Lack of Functoriality: Omitted.

---

[19]Here are some examples:

1. When $X = \emptyset$, we have an isomorphism of groups between $\mathcal{P}(\emptyset)$ and the trivial group:
$$\left(\mathcal{P}(\emptyset), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\emptyset)}\right) \cong \mathrm{pt}.$$
2. When $X = \mathrm{pt}$, we have an isomorphism of groups between $\mathcal{P}(\mathrm{pt})$ and $\mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\mathrm{pt}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\mathrm{pt})}\right) \cong \mathbb{Z}_{/2}.$$
3. When $X = \{0, 1\}$, we have an isomorphism of groups between $\mathcal{P}(\{0, 1\})$ and $\mathbb{Z}_{/2} \times \mathbb{Z}_{/2}$:
$$\left(\mathcal{P}(\{0, 1\}), \triangle, \emptyset, \mathrm{id}_{\mathcal{P}(\{0,1\})}\right) \cong \mathbb{Z}_{/2} \times \mathbb{Z}_{/2}.$$

[20] *Warning:* The analogous statement replacing intersections by unions (i.e. that the quintuple $(\mathcal{P}(X), \triangle, \cup, \emptyset, X)$ is a ring) is false, however. See [Pro24ba] for a proof.
END TEXTDBEND

Item 2, Via Unions and Intersections: See [Pro24r].
Item 3, Associativity: See [Pro24as].
Item 4, Commutativity: See [Pro24at].
Item 5, Unitality: This follows from Item 4 and [Pro24ax].
Item 6, Invertibility: See [Pro24az].
Item 7, Interaction With Unions: See [Pro24bg].
Item 8, Interaction With Complements I: See [Pro24aw].
Item 9, Interaction With Complements II: This follows from Item 4 and [Pro24bb].
Item 10, Interaction With Complements III: See [Pro24au].
Item 11, "Transitivity": We have

$$\begin{aligned}(U \triangle V) \triangle (V \triangle W) &= U \triangle (V \triangle (V \triangle W)) && \text{(by Item 3)}\\ &= U \triangle ((V \triangle V) \triangle W) && \text{(by Item 3)}\\ &= U \triangle (\emptyset \triangle W) && \text{(by Item 6)}\\ &= U \triangle W && \text{(by Item 5)}\end{aligned}$$

Item 12, The Triangle Inequality for Symmetric Differences: This follows from Items 2 and 11.
Item 13, Distributivity Over Intersections: See [Pro24u].
Item 14, Interaction With Characteristic Functions: See [Pro24j].
Item 15, Bijectivity: Clear.
Item 16, Interaction With Powersets and Groups: Item 16a follows from[21] Items 3 to 6, while Item 3b follows from Item 6.
Item 4, Interaction With Powersets and Vector Spaces I: Clear.
Item 5, Interaction With Powersets and Vector Spaces II: Omitted.
Item 6, Interaction With Powersets and Rings: This follows from Items 8 and 11 of Proposition 2.3.9.1.2 and Items 13 and 16.[22] □

005W 2.4 Powersets

005X 2.4.1 Characteristic Functions

Let X be a set.

005Y **Definition 2.4.1.1.1.** Let $U \subset X$ and let $x \in X$.

005Z 1. The **characteristic function of U** [23] is the function[24]

$$\chi_U \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

[21] Reference: [Pro24av].
[22] *Reference:* [Pro24ay].
[23] *Further Terminology:* Also called the **indicator function of** U.
[24] *Further Notation:* Also written $\chi_X(U, -)$ or $\chi_X(-, U)$.

defined by

$$\chi_U(x) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x \in U, \\ \mathsf{false} & \text{if } x \notin U \end{cases}$$

for each $x \in X$.

0060 2. The **characteristic function of** x is the function [25]

$$\chi_x \colon X \to \{\mathsf{t}, \mathsf{f}\}$$

defined by

$$\chi_x \overset{\text{def}}{=} \chi_{\{x\}},$$

i.e. by

$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $y \in X$.

0061 3. The **characteristic relation on** X [26] is the relation[27]

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

on X defined by[28]

$$\chi_X(x, y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$.

0062 4. The **characteristic embedding** [29] **of** X **into** $\mathcal{P}(X)$ is the function

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

defined by

$$\chi_{(-)}(x) \overset{\text{def}}{=} \chi_x$$

for each $x \in X$.

[25] *Further Notation:* Also written χ^x, $\chi_X(x, -)$, or $\chi_X(-, x)$.
[26] *Further Terminology:* Also called the **identity relation on** X.
[27] *Further Notation:* Also written $\chi_{-_2}^{-_1}$, or $\sim_{\text{id}}$ in the context of relations.
[28] As a subset of $X \times X$, the relation χ_X corresponds to the diagonal $\Delta_X \subset X \times X$ of X.
[29] The name "characteristic *embedding*" comes from the fact that there is an analogue of fully faithfulness for $\chi_{(-)}$: given a set X, we have

$$\mathrm{Hom}_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y),$$

for each $x, y \in X$.

0063 **Remark 2.4.1.1.2.** The definitions in Definition 2.4.1.1.1 are decategorifications of co/presheaves, representable co/presheaves, Hom profunctors, and the Yoneda embedding:[30]

0064 1. A function
$$f\colon X \to \{\mathsf{t},\mathsf{f}\}$$
is a decategorification of a presheaf
$$\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets},$$
with the characteristic functions χ_U of the subsets of X being the primordial examples (and, in fact, all examples) of these.

0065 2. The characteristic function
$$\chi_x\colon X \to \{\mathsf{t},\mathsf{f}\}$$
of an *element* x of X is a decategorification of the representable presheaf
$$h_X\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Sets}$$
of an *object* x of a category $\mathcal{C}$.

0066 3. The characteristic relation
$$\chi_X(-_1,-_2)\colon X \times X \to \{\mathsf{t},\mathsf{f}\}$$

[30] These statements can be made precise by using the embeddings
$$(-)_{\mathsf{disc}}\colon \mathsf{Sets} \hookrightarrow \mathsf{Cats},$$
$$(-)_{\mathsf{disc}}\colon \{\mathsf{t},\mathsf{f}\}_{\mathsf{disc}} \hookrightarrow \mathsf{Sets}$$
of sets into categories and of classical truth values into sets.

For instance, in this approach the characteristic function
$$\chi_x\colon X \to \{\mathsf{t},\mathsf{f}\}$$
of an element x of X, defined by
$$\chi_x(y) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$
for each $y \in X$, is recovered as the representable presheaf
$$\mathrm{Hom}_{X_{\mathsf{disc}}}(-,x)\colon X_{\mathsf{disc}} \to \mathsf{Sets}$$
of the corresponding object x of X_{disc}, defined on objects by
$$\mathrm{Hom}_{X_{\mathsf{disc}}}(y,x) \overset{\text{def}}{=} \begin{cases} \mathrm{pt} & \text{if } x = y, \\ \emptyset & \text{if } x \neq y \end{cases}$$
for each $y \in \mathrm{Obj}(X_{\mathsf{disc}})$.

of X is a decategorification of the Hom profunctor

$$\mathrm{Hom}_{\mathcal{C}}(-_1, -_2)\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$$

of a category $\mathcal{C}$.

0067 4. The characteristic embedding

$$\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$ is a decategorification of the Yoneda embedding

$$\text{よ}\colon \mathcal{C}^{\mathsf{op}} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

of a category $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$.

0068 5. There is also a direct parallel between unions and colimits:

- An element of $\mathcal{P}(X)$ is a union of elements of X, viewed as one-point subsets $\{x\} \in \mathcal{P}(A)$.
- An object of $\mathsf{PSh}(\mathcal{C})$ is a colimit of objects of $\mathcal{C}$, viewed as representable presheaves $h_X \in \mathrm{Obj}(\mathsf{PSh}(\mathcal{C}))$.

0069 **Proposition 2.4.1.1.3.** Let X be a set.

1. *The Inclusion of Characteristic Relations Associated to a Function.*
006A Let $f\colon A \to B$ be a function. We have an inclusion[31]

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[rr, "f\times f"] \arrow[dr, "\chi_A"'] & \arrow[d, phantom, "\supset"] & B \times B \arrow[dl, "\chi_B"] \\ & \{\mathsf{t}, \mathsf{f}\}. & \end{tikzcd}$$

006B 2. *Interaction With Unions I.* We have

$$\chi_{U\cup V} = \max(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

006C 3. *Interaction With Unions II.* We have

$$\chi_{U\cup V} = \chi_U + \chi_V - \chi_{U\cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

[31]This is the 0-categorical version of Definition 8.4.4.1.1.

006D 4. *Interaction With Intersections I.* We have

$$\chi_{U\cap V} = \chi_U \chi_V$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

006E 5. *Interaction With Intersections II.* We have

$$\chi_{U\cap V} = \min(\chi_U, \chi_V)$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

006F 6. *Interaction With Differences.* We have

$$\chi_{U\setminus V} = \chi_U - \chi_{U\cap V}$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

006G 7. *Interaction With Complements.* We have

$$\chi_{U^{\mathsf{c}}} = 1 - \chi_U$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U \in \mathcal{P}(X)$.

006H 8. *Interaction With Symmetric Differences.* We have

$$\chi_{U\triangle V} = \chi_U + \chi_V - 2\chi_{U\cap V}$$

and thus, in particular, we have

$$\chi_{U\triangle V} \equiv \chi_U + \chi_V \mod 2$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$ and each $U, V \in \mathcal{P}(X)$.

9. *Interaction Between the Characteristic Embedding and Morphisms.*
006J Let $f \colon X \to Y$ be a map of sets. The diagram

$$f_* \circ \chi_X = \chi_{X'} \circ f, \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\chi_X"'] & X' \arrow[d, "\chi_{X'}"] \\ \mathcal{P}(X) \arrow[r, "f_*"'] & \mathcal{P}(X'). \end{tikzcd}$$

commutes.

Proof. Item 1, *The Inclusion of Characteristic Relations Associated to a Function*: The inclusion $\chi_B(f(a), f(b)) \subset \chi_A(a, b)$ is equivalent to the

statement "if $a = b$, then $f(a) = f(b)$", which is true.
Item 2, Interaction With Unions I: This is a repetition of Item 8 of Proposition 2.3.7.1.2 and is proved there.
Item 3, Interaction With Unions II: This is a repetition of Item 9 of Proposition 2.3.7.1.2 and is proved there.
Item 4, Interaction With Intersections I: This is a repetition of Item 9 of Proposition 2.3.9.1.2 and is proved there.
Item 5, Interaction With Intersections II: This is a repetition of Item 10 of Proposition 2.3.9.1.2 and is proved there.
Item 6, Interaction With Differences: This is a repetition of Item 15 of Proposition 2.3.10.1.2 and is proved there.
Item 7, Interaction With Complements: This is a repetition of Item 4 of Proposition 2.3.11.1.2 and is proved there.
Item 8, Interaction With Symmetric Differences: This is a repetition of Item 14 of Proposition 2.3.12.1.2 and is proved there.
Item 9, Interaction Between the Characteristic Embedding and Morphisms: Indeed, we have

$$\begin{aligned}[f_* \circ \chi_X](x) &\overset{\text{def}}{=} f_*(\chi_X(x)) \\ &\overset{\text{def}}{=} f_*(\{x\}) \\ &= \{f(x)\} \\ &\overset{\text{def}}{=} \chi_{X'}(f(x)) \\ &\overset{\text{def}}{=} [\chi_{X'} \circ f](x),\end{aligned}$$

for each $x \in X$, showing the desired equality. □

006K 2.4.2 The Yoneda Lemma for Sets

Let X be a set and let $U \subset X$ be a subset of X.

006L **Proposition 2.4.2.1.1.** We have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_U) = \chi_U(x)$$

for each $x \in X$, giving an equality of functions

$$\chi_{\mathcal{P}(X)}\Big(\chi_{(-)}, \chi_U\Big) = \chi_U.$$

Proof. Clear. □

006M **Corollary 2.4.2.1.2.** The characteristic embedding is fully faithful, i.e., we have

$$\chi_{\mathcal{P}(X)}(\chi_x, \chi_y) = \chi_X(x, y)$$

for each $x, y \in X$.

Proof. This follows from Proposition 2.4.2.1.1. □

006N 2.4.3 Powersets

Let X be a set.

006P **Definition 2.4.3.1.1.** The **powerset of** X is the set $\mathcal{P}(X)$ defined by

$$\mathcal{P}(X) \stackrel{\text{def}}{=} \{U \in P \mid U \subset X\},$$

where P is the set in the axiom of powerset, **??** of **??**.

006Q **Remark 2.4.3.1.2.** The powerset of a set is a decategorification of the category of presheaves of a category: while[32]

- The powerset of a set X is equivalently (Items 1 and 2 of Proposition 2.4.3.1.6) the set

$$\mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

  of functions from X to the set $\{\mathsf{t}, \mathsf{f}\}$ of classical truth values.

- The category of presheaves on a category $\mathcal{C}$ is the category

$$\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Sets})$$

  of functors from $\mathcal{C}^{\mathsf{op}}$ to the category Sets of sets.

006R **Proposition 2.4.3.1.3.** Let X be a set.

006S 1. *Co/Completeness.* The (posetal) category (associated to) $(\mathcal{P}(X), \subset)$ is complete and cocomplete:

   (a) *Products.* The products in $\mathcal{P}(X)$ are given by intersection of subsets.

   (b) *Coproducts.* The coproducts in $\mathcal{P}(X)$ are given by union of subsets.

   (c) *Co/Equalisers.* Being a posetal category, $\mathcal{P}(X)$ only has at

[32] This parallel is based on the following comparison:

- A category is enriched over the category

$$\mathsf{Sets} \stackrel{\text{def}}{=} \mathsf{Cats}_0$$

  of sets (i.e. "0-categories"), with presheaves taking values on it.

- A set is enriched over the set

$$\{\mathsf{t}, \mathsf{f}\} \stackrel{\text{def}}{=} \mathsf{Cats}_{-1}$$

  of classical truth values (i.e. "(-1)-categories"), with characteristic functions taking values on it.

most one morphisms between any two objects, so co/equalisers are trivial.

006T 2. *Cartesian Closedness.* The category $\mathcal{P}(X)$ is Cartesian closed with internal Hom

$$\mathbf{Hom}_{\mathcal{P}(X)}(-_1, -_2) \colon \mathcal{P}(X)^{\mathsf{op}} \times \mathcal{P}(X) \to \mathcal{P}(X)$$

given by[33]

$$\mathbf{Hom}_{\mathcal{P}(X)}(U, V) \stackrel{\text{def}}{=} (X \setminus U) \cup V$$

for each $U, V \in \mathrm{Obj}(\mathcal{P}(X))$.

Proof. *Item 1, Co/Completeness*: Clear.
Item 2, Cartesian Closedness: This follows from Item 2 of Proposition 2.3.9.1.2. □

006U **Proposition 2.4.3.1.4.** Let X be a set.

006V 1. *Functoriality I.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_* \colon \mathsf{Sets} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_*(A) \stackrel{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

of $\mathcal{P}_*$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

$$\mathcal{P}_*(f) \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\mathcal{P}_*(f) \stackrel{\text{def}}{=} f_*,$$

as in Definition 2.4.4.1.1.

[33]For intuition regarding the expression defining $\mathbf{Hom}_{\mathcal{P}(X)}(U, V)$, see Remark 2.3.9.1.3.

006W 2. *Functoriality II.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}^{-1} \colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}^{-1}(A) \overset{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}^{-1}_{A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A))$$

  of $\mathcal{P}^{-1}$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

$$\mathcal{P}^{-1}(f) \colon \mathcal{P}(B) \to \mathcal{P}(A)$$

  defined by

$$\mathcal{P}^{-1}(f) \overset{\text{def}}{=} f^{-1},$$

  as in Definition 2.4.5.1.1.

006X 3. *Functoriality III.* The assignment $X \mapsto \mathcal{P}(X)$ defines a functor

$$\mathcal{P}_! \colon \mathsf{Sets} \to \mathsf{Sets},$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathcal{P}_!(A) \overset{\text{def}}{=} \mathcal{P}(A).$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on morphisms

$$\mathcal{P}_{!|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B))$$

  of $\mathcal{P}_!$ at (A, B) is the map defined by by sending a map of sets $f \colon A \to B$ to the map

$$\mathcal{P}_!(f) \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

  defined by

$$\mathcal{P}_!(f) \overset{\text{def}}{=} f_!,$$

  as in Definition 2.4.6.1.1.

006Y 4. *Adjointness I.* We have an adjunction

$$\left(\mathcal{P}^{-1} \dashv \mathcal{P}^{-1,\mathsf{op}}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "\mathcal{P}^{-1}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}, \arrow[l, bend left, "\mathcal{P}^{-1,\mathsf{op}}"] \end{tikzcd}$$

witnessed by a bijection

$$\underbrace{\mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(A), B)}_{\overset{\text{def}}{=} \mathsf{Sets}(B, \mathcal{P}(A))} \cong \mathsf{Sets}(A, \mathcal{P}(B)),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathsf{Sets}^{\mathsf{op}})$.

006Z 5. *Adjointness II.* We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathrm{Rel})$, where Gr is the graph functor of Item 1 of Proposition 6.3.1.1.2 and $\mathcal{P}_*$ is the functor of Proposition 6.4.5.1.1.

Proof. Item 1, Functoriality I: This follows from Items 3 and 4 of Proposition 2.4.4.1.5.
Item 2, Functoriality II: This follows Items 3 and 4 of Proposition 2.4.5.1.4.
Item 3, Functoriality III: This follows Items 3 and 4 of Proposition 2.4.6.1.7.
Item 4, Adjointness I: We have

$$\begin{aligned} \mathsf{Sets}^{\mathsf{op}}(\mathcal{P}(A), B) &\overset{\text{def}}{=} \mathsf{Sets}(B, \mathcal{P}(A)) \\ &\cong \mathsf{Sets}(B, \mathsf{Sets}(A, \{\mathsf{t}, \mathsf{f}\})) \\ &\qquad\qquad \text{(by Item 1 of Proposition 2.4.3.1.6)} \\ &\cong \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) \\ &\qquad\qquad \text{(by Item 2 of Proposition 2.1.3.1.2)} \\ &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) \\ &\qquad\qquad \text{(by Item 2 of Proposition 2.1.3.1.2)} \\ &\cong \mathsf{Sets}(A, \mathcal{P}(B)) \quad \text{(by Item 1 of Proposition 2.4.3.1.6)} \end{aligned}$$

with all bijections natural in A and B (where we use Item 2 of Proposition 2.4.3.1.6 here).

Item 5, Adjointness II: We have

$$\begin{aligned}
\mathrm{Rel}(\mathrm{Gr}(A), B) &\cong \mathcal{P}(A \times B) \\
&\cong \mathsf{Sets}(A \times B, \{\mathsf{t}, \mathsf{f}\}) \\
&\qquad\qquad \text{(by Item 1 of Proposition 2.4.3.1.6)} \\
&\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{t}, \mathsf{f}\})) \\
&\qquad\qquad \text{(by Item 2 of Proposition 2.1.3.1.2)} \\
&\cong \mathsf{Sets}(A, \mathcal{P}(B)) \quad \text{(by Item 1 of Proposition 2.4.3.1.6)}
\end{aligned}$$

with all bijections natural in A (where we use Item 2 of Proposition 2.4.3.1.6 here). Explicitly, this isomorphism is given by sending a relation $R\colon \mathrm{Gr}(A) \mathrel{\ooalign{$\rightarrow$\cr\hidewidth$+$\hidewidth}} B$ to the map $R^{\dagger}\colon A \to \mathcal{P}(B)$ sending a to the subset $R(a)$ of B, as in Remark 5.1.1.1.4.

Naturality in B is then the statement that given a relation $R\colon B \mathrel{\ooalign{$\rightarrow$\cr\hidewidth$+$\hidewidth}} B'$, the diagram

$$\begin{tikzcd}
\mathrm{Rel}(\mathrm{Gr}(A), B) \arrow[r, "R\diamond -"] \arrow[d, dashed, "\wr"] & \mathrm{Rel}(\mathrm{Gr}(A), B') \arrow[d, dashed, "\wr"'] \\
\mathsf{Sets}(A, \mathcal{P}(B)) \arrow[r, "R_*"'] & \mathsf{Sets}(A, \mathcal{P}(B'))
\end{tikzcd}$$

commutes, which follows from Remark 6.4.1.1.2. □

0070 **Proposition 2.4.3.1.5.** Let X be a set.

0071 1. *Symmetric Strong Monoidality With Respect to Coproducts I.* The powerset functor $\mathcal{P}_*$ of Item 1 of Proposition 2.4.3.1.4 has a symmetric strong monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}_*^{\coprod}, \mathcal{P}_{*|\mathbb{1}}^{\coprod}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \textstyle\coprod, \emptyset)$$

being equipped with isomorphisms

$$\mathcal{P}_{*|X,Y}^{\coprod} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \textstyle\coprod Y),$$
$$\mathcal{P}_{*|\mathbb{1}}^{\coprod} \colon \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

0072 2. *Symmetric Strong Monoidality With Respect to Coproducts II.* The powerset functor $\mathcal{P}^{-1}$ of Item 2 of Proposition 2.4.3.1.4 has a symmetric strong monoidal structure

$$\left(\mathcal{P}^{-1}, \mathcal{P}^{-1|\coprod}, \mathcal{P}_{\mathbb{1}}^{-1|\coprod}\right) \colon (\mathsf{Sets}^{\mathsf{op}}, \times^{\mathsf{op}}, \mathrm{pt}) \to (\mathsf{Sets}, \textstyle\coprod, \emptyset)$$

being equipped with isomorphisms

$$\mathcal{P}^{-1|\coprod}_{X,Y} : \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}^{-1|\coprod}_{\mathbb{1}} : \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

0073 3. *Symmetric Strong Monoidality With Respect to Coproducts III.* The powerset functor $\mathcal{P}_!$ of Item 3 of Proposition 2.4.3.1.4 has a symmetric strong monoidal structure

$$\left(\mathcal{P}_!, \mathcal{P}^{\coprod}_!, \mathcal{P}^{\coprod}_{!|\mathbb{1}}\right) : (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \coprod, \emptyset)$$

being equipped with isomorphisms

$$\mathcal{P}^{\coprod}_{!|X,Y} : \mathcal{P}(X) \times \mathcal{P}(Y) \xrightarrow{\cong} \mathcal{P}(X \coprod Y),$$
$$\mathcal{P}^{\coprod}_{!|\mathbb{1}} : \mathrm{pt} \xrightarrow{\cong} \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

0074 4. *Symmetric Lax Monoidality With Respect to Products I.* The powerset functor $\mathcal{P}_*$ of Item 1 of Proposition 2.4.3.1.4 has a symmetric lax monoidal structure

$$\left(\mathcal{P}_*, \mathcal{P}^{\otimes}_*, \mathcal{P}^{\otimes}_{*|\mathbb{1}}\right) : (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with morphisms

$$\mathcal{P}^{\times}_{*|X,Y} : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}^{\times}_{*|\mathbb{1}} : \mathrm{pt} \to \mathcal{P}(\mathrm{pt}),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, where

- The map $\mathcal{P}^{\times}_{*|X,Y}$ is given by

$$\mathcal{P}^{\times}_{*|X,Y}(U, V) \stackrel{\text{def}}{=} U \times V$$

  for each $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$,
- The map $\mathcal{P}^{\times}_{*|\mathbb{1}}$ is given by

$$\mathcal{P}^{\times}_{*|\mathbb{1}}(\star) = \mathrm{pt}.$$

0075 5. *Symmetric Lax Monoidality With Respect to Products II.* The powerset functor $\mathcal{P}^{-1}$ of Item 2 of Proposition 2.4.3.1.4 has a symmetric lax monoidal structure

$$\left(\mathcal{P}^{-1}, \mathcal{P}^{-1|\otimes}, \mathcal{P}^{-1|\otimes}_{\mathbb{1}}\right) \colon (\mathsf{Sets}^{\mathsf{op}}, \times^{\mathsf{op}}, \mathrm{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with morphisms

$$\mathcal{P}^{-1|\times}_{X,Y} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}^{\times}_{\mathbb{1}} \colon \mathrm{pt} \to \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, defined as in Item 4.

0076 6. *Symmetric Lax Monoidality With Respect to Products III.* The powerset functor $\mathcal{P}_!$ of Item 3 of Proposition 2.4.3.1.4 has a symmetric lax monoidal structure

$$\left(\mathcal{P}_!, \mathcal{P}^{\otimes}_!, \mathcal{P}^{\otimes}_{!|\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt})$$

being equipped with morphisms

$$\mathcal{P}^{\times}_{!|X,Y} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$
$$\mathcal{P}^{\times}_{!|\mathbb{1}} \colon \mathrm{pt} \to \mathcal{P}(\emptyset),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, defined as in Item 4.

Proof. Item 1, Symmetric Strong Monoidality With Respect to Coproducts I: The isomorphism

$$\mathcal{P}^{\coprod}_{*|X,Y} \colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \textstyle\coprod Y)$$

is given by sending $(U, V) \in \mathcal{P}(X) \times \mathcal{P}(Y)$ to $U \coprod V$, with inverse given by sending a subset S of $X \coprod Y$ to the pair $(S_X, S_Y) \in \mathcal{P}(X) \times \mathcal{P}(Y)$ with

$$S_X \stackrel{\mathrm{def}}{=} \{x \in X \mid (0, x) \in S\}$$
$$S_Y \stackrel{\mathrm{def}}{=} \{y \in Y \mid (1, y) \in S\}.$$

The isomorphism $\mathrm{pt} \cong \mathcal{P}(\emptyset)$ is given by $\star \mapsto \emptyset \in \mathcal{P}(\emptyset)$.

Naturality for the isomorphism $\mathcal{P}^{\coprod}_{*|X,Y}$ is the statement that, given maps of sets $f \colon X \to X'$ and $g \colon Y \to Y'$, the diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(Y) \arrow[r, "f_* \times g_*"] \arrow[d, dashed, "\wr"] & \mathcal{P}(X') \times \mathcal{P}(Y') \arrow[d, dashed, "\wr"'] \\
\mathcal{P}(X \coprod Y) \arrow[r, "(f \coprod g)_*"'] & \mathcal{P}(X' \coprod Y')
\end{tikzcd}$$

commutes, which is clear, as it acts on elements as

$$\begin{tikzcd}
(U,V) \arrow[r, mapsto] \arrow[d, mapsto] & (f_*(U), g_*(V)) \arrow[d, mapsto] \\
U \coprod V \arrow[r, mapsto] & (f \coprod g)_*(U \coprod V) = f_*(U) \coprod g_*(V),
\end{tikzcd}$$

where we are using Item 7 of Proposition 2.4.4.1.4.

Finally, monoidality, unity, and symmetry of $\mathcal{P}_*$ as a monoidal functor follow by checking the commutativity of the relevant diagrams on elements.

Item 2, Symmetric Strong Monoidality With Respect to Coproducts II: The proof is similar to Item 1, and is hence omitted.

Item 3, Symmetric Strong Monoidality With Respect to Coproducts III: The proof is similar to Item 1, and is hence omitted.

Item 4, Symmetric Lax Monoidality With Respect to Products I: Naturality for the morphism $\mathcal{P}^{\times}_{*|X,Y}$ is the statement that, given maps of sets $f\colon X \to X'$ and $g\colon Y \to Y'$, the diagram

$$\begin{tikzcd}
\mathcal{P}(X) \times \mathcal{P}(Y) \arrow[r, "f_* \times g_*"] \arrow[d, dashed, "\wr"] & \mathcal{P}(X') \times \mathcal{P}(Y') \arrow[d, dashed, "\wr"'] \\
\mathcal{P}(X \times Y) \arrow[r, "(f \times g)_*"'] & \mathcal{P}(X' \times Y')
\end{tikzcd}$$

commutes, which is clear, as it acts on elements as

$$\begin{tikzcd}
(U,V) \arrow[r, mapsto] \arrow[d, mapsto] & (f_*(U), g_*(V)) \arrow[d, mapsto] \\
U \times V \arrow[r, mapsto] & (f \times g)_*(U \times V) = f_*(U) \times g_*(V),
\end{tikzcd}$$

where we are using Item 8 of Proposition 2.4.4.1.4.

Finally, monoidality, unity, and symmetry of $\mathcal{P}_*$ as a monoidal functor follow by checking the commutativity of the relevant diagrams on elements.

Item 5, Symmetric Lax Monoidality With Respect to Products II: The proof is similar to Item 4, and is hence omitted.

Item 6, Symmetric Lax Monoidality With Respect to Products III: The proof is similar to Item 4, and is hence omitted. □

0077 **Proposition 2.4.3.1.6.** Let X be a set.

0078 1. *Powersets as Sets of Functions I.* The assignment $U \mapsto \chi_U$ defines a bijection

$$\chi_{(-)}\colon \mathcal{P}(X) \xrightarrow{\cong} \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\}),$$

for each $X \in \mathrm{Obj}(\mathsf{Sets})$.

0079 2. *Powersets as Sets of Functions II.* The bijection

$$\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of Item 1 is natural in $X \in \mathrm{Obj}(\mathsf{Sets})$, refining to a natural isomorphism of functors

$$\mathcal{P}^{-1} \cong \mathsf{Sets}(-, \{\mathsf{t}, \mathsf{f}\}).$$

007A 3. *Powersets as Sets of Relations.* We have bijections

$$\begin{aligned}\mathcal{P}(X) &\cong \mathrm{Rel}(\mathrm{pt}, X),\\ \mathcal{P}(X) &\cong \mathrm{Rel}(X, \mathrm{pt}),\end{aligned}$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Powersets as Sets of Functions I*: Indeed, the inverse of $\chi_{(-)}$ is given by sending a function $f\colon X \to \{\mathsf{t}, \mathsf{f}\}$ to the subset U_f of $\mathcal{P}(X)$ defined by

$$U_f \overset{\text{def}}{=} \{x \in X \mid f(x) = \mathsf{true}\},$$

i.e. by $U_f = f^{-1}(\mathsf{true})$. That $\chi_{(-)}$ and $f \mapsto U_f$ are inverses is then straightforward to check.

Item 2, Powersets as Sets of Functions II: We need to check that, given a function $f\colon X \to Y$, the diagram

$$\begin{tikzcd}
\mathcal{P}(Y) \arrow[r, "f^{-1}"] \arrow[d, dashed, "\chi_{(-)}"', "\wr"] & \mathcal{P}(X) \arrow[d, dashed, "\chi_{(-)}", "\wr"'] \\
\mathsf{Sets}(Y, \{\mathsf{t}, \mathsf{f}\}) \arrow[r, "f^*"'] & \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})
\end{tikzcd}$$

commutes, i.e. that for each $V \in \mathcal{P}(Y)$, we have

$$\chi_V \circ f = \chi_{f^{-1}(V)}.$$

And indeed, we have

$$
\begin{aligned}
[\chi_V \circ f](v) &\overset{\text{def}}{=} \chi_V(f(v)) \\
&= \begin{cases} \mathsf{true} & \text{if } f(v) \in V, \\ \mathsf{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathsf{true} & \text{if } v \in f^{-1}(V), \\ \mathsf{false} & \text{otherwise} \end{cases} \\
&\overset{\text{def}}{=} \chi_{f^{-1}(V)}(v)
\end{aligned}
$$

for each $v \in V$.
Item 3, Powersets as Sets of Relations: Indeed, we have

$$
\begin{aligned}
\mathrm{Rel}(\mathrm{pt}, X) &\overset{\text{def}}{=} \mathcal{P}(\mathrm{pt} \times X) \\
&\cong \mathcal{P}(X)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Rel}(X, \mathrm{pt}) &\overset{\text{def}}{=} \mathcal{P}(X \times \mathrm{pt}) \\
&\cong \mathcal{P}(X),
\end{aligned}
$$

where we have used Item 4 of Proposition 2.1.3.1.2. □

007B **Remark 2.4.3.1.7.** The bijection

$$\mathcal{P}(X) \cong \mathsf{Sets}(X, \{\mathsf{t}, \mathsf{f}\})$$

of Item 1 of Proposition 2.4.3.1.6, which

- Takes a subset $U \hookrightarrow X$ of X and *straightens* it to a function $\chi_U \colon X \to \{\mathsf{true}, \mathsf{false}\}$;
- Takes a function $f \colon X \to \{\mathsf{true}, \mathsf{false}\}$ and *unstraightens* it to a subset $f^{-1}(\mathsf{true}) \hookrightarrow X$ of X;

may be viewed as the (-1)-categorical version of the un/straightening isomorphism for indexed and fibred sets

$$\underbrace{\mathsf{FibSets}(X)}_{\overset{\text{def}}{=} \mathsf{Sets}_{/X}} \cong \underbrace{\mathsf{ISets}(X)}_{\overset{\text{def}}{=} \mathsf{Fun}(X_{\mathsf{disc}}, \mathsf{Sets})}$$

of **??**, where we view:

- Subsets $U \hookrightarrow X$ as analogous to X-fibred sets $\phi_X \colon A \to X$.
- Functions $f \colon X \to \{\mathsf{t}, \mathsf{f}\}$ as analogous to X-indexed sets $A \colon X_{\mathsf{disc}} \to \mathsf{Sets}$.

007C **Proposition 2.4.3.1.8.** Let X be a set.

007D 1. *Universal Property.* The pair $\left(\mathcal{P}(X), \chi_{(-)}\right)$ consisting of

   - The powerset $\mathcal{P}(X)$ of X;
   - The characteristic embedding $\chi_{(-)}\colon X \hookrightarrow \mathcal{P}(X)$ of X into $\mathcal{P}(X)$;

   satisfies the following universal property:

   (★) Given another pair (Y, f) consisting of
   - A cocomplete poset $(Y, \preceq)$;
   - A function $f\colon X \to Y$;

   there exists a unique cocontinuous morphism of posets

$$(\mathcal{P}(X), \subset) \xrightarrow{\exists!} (Y, \preceq)$$

   making the diagram

$$\begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\exists!"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

   commute.

007E 2. *Adjointness.* We have an adjunction[34]

$$(\mathcal{P} \dashv 忘)\colon \quad \mathsf{Sets} \underset{忘}{\overset{\mathcal{P}}{\rightleftarrows}} \mathsf{Pos}^{\mathsf{cocomp.}},$$

   witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \cong \mathsf{Sets}(X, Y),$$

   natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, \preceq) \in \mathrm{Obj}(\mathsf{Pos}^{\mathsf{cocomp.}})$, where the maps witnessing this bijection are given by

   - The map

$$\chi_X^*\colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

[34] In this sense, $\mathcal{P}(A)$ is the free cocompletion of A. (Note that, despite its name, however, this is not an idempotent operation, as we have $\mathcal{P}(\mathcal{P}(A)) \neq \mathcal{P}(A)$.)

defined by

$$\chi_X^*(f) \stackrel{\text{def}}{=} f \circ \chi_X,$$

i.e. by sending a cocontinuous morphism of posets $f \colon \mathcal{P}(X) \to Y$ to the composition

$$X \overset{\chi_X}{\hookrightarrow} \mathcal{P}(X) \xrightarrow{f} Y.$$

- The map

  $$\mathrm{Lan}_{\chi_X} \colon \mathsf{Sets}(X,Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq))$$

  is given by sending a function $f \colon X \to Y$ to its left Kan extension along χ_X,

  $$\mathrm{Lan}_{\chi_X}(f) \colon \mathcal{P}(X) \to Y, \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\mathrm{Lan}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & Y. \end{tikzcd}$$

  Moreover, $\mathrm{Lan}_{\chi_X}(f)$ can be explicitly computed by

  $$\begin{aligned} [\mathrm{Lan}_{\chi_X}(f)](U) &\cong \int^{x \in X} \chi_{\mathcal{P}(X)}(\chi_x, U) \odot f(x) \\ &\cong \int^{x \in X} \chi_U(x) \odot f(x) \qquad \text{(by Proposition 2.4.2.1.1)} \\ &\cong \bigvee_{x \in X} (\chi_U(x) \odot f(x)) \end{aligned}$$

  for each $U \in \mathcal{P}(X)$, where:

  - $\bigvee$ is the join in $(Y,\preceq)$.
  - We have

    $$\begin{aligned} \mathsf{true} \odot f(x) &\stackrel{\text{def}}{=} f(x), \\ \mathsf{false} \odot f(x) &\stackrel{\text{def}}{=} \varnothing_Y, \end{aligned}$$

    where $\varnothing_Y$ is the minimal element of $(Y,\preceq)$.

Proof. *Item 1, Universal Property*: This is a rephrasing of Item 2.
Item 2, Adjointness: We claim we have adjunction $\mathcal{P} \dashv 忘$, witnessed by a bijection

$$\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X),\subset),(Y,\preceq)) \cong \mathsf{Sets}(X,Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y,\preceq) \in \mathrm{Obj}(\mathsf{Pos}^{\mathsf{cocomp.}})$.

- *Map I.* We define a map

$$\Phi_{X,Y} \colon \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \to \mathsf{Sets}(X, Y)$$

as in the statement, by

$$\Phi_{X,Y}(f) \overset{\text{def}}{=} f \circ \chi_X$$

for each $f \in \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$.

- *Map II.* We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))$$

as in the statement, by

$$\Psi_{X,Y}(f) \overset{\text{def}}{=} \operatorname{Lan}_{\chi_X}(f), \qquad \begin{tikzcd} & \mathcal{P}(X) \arrow[d, dashed, "\operatorname{Lan}_{\chi_X}(f)"] \\ X \arrow[ur, "\chi_X"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & Y, \end{tikzcd}$$

for each $f \in \mathsf{Sets}(X, Y)$.

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq))}.$$

Indeed, given a cocontinuous morphism of posets

$$\xi \colon (\mathcal{P}(X), \subset) \to (Y, \preceq),$$

we have

$$\begin{aligned} [\Psi_{X,Y} \circ \Phi_{X,Y}](\xi) &\overset{\text{def}}{=} \Psi_{X,Y}(\Phi_{X,Y}(\xi)) \\ &\overset{\text{def}}{=} \Psi_{X,Y}(\xi \circ \chi_X) \\ &\overset{\text{def}}{=} \operatorname{Lan}_{\chi_X}(\xi \circ \chi_X) \\ &\cong \bigvee_{x \in X} \chi_{(-)}(x) \odot \xi(\chi_X(x)) \\ &\overset{\text{clm}}{=} \xi, \end{aligned}$$

where indeed

$$\begin{aligned}\left[\bigvee_{x\in X}\chi_{(-)}(x)\odot\xi(\chi_X(x))\right](U) &\overset{\text{def}}{=} \bigvee_{x\in X}\chi_U(x)\odot\xi(\chi_X(x))\\ &= \left(\bigvee_{x\in U}\chi_U(x)\odot\xi(\chi_X(x))\right)\vee\left(\bigvee_{x\in X\setminus U}\chi_U(x)\odot\xi(\chi_X(x))\right)\\ &= \left(\bigvee_{x\in U}\xi(\chi_X(x))\right)\vee\left(\bigvee_{x\in X\setminus U}\varnothing_Y\right)\\ &= \bigvee_{x\in U}\xi(\chi_X(x))\\ &\overset{(\dagger)}{=} \xi\left(\bigvee_{x\in U}\chi_X(x)\right)\\ &= \xi(U)\end{aligned}$$

for each $U\in\mathcal{P}(X)$, where we have used that ξ is cocontinuous for the equality $\overset{(\dagger)}{=}$.

- *Invertibility II.* We claim that

$$\Phi_{X,Y}\circ\Psi_{X,Y}=\mathrm{id}_{\mathsf{Sets}(X,Y)}.$$

Indeed, given a function $f\colon X\to Y$, we have

$$\begin{aligned}[\Phi_{X,Y}\circ\Psi_{X,Y}](f) &\overset{\text{def}}{=} \Phi_{X,Y}(\Psi_{X,Y}(f))\\ &\overset{\text{def}}{=} \Phi_{X,Y}(\mathrm{Lan}_{\chi_X}(f))\\ &\overset{\text{def}}{=} \mathrm{Lan}_{\chi_X}(f)\circ\chi_X\\ &\overset{\text{clm}}{=} f,\end{aligned}$$

where indeed

$$\begin{aligned}[\mathrm{Lan}_{\chi_X}(f)\circ\chi_X](x) &\overset{\text{def}}{=} \bigvee_{y\in X}\chi_{\{x\}}(y)\odot f(y)\\ &= \Big(\chi_{\{x\}}(x)\odot f(x)\Big)\vee\left(\bigvee_{y\in X\setminus\{x\}}\chi_{\{x\}}(y)\odot f(y)\right)\\ &= f(x)\vee\left(\bigvee_{y\in X\setminus\{x\}}\varnothing_Y\right)\\ &= f(x)\vee\varnothing_Y\\ &= f(x)\end{aligned}$$

for each $x\in X$.

- *Naturality for Φ, Part I.* We need to show that, given a function $f\colon X \to X'$, the diagram

$$\begin{tikzcd}
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X'), \subset), (Y, \preceq)) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "\mathcal{P}_*(f)^*"'] & \mathsf{Sets}(X', Y) \arrow[d, "f^*"] \\
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X, Y)
\end{tikzcd}$$

  commutes. Indeed, given a cocontinuous morphism of posets

$$\xi\colon (\mathcal{P}(X'), \subset) \to (Y, \preceq),$$

  we have

$$\begin{aligned}
[\Phi_{X,Y} \circ \mathcal{P}_*(f)^*](\xi) &\stackrel{\text{def}}{=} \Phi_{X,Y}(\mathcal{P}_*(f)^*(\xi)) \\
&\stackrel{\text{def}}{=} \Phi_{X,Y}(\xi \circ f_*) \\
&\stackrel{\text{def}}{=} (\xi \circ f_*) \circ \chi_X \\
&= \xi \circ (f_* \circ \chi_X) \\
&\stackrel{(\dagger)}{=} \xi \circ (\chi_{X'} \circ f) \\
&= (\xi \circ \chi_{X'}) \circ f \\
&\stackrel{\text{def}}{=} \Phi_{X',Y}(\xi) \circ f \\
&\stackrel{\text{def}}{=} f^*(\Phi_{X',Y}(\xi)) \\
&\stackrel{\text{def}}{=} [f^* \circ \Phi_{X',Y}](\xi),
\end{aligned}$$

  where we have used Item 9 of Proposition 2.4.1.1.3 for the equality $\stackrel{(\dagger)}{=}$ above.

- *Naturality for Φ, Part II.* We need to show that, given a cocontinuous morphism of posets

$$g\colon (Y, \preceq_Y) \to (Y', \preceq_{Y'}),$$

  the diagram

$$\begin{tikzcd}
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y, \preceq)) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}(X, Y) \arrow[d, "g_*"] \\
\mathsf{Pos}^{\mathsf{cocomp.}}((\mathcal{P}(X), \subset), (Y', \preceq)) \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X, Y')
\end{tikzcd}$$

  commutes. Indeed, given a cocontinuous morphism of posets

$$\xi\colon (\mathcal{P}(X), \subset) \to (Y, \preceq),$$

we have

$$\begin{aligned}[\Phi_{X,Y'} \circ g_*](\xi) &\overset{\text{def}}{=} \Phi_{X,Y'}(g_*(\xi))\\ &\overset{\text{def}}{=} \Phi_{X,Y'}(g \circ \xi)\\ &\overset{\text{def}}{=} (g \circ \xi) \circ \chi_X\\ &= g \circ (\xi \circ \chi_X)\\ &\overset{\text{def}}{=} g \circ (\Phi_{X,Y}(\xi))\\ &\overset{\text{def}}{=} g_*(\Phi_{X,Y}(\xi))\\ &\overset{\text{def}}{=} [g_* \circ \Phi_{X,Y}](\xi).\end{aligned}$$

- *Naturality for Ψ.* Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Item 2 of Proposition 8.8.6.1.2 that Ψ is also natural in each argument.

This finishes the proof. □

007F 2.4.4 Direct Images

Let A and B be sets and let $f : A \to B$ be a function.

007G **Definition 2.4.4.1.1.** The **direct image function associated to** f is the function

$$f_* : \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[35,36]

$$\begin{aligned} f_*(U) &\overset{\text{def}}{=} f(U)\\ &\overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l}\text{there exists some } a \in U\\ \text{such that } b = f(a)\end{array} \right\}\\ &= \{f(a) \in B \mid a \in U\}\end{aligned}$$

for each $U \in \mathcal{P}(A)$.

007H **Notation 2.4.4.1.2.** Sometimes one finds the notation

$$\exists_f : \mathcal{P}(A) \to \mathcal{P}(B)$$

for f_*. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \exists_f(U)$.

[35] *Further Terminology:* The set $f(U)$ is called the **direct image of U by f**.

[36] We also have

$$f_*(U) = B \setminus f_!(A \setminus U);$$

- There exists some $a \in U$ such that $f(a) = b$.

007J **Remark 2.4.4.1.3.** Identifying subsets of A with functions from A to $\{\mathsf{true}, \mathsf{false}\}$ via Items 1 and 2 of Proposition 2.4.3.1.6, we see that the direct image function associated to f is equivalently the function

$$f_* \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by

$$\begin{aligned}
f_*(\chi_U) &\overset{\text{def}}{=} \operatorname{Lan}_f(\chi_U) \\
&= \operatorname{colim}\left(\left(f \overset{\rightarrow}{\times} \underline{(-_1)}\right) \overset{\text{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{t}, \mathsf{f}\}\right) \\
&= \operatorname*{colim}_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\
&= \bigvee_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)),
\end{aligned}$$

where we have used **??** for the second equality. In other words, we have

$$\begin{aligned}
[f_*(\chi_U)](b) &= \bigvee_{\substack{a \in A \\ f(a) = b}} (\chi_U(a)) \\
&= \begin{cases} \mathsf{true} & \text{if there exists some } a \in A \text{ such} \\ & \text{that } f(a) = b \text{ and } a \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathsf{true} & \text{if there exists some } a \in U \\ & \text{such that } f(a) = b, \\ \mathsf{false} & \text{otherwise} \end{cases}
\end{aligned}$$

for each $b \in B$.

007K **Proposition 2.4.4.1.4.** Let $f \colon A \to B$ be a function.

007L 1. *Functoriality.* The assignment $U \mapsto f_*(U)$ defines a functor

$$f_* \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_*](U) \overset{\text{def}}{=} f_*(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  ($\star$) If $U \subset V$, then $f_*(U) \subset f_*(V)$.

007M 2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon\quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "f_*"] \arrow[r, bend right, "f_!"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{P}(B) \arrow[l, "f^{-1}"'] \end{tikzcd},$$

witnessed by bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right),$$
$$\mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
   i. We have $f_*(U) \subset V$.
   ii. We have $U \subset f^{-1}(V)$.

(b) The following conditions are equivalent:
   i. We have $f^{-1}(U) \subset V$.
   ii. We have $U \subset f_!(V)$.

007N 3. *Preservation of Colimits.* We have an equality of sets

$$f_*\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f_*(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$f_*(U) \cup f_*(V) = f_*(U \cup V),$$
$$f_*(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

007P 4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$f_*\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} f_*(U_i),$$

see Item 9 of Proposition 2.4.4.1.4.

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$f_*(U \cap V) \subset f_*(U) \cap f_*(V),$$
$$f_*(A) \subset B,$$

natural in $U, V \in \mathcal{P}(A)$.

007Q 5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$f_{*|U,V}^{\otimes} : f_*(U) \cup f_*(V) \xrightarrow{=} f_*(U \cup V),$$
$$f_{*|\mathbb{1}}^{\otimes} : \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

007R 6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(f_*, f_*^{\otimes}, f_{*|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$f_{*|U,V}^{\otimes} : f_*(U \cap V) \hookrightarrow f_*(U) \cap f_*(V),$$
$$f_{*|\mathbb{1}}^{\otimes} : f_*(A) \hookrightarrow B,$$

natural in $U, V \in \mathcal{P}(A)$.

007S 7. *Interaction With Coproducts.* Let $f : A \to A'$ and $g : B \to B'$ be maps of sets. We have

$$(f \textstyle\coprod g)_*(U \textstyle\coprod V) = f_*(U) \textstyle\coprod g_*(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

007T 8. *Interaction With Products.* Let $f : A \to A'$ and $g : B \to B'$ be maps of sets. We have

$$(f \times g)_*(U \times V) = f_*(U) \times g_*(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

007U 9. *Relation to Direct Images With Compact Support.* We have

$$f_*(U) = B \setminus f_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Triple Adjointness: This follows from Remark 2.4.4.1.3, Remark 2.4.5.1.2, Remark 2.4.6.1.3, and **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and **??** of **??**.[37]
Item 4, Oplax Preservation of Limits: The inclusion $f_*(A) \subset B$ is clear. See [Pro24s] for the other inclusions.
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Interaction With Coproducts: Clear.
Item 8, Interaction With Products: Clear.
Item 9, Relation to Direct Images With Compact Support: Applying Item 9 of Proposition 2.4.6.1.6 to $A \setminus U$, we have

$$\begin{aligned} f_!(A \setminus U) &= B \setminus f_*(A \setminus (A \setminus U)) \\ &= B \setminus f_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} f_*(U) &= B \setminus (B \setminus f_*(U)), \\ &= B \setminus f_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

007V **Proposition 2.4.4.1.5.** Let $f \colon A \to B$ be a function.

007W 1. *Functionality I.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

007X 2. *Functionality II.* The assignment $f \mapsto f_*$ defines a function

$$(-)_{*|A,B} \colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

007Y 3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_* = \mathrm{id}_{\mathcal{P}(A)}.$$

[37] See also [Pro24t].

007Z 4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathcal{P}(B) \arrow[d, "g_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: This follows from Remark 2.4.4.1.3 and **??** of **??**.
Item 4, Interaction With Composition: This follows from Remark 2.4.4.1.3 and **??** of **??**. □

0080 2.4.5 Inverse Images

Let A and B be sets and let $f\colon A \to B$ be a function.

0081 **Definition 2.4.5.1.1.** The **inverse image function associated to** f is the function[38]

$$f^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[39]

$$f^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid \text{we have } f(a) \in V\}$$

for each $V \in \mathcal{P}(B)$.

0082 **Remark 2.4.5.1.2.** Identifying subsets of B with functions from B to $\{\mathsf{true}, \mathsf{false}\}$ via Items 1 and 2 of Proposition 2.4.3.1.6, we see that the inverse image function associated to f is equivalently the function

$$f^*\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by

$$f^*(\chi_V) \stackrel{\text{def}}{=} \chi_V \circ f$$

for each $\chi_V \in \mathcal{P}(B)$, where $\chi_V \circ f$ is the composition

$$A \xrightarrow{f} B \xrightarrow{\chi_V} \{\mathsf{true}, \mathsf{false}\}$$

in Sets.

[38] *Further Notation:* Also written $f^*\colon \mathcal{P}(B) \to \mathcal{P}(A)$.
[39] *Further Terminology:* The set $f^{-1}(V)$ is called the **inverse image of** V **by** f.

0083 **Proposition 2.4.5.1.3.** Let $f\colon A \to B$ be a function.

0084 1. *Functoriality.* The assignment $V \mapsto f^{-1}(V)$ defines a functor

$$f^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have
$$\left[f^{-1}\right](V) \stackrel{\text{def}}{=} f^{-1}(V).$$
- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  ($\star$) If $U \subset V$, then $f^{-1}(U) \subset f^{-1}(V)$.

0085 2. *Triple Adjointness.* We have a triple adjunction

$$\left(f_* \dashv f^{-1} \dashv f_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend right=50, "f_!"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{P}(B), \arrow[l, "f^{-1}" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}\left(U, f^{-1}(V)\right),$$
$$\mathrm{Hom}_{\mathcal{P}(A)}\left(f^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:
  i. We have $f_*(U) \subset V$;
  ii. We have $U \subset f^{-1}(V)$;

(b) The following conditions are equivalent:
  i. We have $f^{-1}(U) \subset V$.
  ii. We have $U \subset f_!(V)$.

0086 3. *Preservation of Colimits.* We have an equality of sets

$$f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V),$$
$$f^{-1}(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(B)$.

0087 4. *Preservation of Limits.* We have an equality of sets

$$f^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f^{-1}(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(V) &= f^{-1}(U \cap V), \\ f^{-1}(B) &= A, \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

0088 5. *Symmetric Strict Monoidality With Respect to Unions.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f^{-1,\otimes}_{\mathbb{1}}\right) : (\mathcal{P}(B), \cup, \emptyset) \to (\mathcal{P}(A), \cup, \emptyset),$$

being equipped with equalities

$$\begin{aligned} f^{-1,\otimes}_{U,V} &: f^{-1}(U) \cup f^{-1}(V) \xrightarrow{=} f^{-1}(U \cup V), \\ f^{-1,\otimes}_{\mathbb{1}} &: \emptyset \xrightarrow{=} f^{-1}(\emptyset), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

0089 6. *Symmetric Strict Monoidality With Respect to Intersections.* The inverse image function of Item 1 has a symmetric strict monoidal structure

$$\left(f^{-1}, f^{-1,\otimes}, f^{-1,\otimes}_{\mathbb{1}}\right) : (\mathcal{P}(B), \cap, B) \to (\mathcal{P}(A), \cap, A),$$

being equipped with equalities

$$\begin{aligned} f^{-1,\otimes}_{U,V} &: f^{-1}(U) \cap f^{-1}(V) \xrightarrow{=} f^{-1}(U \cap V), \\ f^{-1,\otimes}_{\mathbb{1}} &: A \xrightarrow{=} f^{-1}(B), \end{aligned}$$

natural in $U, V \in \mathcal{P}(B)$.

008A 7. *Interaction With Coproducts.* Let $f : A \to A'$ and $g : B \to B'$ be maps of sets. We have

$$(f \textstyle\coprod g)^{-1}(U' \textstyle\coprod V') = f^{-1}(U') \textstyle\coprod g^{-1}(V')$$

for each $U' \in \mathcal{P}(A')$ and each $V' \in \mathcal{P}(B')$.

008B 8. *Interaction With Products.* Let $f\colon A \to A'$ and $g\colon B \to B'$ be maps of sets. We have

$$(f \times g)^{-1}(U' \times V') = f^{-1}(U') \times g^{-1}(V')$$

for each $U' \in \mathcal{P}(A')$ and each $V' \in \mathcal{P}(B')$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Triple Adjointness: This follows from Remark 2.4.4.1.3, Remark 2.4.5.1.2, Remark 2.4.6.1.3, and **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and **??** of **??**.[40]
Item 4, Preservation of Limits: This follows from Item 2 and **??** of **??**.[41]
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Interaction With Coproducts: Clear.
Item 8, Interaction With Products: Clear. □

008C **Proposition 2.4.5.1.4.** Let $f\colon A \to B$ be a function.

008D 1. *Functionality I.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(B), \mathcal{P}(A)).$$

008E 2. *Functionality II.* The assignment $f \mapsto f^{-1}$ defines a function

$$(-)^{-1}_{A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(B), \subset), (\mathcal{P}(A), \subset)).$$

008F 3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{id}_A^{-1} = \mathrm{id}_{\mathcal{P}(A)}.$$

008G 4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "g^{-1}"] \arrow[dr, "(g\circ f)^{-1}"'] & \mathcal{P}(B) \arrow[d, "f^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

[40] See also [Pro24ac].
[41] See also [Pro24ab].

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: This follows from Remark 2.4.5.1.2 and Item 5 of Proposition 8.1.6.1.2.
Item 4, Interaction With Composition: This follows from Remark 2.4.5.1.2 and Item 2 of Proposition 8.1.6.1.2. □

008H 2.4.6 Direct Images With Compact Support

Let A and B be sets and let $f\colon A \to B$ be a function.

008J **Definition 2.4.6.1.1.** The **direct image with compact support function associated to** f is the function

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[42,43]

$$\begin{aligned} f_!(U) &\overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A\text{, if we have} \\ f(a) = b\text{, then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; \text{we have } f^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

008K **Notation 2.4.6.1.2.** Sometimes one finds the notation

$$\forall_f \colon \mathcal{P}(A) \to \mathcal{P}(B)$$

for f_*. This notation comes from the fact that the following statements are equivalent, where $b \in B$ and $U \in \mathcal{P}(A)$:

- We have $b \in \forall_f(U)$.
- For each $a \in A$, if $b = f(a)$, then $a \in U$.

008L **Remark 2.4.6.1.3.** Identifying subsets of A with functions from A to $\{\mathsf{true}, \mathsf{false}\}$ via Items 1 and 2 of Proposition 2.4.3.1.6, we see that the direct image with compact support function associated to f is equivalently the function

$$f_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

[42] *Further Terminology:* The set $f_!(U)$ is called the **direct image with compact support of U by f**.
[43] We also have

$$f_!(U) = B \setminus f_*(A \setminus U);$$

see Item 9 of Proposition 2.4.6.1.6.

defined by

$$
\begin{aligned}
f_!(\chi_U) &\stackrel{\text{def}}{=} \mathrm{Ran}_f(\chi_U) \\
&= \lim\left(\left(\underline{(-_1)} \overset{\rightarrow}{\times} f\right) \overset{\mathrm{pr}}{\twoheadrightarrow} A \xrightarrow{\chi_U} \{\mathsf{true}, \mathsf{false}\}\right) \\
&= \lim_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)) \\
&= \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)).
\end{aligned}
$$

where we have used **??** for the second equality. In other words, we have

$$
\begin{aligned}
[f_!(\chi_U)](b) &= \bigwedge_{\substack{a \in A \\ f(a) = b}} (\chi_U(a)) \\
&= \begin{cases} \mathsf{true} & \text{if, for each } a \in A \text{ such that} \\ & f(a) = b, \text{ we have } a \in U, \\ \mathsf{false} & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathsf{true} & \text{if } f^{-1}(b) \subset U \\ \mathsf{false} & \text{otherwise} \end{cases}
\end{aligned}
$$

for each $b \in B$.

008M **Definition 2.4.6.1.4.** Let U be a subset of A.[44,45]

1. The **image part of the direct image with compact support**

[44] Note that we have

$$f_!(U) = f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U),$$

as

$$
\begin{aligned}
f_!(U) &= f_!(U) \cap B \\
&= f_!(U) \cap (\mathrm{Im}(f) \cup (B \setminus \mathrm{Im}(f))) \\
&= (f_!(U) \cap \mathrm{Im}(f)) \cup (f_!(U) \cap (B \setminus \mathrm{Im}(f))) \\
&\stackrel{\text{def}}{=} f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U).
\end{aligned}
$$

[45] In terms of the meet computation of $f_!(U)$ of Remark 2.4.6.1.3, namely

$$f_!(\chi_U) = \bigwedge_{\substack{a \in A \\ f(a) = -_1}} (\chi_U(a)),$$

we see that $f_{!,\mathrm{im}}$ corresponds to meets indexed over nonempty sets, while $f_{!,\mathrm{cp}}$ corresponds to meets indexed over the empty set.

008N $f_!(U)$ **of** U is the set $f_{!,\mathrm{im}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\overset{\text{def}}{=} f_!(U) \cap \mathrm{Im}(f) \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{we have } f^{-1}(b) \subset \\ U \text{ and } f^{-1}(b) \neq \emptyset \end{array} \right\}. \end{aligned}$$

2. The **complement part of the direct image with compact**
008P **support** $f_!(U)$ **of** U is the set $f_{!,\mathrm{cp}}(U)$ defined by

$$\begin{aligned} f_{!,\mathrm{cp}}(U) &\overset{\text{def}}{=} f_!(U) \cap (B \setminus \mathrm{Im}(f)) \\ &= B \setminus \mathrm{Im}(f) \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{we have } f^{-1}(b) \subset \\ U \text{ and } f^{-1}(b) = \emptyset \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; f^{-1}(b) = \emptyset \right\}. \end{aligned}$$

008Q **Example 2.4.6.1.5.** Here are some examples of direct images with compact support.

1. *The Multiplication by Two Map on the Natural Numbers.* Consider the function $f\colon \mathbb{N} \to \mathbb{N}$ given by

$$f(n) \overset{\text{def}}{=} 2n$$

for each $n \in \mathbb{N}$. Since f is injective, we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U) \\ f_{!,\mathrm{cp}}(U) &= \{\text{odd natural numbers}\} \end{aligned}$$

for any $U \subset \mathbb{N}$.

2. *Parabolas.* Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \overset{\text{def}}{=} x^2$$

for each $x \in \mathbb{R}$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}$. Moreover, since $f^{-1}(x) = \{-\sqrt{x}, \sqrt{x}\}$, we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([0,1]) &= \{0\}, \\ f_{!,\mathrm{im}}([-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}([1,2]) &= \emptyset, \\ f_{!,\mathrm{im}}([-2,-1] \cup [1,2]) &= [1,4]. \end{aligned}$$

3. *Circles.* Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \overset{\text{def}}{=} x^2 + y^2$$

for each $(x,y) \in \mathbb{R}^2$. We have

$$f_{!,\mathrm{cp}}(U) = \mathbb{R}_{<0}$$

for any $U \subset \mathbb{R}^2$, and since

$$f^{-1}(r) = \begin{cases} \text{a circle of radius } r \text{ about the origin} & \text{if } r > 0, \\ \{(0,0)\} & \text{if } r = 0, \\ \emptyset & \text{if } r < 0, \end{cases}$$

we have e.g.:

$$\begin{aligned} f_{!,\mathrm{im}}([-1,1] \times [-1,1]) &= [0,1], \\ f_{!,\mathrm{im}}(([-1,1] \times [-1,1]) \setminus [-1,1] \times \{0\}) &= \emptyset. \end{aligned}$$

008R **Proposition 2.4.6.1.6.** Let $f\colon A \to B$ be a function.

008S 1. *Functoriality.* The assignment $U \mapsto f_!(U)$ defines a functor

$$f_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[f_!](U) \overset{\text{def}}{=} f_!(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  ($\star$) If $U \subset V$, then $f_!(U) \subset f_!(V)$.

008T 2. *Triple Adjointness.* We have a triple adjunction

$$\Big(f_* \dashv f^{-1} \dashv f_!\Big)\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left=50, "f_*"] \arrow[r, bend right=50, "f_!"'] & \mathcal{P}(B), \arrow[l, "f^{-1}"'] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}(B)}(f_*(U), V) &\cong \mathrm{Hom}_{\mathcal{P}(A)}\Big(U, f^{-1}(V)\Big), \\ \mathrm{Hom}_{\mathcal{P}(A)}\Big(f^{-1}(U), V\Big) &\cong \mathrm{Hom}_{\mathcal{P}(A)}(U, f_!(V)), \end{aligned}$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$ and (respectively) $V \in \mathcal{P}(A)$ and $U \in \mathcal{P}(B)$, i.e. where:

(a) The following conditions are equivalent:

i. We have $f_*(U) \subset V$.

ii. We have $U \subset f^{-1}(V)$.

(b) The following conditions are equivalent:

i. We have $f^{-1}(U) \subset V$.

ii. We have $U \subset f_!(V)$.

008U 3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i\in I} f_!(U_i) \subset f_!\left(\bigcup_{i\in I} U_i\right),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V),$$
$$\emptyset \hookrightarrow f_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

008V 4. *Preservation of Limits.* We have an equality of sets

$$f_!\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f_!(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$f^{-1}(U \cap V) = f_!(U) \cap f^{-1}(V),$$
$$f_!(A) = B,$$

natural in $U, V \in \mathcal{P}(A)$.

008W 5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$f_{!|U,V}^{\otimes} \colon f_!(U) \cup f_!(V) \hookrightarrow f_!(U \cup V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon \emptyset \hookrightarrow f_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

008X 6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(f_!, f_!^{\otimes}, f_{!|\mathbb{1}}^{\otimes}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$f_{!|U,V}^{\otimes} \colon f_!(U \cap V) \xrightarrow{=} f_!(U) \cap f_!(V),$$
$$f_{!|\mathbb{1}}^{\otimes} \colon f_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

008Y 7. *Interaction With Coproducts.* Let $f \colon A \to A'$ and $g \colon B \to B'$ be maps of sets. We have

$$\left(f \coprod g\right)_!\left(U \coprod V\right) = f_!(U) \coprod g_!(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

008Z 8. *Interaction With Products.* Let $f \colon A \to A'$ and $g \colon B \to B'$ be maps of sets. We have

$$(f \times g)_!(U \times V) = f_!(U) \times g_!(V)$$

for each $U \in \mathcal{P}(A)$ and each $V \in \mathcal{P}(B)$.

0090 9. *Relation to Direct Images.* We have

$$f_!(U) = B \setminus f_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

0091 10. *Interaction With Injections.* If f is injective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &= f_*(U), \\ f_{!,\mathrm{cp}}(U) &= B \setminus \mathrm{Im}(f), \\ f_!(U) &= f_{!,\mathrm{im}}(U) \cup f_{!,\mathrm{cp}}(U) \\ &= f_*(U) \cup (B \setminus \mathrm{Im}(f)) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

0092 11. *Interaction With Surjections.* If f is surjective, then we have

$$\begin{aligned} f_{!,\mathrm{im}}(U) &\subset f_*(U), \\ f_{!,\mathrm{cp}}(U) &= \emptyset, \\ f_!(U) &\subset f_*(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Triple Adjointness: This follows from Remark 2.4.4.1.3, Remark 2.4.5.1.2, Remark 2.4.6.1.3, and **??** of **??**.
Item 3, Lax Preservation of Colimits: Omitted.
Item 4, Preservation of Limits: This follows from Item 2 and **??** of **??**.
Item 5, Symmetric Lax Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Interaction With Coproducts: Clear.
Item 8, Interaction With Products: Clear.
Item 9, Relation to Direct Images: We claim that $f_!(U) = B \setminus f_*(A \setminus U)$.

- *The First Implication.* We claim that
$$f_!(U) \subset B \setminus f_*(A \setminus U).$$
Let $b \in f_!(U)$. We need to show that $b \notin f_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $f(a) = b$.
This is indeed the case, as otherwise we would have $a \in f^{-1}(b)$ and $a \notin U$, contradicting $f^{-1}(b) \subset U$ (which holds since $b \in f_!(U)$).
Thus $b \in B \setminus f_*(A \setminus U)$.
- *The Second Implication.* We claim that
$$B \setminus f_*(A \setminus U) \subset f_!(U).$$
Let $b \in B \setminus f_*(A \setminus U)$. We need to show that $b \in f_!(U)$, i.e. that $f^{-1}(b) \subset U$.
Since $b \notin f_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b = f(a)$, and hence $f^{-1}(b) \subset U$.
Thus $b \in f_!(U)$.

This finishes the proof of Item 9.
Item 10, Interaction With Injections: Clear.
Item 11, Interaction With Surjections: Clear. □

0093 **Proposition 2.4.6.1.7.** Let $f\colon A \to B$ be a function.

0094 1. *Functionality I.* The assignment $f \mapsto f_!$ defines a function
$$(-)_{!|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

0095 2. *Functionality II.* The assignment $f \mapsto f_!$ defines a function
$$(-)_{!|A,B}\colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

0096 3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)}.$$

0097 4. *Interaction With Composition.* For each pair of composable functions $f\colon A \to B$ and $g\colon B \to C$, we have

$$(g \circ f)_! = g_! \circ f_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "f_!"] \arrow[dr, "(g\circ f)_!"'] & \mathcal{P}(B) \arrow[d, "g_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1*, *Functionality I*: Clear.
Item 2, *Functionality II*: Clear.
Item 3, *Interaction With Identities*: This follows from Remark 2.4.6.1.3 and **??** of **??**.
Item 4, *Interaction With Composition*: This follows from Remark 2.4.6.1.3 and **??** of **??**. □

Appendices

2.A Other Chapters

Chapter 3

Pointed Sets

0098 This chapter contains some foundational material on pointed sets.

Contents

0099 3.1 Pointed Sets

009A 3.1.1 Foundations

009B **Definition 3.1.1.1.1.** A **pointed set**[1] is equivalently:

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A pointed object in $(\mathsf{Sets}, \mathrm{pt})$.

009C **Remark 3.1.1.1.2.** In detail, a **pointed set** is a pair (X, x_0) consisting of:

- *The Underlying Set.* A set X, called the **underlying set of** (X, x_0).
- *The Basepoint.* A morphism

$$[x_0] : \mathrm{pt} \to X$$

  in Sets, determining an element $x_0 \in X$, called the **basepoint of** X.

009D **Example 3.1.1.1.3.** The 0**-sphere**[2] is the pointed set $(S^0, 0)$[3] consisting of:

- *The Underlying Set.* The set S^0 defined by

$$S^0 \stackrel{\text{def}}{=} \{0, 1\}.$$

- *The Basepoint.* The element 0 of S^0.

009E **Example 3.1.1.1.4.** The **trivial pointed set** is the pointed set $(\mathrm{pt}, \star)$ consisting of:

- *The Underlying Set.* The punctual set $\mathrm{pt} \stackrel{\text{def}}{=} \{\star\}$.
- *The Basepoint.* The element $\star$ of pt.

009F **Example 3.1.1.1.5.** The **underlying pointed set** of a semimodule (M, α_M) is the pointed set $(M, 0_M)$.

009G **Example 3.1.1.1.6.** The **underlying pointed set** of a module (M, α_M) is the pointed set $(M, 0_M)$.

[1] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, pointed sets are viewed as $\mathbb{F}_1$**-modules**.

[2] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the 0-sphere is viewed as the **underlying pointed set of the field with one element**.

[3] *Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, S^0 is also denoted $(\mathbb{F}_1, 0)$.

009H 3.1.2 Morphisms of Pointed Sets

009J **Definition 3.1.2.1.1.** A **morphism of pointed sets**[4,5] is equivalently:

- A morphism of $\mathbb{E}_0$-monoids in $(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$.
- A morphism of pointed objects in $(\mathsf{Sets}, \mathrm{pt})$.

009K **Remark 3.1.2.1.2.** In detail, a **morphism of pointed sets** $f\colon (X, x_0) \to (Y, y_0)$ is a morphism of sets $f\colon X \to Y$ such that the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[dr, "{[y_0]}"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

commutes, i.e. such that

$$f(x_0) = y_0.$$

009L 3.1.3 The Category of Pointed Sets

009M **Definition 3.1.3.1.1.** The **category of pointed sets** is the category Sets_* defined equivalently as

- The homotopy category of the ∞-category $\mathsf{Mon}_{\mathbb{E}_0}(\mathrm{N}_\bullet(\mathsf{Sets}), \mathrm{pt})$ of **??**;
- The category Sets_* of **??**.

009N **Remark 3.1.3.1.2.** In detail, the **category of pointed sets** is the category Sets_* where

- *Objects.* The objects of Sets_* are pointed sets;
- *Morphisms.* The morphisms of Sets_* are morphisms of pointed sets;
- *Identities.* For each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the unit map

$$\mathbb{1}^{\mathsf{Sets}_*}_{(X,x_0)}\colon \mathrm{pt} \to \mathsf{Sets}_*((X, x_0), (X, x_0))$$

of Sets_* at (X, x_0) is defined by[6]

$$\mathrm{id}^{\mathsf{Sets}_*}_{(X,x_0)} \overset{\text{def}}{=} \mathrm{id}_X;$$

[4] *Further Terminology:* Also called a **pointed function**.

[5] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, morphisms of pointed sets are also called **morphism of $\mathbb{F}_1$-modules**.

[6] Note that id_X is indeed a morphism of pointed sets, as we have $\mathrm{id}_X(x_0) = x_0$.

- *Composition.* For each $(X,x_0),(Y,y_0),(Z,z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the composition map

$$\circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} \colon \mathsf{Sets}_*((Y,y_0),(Z,z_0)) \times \mathsf{Sets}_*((X,x_0),(Y,y_0)) \to \mathsf{Sets}_*((X,x_0),(Z,z_0))$$

  of Sets_* at $((X,x_0),(Y,y_0),(Z,z_0))$ is defined by[7]

$$g \circ^{\mathsf{Sets}_*}_{(X,x_0),(Y,y_0),(Z,z_0)} f \stackrel{\text{def}}{=} g \circ f.$$

009P 3.1.4 Elementary Properties of Pointed Sets

009Q **Proposition 3.1.4.1.1.** Let (X,x_0) be a pointed set.

009R 1. *Completeness.* The category Sets_* of pointed sets and morphisms between them is complete, having in particular:

009S (a) Products, described as in Definition 3.2.3.1.1;

009T (b) Pullbacks, described as in Definition 3.2.4.1.1;

009U (c) Equalisers, described as in Definition 3.2.5.1.1.

009V 2. *Cocompleteness.* The category Sets_* of pointed sets and morphisms between them is cocomplete, having in particular:

009W (a) Coproducts, described as in Definition 3.3.3.1.1;

009X (b) Pushouts, described as in Definition 3.3.4.1.1;

009Y (c) Coequalisers, described as in Definition 3.3.5.1.1.

009Z 3. *Failure To Be Cartesian Closed.* The category Sets_* is not Cartesian closed.[8]

[7]Note that the composition of two morphisms of pointed sets is indeed a morphism of pointed sets, as we have

$$\begin{aligned} g(f(x_0)) &= g(y_0) \\ &= z_0, \end{aligned}$$

or

$$\begin{tikzcd} & \mathrm{pt} \arrow[dl, "{[x_0]}"'] \arrow[d, "{[y_0]}"] \arrow[dr, "{[z_0]}"] & \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

in terms of diagrams.

[8]The category Sets_* does admit monoidal closed structures however; see Tensor Products of Pointed Sets.

00A0 4. *Morphisms From the Monoidal Unit.* We have a bijection of sets[9]

$$\mathsf{Sets}_*\left(S^0, X\right) \cong X,$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, internalising also to an isomorphism of pointed sets

$$\mathbf{Sets}_*\left(S^0, X\right) \cong (X, x_0),$$

again natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00A1 5. *Relation to Partial Functions.* We have an equivalence of categories[10]

$$\mathsf{Sets}_* \overset{\text{eq.}}{\cong} \mathsf{Sets}^{\text{part.}}$$

between the category of pointed sets and pointed functions between them and the category of sets and partial functions between them, where:

(a) *From Pointed Sets to Sets With Partial Functions.* The equivalence

$$\xi \colon \mathsf{Sets}_* \xrightarrow{\cong} \mathsf{Sets}^{\text{part.}}$$

sends:

i. A pointed set (X, x_0) to X.

ii. A pointed function

$$f \colon (X, x_0) \to (Y, y_0)$$

to the partial function

$$\xi_f \colon X \to Y$$

defined on $f^{-1}(Y \setminus y_0)$ and given by

$$\xi_f(x) \stackrel{\text{def}}{=} f(x)$$

for each $x \in f^{-1}(Y \setminus y_0)$.

[9] In other words, the forgetful functor

$$忘 \colon \mathsf{Sets}_* \to \mathsf{Sets}$$

defined on objects by sending a pointed set to its underlying set is corepresentable by S^0.

[10] *Warning:* This is not an isomorphism of categories, only an equivalence.
END TEXTDBEND

(b) *From Sets With Partial Functions to Pointed Sets.* The equivalence

$$\xi^{-1}\colon \mathsf{Sets}^{\text{part.}} \xrightarrow{\cong} \mathsf{Sets}_*$$

sends:

i. A set X is to the pointed set $(X, \star)$ with $\star$ an element that is not in X.

ii. A partial function

$$f\colon X \to Y$$

defined on $U \subset X$ to the pointed function

$$\xi_f^{-1}\colon (X, x_0) \to (Y, y_0)$$

defined by

$$\xi_f(x) \overset{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in U, \\ y_0 & \text{otherwise.} \end{cases}$$

for each $x \in X$.

Proof. *Item 1, Completeness*: This follows from (the proofs) of Definitions 3.2.3.1.1, 3.2.4.1.1 and 3.2.5.1.1 and **??**.
Item 2, Cocompleteness: This follows from (the proofs) of Definitions 3.3.3.1.1, 3.3.4.1.1 and 3.3.5.1.1 and **??**.
Item 3, Failure To Be Cartesian Closed: See [`MSE 2855868`].
Item 4, Morphisms From the Monoidal Unit: Since a morphism from S^0 to a pointed set (X, x_0) sends $0 \in S^0$ to x_0 and then can send $1 \in S^0$ to any element of X, we obtain a bijection between pointed maps $S^0 \to X$ and the elements of X.
The isomorphism then

$$\mathsf{Sets}_*\left(S^0, X\right) \cong (X, x_0)$$

follows by noting that $\Delta_{x_0}\colon S^0 \to X$, the basepoint of $\mathsf{Sets}_*(S^0, X)$, corresponds to the pointed map $S^0 \to X$ picking the element x_0 of X, and thus we see that the bijection between pointed maps $S^0 \to X$ and elements of X is compatible with basepoints, lifting to an isomorphism of pointed sets.
Item 5, Relation to Partial Functions: See [`MSE 884460`]. □

00A2 3.2 Limits of Pointed Sets

00A3 3.2.1 The Terminal Pointed Set

00A4 **Definition 3.2.1.1.1.** The **terminal pointed set** is the pair $\left((\mathrm{pt}, \star), \{!_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
- *The Cone.* The collection of morphisms of pointed sets

$$\{!_X : (X, x_0) \to (\mathrm{pt}, \star)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

$$!_X(x) \overset{\mathrm{def}}{=} \star$$

  for each $x \in X$ and each $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that $(\mathrm{pt}, \star)$ is the terminal object of Sets_*. Indeed, suppose we have a diagram of the form

$$(X, x_0) \qquad (\mathrm{pt}, \star)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (X, x_0) \to (\mathrm{pt}, \star)$$

making the diagram

$$(X, x_0) \underset{\exists!}{\overset{\phi}{\dashrightarrow}} (\mathrm{pt}, \star)$$

commute, namely $!_X$. □

00A5 3.2.2 Products of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i\in I}$ be a family of pointed sets.

00A6 **Definition 3.2.2.1.1.** The **product of** $\{(X_i, x_0^i)\}_{i\in I}$ is the pair $\left(\left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right), \{\mathrm{pr}_i\}_{i\in I}\right)$ consisting of:

- *The Limit.* The pointed set $\left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right)$.
- *The Cone.* The collection

$$\left\{\mathrm{pr}_i : \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right) \to \left(X_i, x_0^i\right)\right\}_{i\in I}$$

  of maps given by

$$\mathrm{pr}_i\left((x_j)_{j\in I}\right) \overset{\mathrm{def}}{=} x_i$$

  for each $(x_j)_{j\in I} \in \prod_{i\in I} X_i$ and each $i \in I$.

Proof. We claim that $\left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right)$ is the categorical product of $\{(X_i, x_0^i)\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} (P,*) \arrow[dr, "p_i"] & \\ \left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right) \arrow[r, "\mathrm{pr}_i"'] & (X_i, x_0^i) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P,*) \to \left(\prod_{i\in I} X_i, \left(x_0^i\right)_{i\in I}\right)$$

making the diagram

$$\begin{tikzcd} (P,*) \arrow[d, dashed, "\phi"', "\exists!"] \arrow[dr, "p_i"] & \\ \left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right) \arrow[r, "\mathrm{pr}_i"'] & (X_i, x_0^i) \end{tikzcd}$$

commute, being uniquely determined by the condition $\mathrm{pr}_i \circ \phi = p_i$ for each $i \in I$ via

$$\phi(x) = (p_i(x))_{i\in I}$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(*) &= (p_i(*))_{i\in I} \\ &= \left(x_0^i\right)_{i\in I}, \end{aligned}$$

where we have used that p_i is a morphism of pointed sets for each $i \in I$. □

00A7 **Proposition 3.2.2.1.2.** Let $\{(X_i, x_0^i)\}_{i\in I}$ be a family of pointed sets.

00A8 1. *Functoriality.* The assignment $\{(X_i, x_0^i)\}_{i\in I} \mapsto \left(\prod_{i\in I} X_i, (x_0^i)_{i\in I}\right)$ defines a functor

$$\prod_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. *Item 1*, *Functoriality*: This follows from **??** of **??**. □

00A9 3.2.3 Products

Let (X, x_0) and (Y, y_0) be pointed sets.

00AA **Definition 3.2.3.1.1.** The **product of** (X, x_0) **and** (Y, y_0) is the pair consisting of:

- *The Limit.* The pointed set $(X \times Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

$$\begin{aligned} \mathrm{pr}_1 &: (X \times Y, (x_0, y_0)) \to (X, x_0), \\ \mathrm{pr}_2 &: (X \times Y, (x_0, y_0)) \to (Y, y_0) \end{aligned}$$

  defined by

$$\begin{aligned} \mathrm{pr}_1(x, y) &\overset{\mathrm{def}}{=} x, \\ \mathrm{pr}_2(x, y) &\overset{\mathrm{def}}{=} y \end{aligned}$$

  for each $(x, y) \in X \times Y$.

Proof. We claim that $(X \times Y, (x_0, y_0))$ is the categorical product of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$\begin{tikzcd} & (P, *) \arrow[ld, bend right, "p_1"'] \arrow[rd, bend left, "p_2"] & \\ (X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & (Y, y_0) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (P, *) \to (X \times Y, (x_0, y_0))$$

making the diagram

$$\begin{tikzcd} & (P, *) \arrow[ld, bend right, "p_1"'] \arrow[rd, bend left, "p_2"] \arrow[d, dashed, "\phi"', "\exists!"] & \\ (X, x_0) & (X \times Y, (x_0, y_0)) \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & (Y, y_0) \end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned} \mathrm{pr}_1 \circ \phi &= p_1, \\ \mathrm{pr}_2 \circ \phi &= p_2 \end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$. Note that this is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(*) &= (p_1(*), p_2(*))\\ &= (x_0, y_0),\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. □

00AB **Proposition 3.2.3.1.2.** Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

00AC 1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \times Y, (x_0, y_0))$$

define functors

$$\begin{aligned}X \times -\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*,\\ - \times Y\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*,\\ -_1 \times -_2\colon \mathsf{Sets}_* \times \mathsf{Sets}_* &\to \mathsf{Sets}_*,\end{aligned}$$

defined in the same way as the functors of Item 1 of Proposition 2.1.3.1.2.

00AD 2. *Associativity.* We have an isomorphism of pointed sets

$$((X \times Y) \times Z, ((x_0, y_0), z_0)) \cong (X \times (Y \times Z), (x_0, (y_0, z_0)))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00AE 3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{aligned}(\mathrm{pt}, \star) \times (X, x_0) &\cong (X, x_0),\\ (X, x_0) \times (\mathrm{pt}, \star) &\cong (X, x_0),\end{aligned}$$

natural in $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00AF 4. *Commutativity.* We have an isomorphism of pointed sets

$$(X \times Y, (x_0, y_0)) \cong (Y \times X, (y_0, x_0)),$$

natural in $(X, x_0), (Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00AG 5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times, (\mathrm{pt}, \star))$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of limits, **??** of **??**.
Item 2, Associativity: This follows from Item 3 of Proposition 2.1.3.1.2.
Item 3, Unitality: This follows from Item 4 of Proposition 2.1.3.1.2.
Item 4, Commutativity: This follows from Item 5 of Proposition 2.1.3.1.2.
Item 5, Symmetric Monoidality: This follows from Item 12 of Proposition 2.1.3.1.2. □

00AH
3.2.4 Pullbacks

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (X, x_0) \to (Z, z_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$ be morphisms of pointed sets.

00AJ **Definition 3.2.4.1.1.** The **pullback of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(X \times_Z Y, (x_0, y_0))$.
- *The Cone.* The morphisms of pointed sets

  $$\begin{aligned} \mathrm{pr}_1 &\colon (X \times_Z Y, (x_0, y_0)) \to (X, x_0), \\ \mathrm{pr}_2 &\colon (X \times_Z Y, (x_0, y_0)) \to (Y, y_0) \end{aligned}$$

  defined by

  $$\begin{aligned} \mathrm{pr}_1(x, y) &\stackrel{\text{def}}{=} x, \\ \mathrm{pr}_2(x, y) &\stackrel{\text{def}}{=} y \end{aligned}$$

  for each $(x, y) \in X \times_Z Y$.

Proof. We claim that $X \times_Z Y$ is the categorical pullback of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pullback diagram commutes, i.e. that we have

$$f \circ \mathrm{pr}_1 = g \circ \mathrm{pr}_2, \qquad \begin{tikzcd} (X \times_Z Y, (x_0, y_0)) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & (Y, y_0) \arrow[d, "g"] \\ (X, x_0) \arrow[r, "f"'] & (Z, z_0). \end{tikzcd}$$

Indeed, given $(x, y) \in X \times_Z Y$, we have

$$\begin{aligned} [f \circ \mathrm{pr}_1](x, y) &= f(\mathrm{pr}_1(x, y)) \\ &= f(x) \\ &= g(y) \\ &= g(\mathrm{pr}_2(x, y)) \\ &= [g \circ \mathrm{pr}_2](x, y), \end{aligned}$$

where $f(x) = g(y)$ since $(x, y) \in X \times_Z Y$. Next, we prove that $X \times_Z Y$ satisfies the universal property of the pullback. Suppose we have a diagram of the form

$$\begin{tikzcd}
(P,*) \arrow[rrd, bend left, "p_2"] \arrow[rdd, bend right, "p_1"'] & & \\
& (X \times_Z Y, (x_0,y_0)) \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, dashed, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\
& (X, x_0) \arrow[r, "f"'] & (Z, z_0)
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi \colon (P,*) \to (X \times_Z Y, (x_0,y_0))$$

making the diagram

$$\begin{tikzcd}
(P,*) \arrow[rrd, bend left, "p_2"] \arrow[rdd, bend right, "p_1"'] \arrow[rd, dashed, "\phi", "\exists!"'] & & \\
& (X \times_Z Y, (x_0,y_0)) \arrow[r, dashed, "\mathrm{pr}_2"] \arrow[d, dashed, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & (Y, y_0) \arrow[d, "g"] \\
& (X, x_0) \arrow[r, "f"'] & (Z, z_0)
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\mathrm{pr}_1 \circ \phi &= p_1, \\
\mathrm{pr}_2 \circ \phi &= p_2
\end{aligned}$$

via

$$\phi(x) = (p_1(x), p_2(x))$$

for each $x \in P$, where we note that $(p_1(x), p_2(x)) \in X \times Y$ indeed lies in $X \times_Z Y$ by the condition

$$f \circ p_1 = g \circ p_2,$$

which gives

$$f(p_1(x)) = g(p_2(x))$$

for each $x \in P$, so that $(p_1(x), p_2(x)) \in X \times_Z Y$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
\phi(*) &= (p_1(*), p_2(*)) \\
&= (x_0, y_0),
\end{aligned}$$

where we have used that p_1 and p_2 are morphisms of pointed sets. $\square$

00AK **Proposition 3.2.4.1.2.** Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

00AL 1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \times_{f,Z,g} Y$ defines a functor

$$-_1 \times_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*) \to \mathsf{Sets}_*,$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet. \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \times_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& X' \times_{Z'} Y' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "g'"] \\
X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\
& X' \arrow[rr, "f'"'] & & Z'
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \times_Z Y, (x_0, y_0)) \xrightarrow{\exists!} (X' \times_{Z'} Y', (x'_0, y'_0))$$

given by

$$\xi(x, y) \overset{\text{def}}{=} (\phi(x), \psi(y))$$

for each $(x, y) \in X \times_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[dd, "g" near start] \arrow[dr, "\psi"] & \\
& X' \times_{Z'} Y' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & Y' \arrow[dd, "g'"] \\
X \arrow[rr, "f" near start] \arrow[dr, "\phi"'] & & Z \arrow[dr, "\chi"] & \\
& X' \arrow[rr, "f'"'] & & Z'
\end{tikzcd}$$

commute.

00AM 2. *Associativity.* Given a diagram

$$\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}$$

in Sets_*, we have isomorphisms of pointed sets

$$(X \times_W Y) \times_V Z \cong (X \times_W Y) \times_Y (Y \times_V Z) \cong X \times_W (Y \times_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=tiny]
& & (X \times_W Y) \times_Y Z \arrow[dl] \arrow[drdr] \arrow[d, phantom, "\lrcorner", rotate=-45, very near start] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner", rotate=-45, very near start] & & & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & (X \times_W Y) \times_Y (Y \times_V Z) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner", rotate=-45, very near start] & & \\
& X \times_W Y \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner", rotate=-45, very near start] & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner", rotate=-45, very near start] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z, \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & X \times_W (Y \times_V Z) \arrow[dldl] \arrow[dr] \arrow[d, phantom, "\lrcorner", rotate=-45, very near start] & & \\
& & & Y \times_V Z \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner", rotate=-45, very near start] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \arrow[dr, "h"'] & & Z. \arrow[dl, "k"] \\
& W & & V &
\end{tikzcd}$$

00AN 3. *Unitality.* We have isomorphisms of pointed sets

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{aligned}
X \times_X A &\cong A, \\
A \times_X X &\cong A,
\end{aligned}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}$$

00AP 4. *Commutativity.* We have an isomorphism of pointed sets

$$\begin{tikzcd}
A \times_X B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & X,
\end{tikzcd}
\qquad
A \times_X B \cong B \times_X A
\qquad
\begin{tikzcd}
B \times_X A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & X.
\end{tikzcd}$$

00AQ 5. *Interaction With Products.* We have an isomorphism of pointed sets

$$X \times_{\mathrm{pt}} Y \cong X \times Y,
\qquad
\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "!_Y"] \\
X \arrow[r, "!_X"'] & \mathrm{pt}.
\end{tikzcd}$$

00AR 6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \times_X, X)$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pullback diagram.
Item 2, Associativity: This follows from Item 2 of Proposition 3.2.4.1.2.
Item 3, Unitality: This follows from Item 3 of Proposition 2.1.4.1.3.
Item 4, Commutativity: This follows from Item 4 of Proposition 2.1.4.1.3.
Item 5, Interaction With Products: This follows from Item 6 of Proposition 2.1.4.1.3.
Item 6, Symmetric Monoidality: This follows from Item 7 of Proposition 2.1.4.1.3. □

00AS 3.2.5 Equalisers

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

00AT **Definition 3.2.5.1.1.** The **equaliser of** (f, g) is the pair consisting of:

- *The Limit.* The pointed set $(\mathrm{Eq}(f, g), x_0)$.
- *The Cone.* The morphism of pointed sets

$$\mathrm{eq}(f, g)\colon (\mathrm{Eq}(f, g), x_0) \hookrightarrow (X, x_0)$$

  given by the canonical inclusion $\mathrm{eq}(f, g) \hookrightarrow \mathrm{Eq}(f, g) \hookrightarrow X$.

Proof. We claim that $(\mathrm{Eq}(f, g), x_0)$ is the categorical equaliser of f and g in Sets_*. First we need to check that the relevant equaliser diagram commutes, i.e. that we have

$$f \circ \mathrm{eq}(f, g) = g \circ \mathrm{eq}(f, g),$$

which indeed holds by the definition of the set $\mathrm{Eq}(f, g)$. Next, we prove that $\mathrm{Eq}(f, g)$ satisfies the universal property of the equaliser. Suppose we have a diagram of the form

$$\begin{tikzcd}
(\mathrm{Eq}(f,g), x_0) \arrow[r, "\mathrm{eq}(f,g)"] & (X, x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y, y_0) \\
(E, *) \arrow[ur, "e"'] & &
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (E, *) \to (\mathrm{Eq}(f, g), x_0)$$

making the diagram

$$
\begin{tikzcd}
(\mathrm{Eq}(f,g),x_0) \arrow[r, hook, "\mathrm{eq}(f,g)"] & (X,x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,y_0) \\
(E,*) \arrow[u, dashed, "\phi", "\exists!"'] \arrow[ur, "e"'] & &
\end{tikzcd}
$$

commute, being uniquely determined by the condition

$$\mathrm{eq}(f,g)\circ\phi = e$$

via

$$\phi(x) = e(x)$$

for each $x \in E$, where we note that $e(x) \in A$ indeed lies in $\mathrm{Eq}(f,g)$ by the condition

$$f \circ e = g \circ e,$$

which gives

$$f(e(x)) = g(e(x))$$

for each $x \in E$, so that $e(x) \in \mathrm{Eq}(f,g)$. Lastly, we note that ϕ is indeed a morphism of pointed sets, as we have

$$
\begin{aligned}
\phi(*) &= e(*) \\
&= x_0,
\end{aligned}
$$

where we have used that e is a morphism of pointed sets. □

00AU **Proposition 3.2.5.1.2.** Let (X,x_0) and (Y,y_0) be pointed sets and let $f,g,h\colon (X,x_0)\to(Y,y_0)$ be morphisms of pointed sets.

00AV 1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{Eq}(f\circ \mathrm{eq}(g,h), g\circ \mathrm{eq}(g,h))}_{=\mathrm{Eq}(f\circ \mathrm{eq}(g,h),h\circ \mathrm{eq}(g,h))} \cong \mathrm{Eq}(f,g,h) \cong \underbrace{\mathrm{Eq}(f\circ \mathrm{eq}(f,g), h\circ \mathrm{eq}(f,g))}_{=\mathrm{Eq}(g\circ \mathrm{eq}(f,g),h\circ \mathrm{eq}(f,g))},$$

   where $\mathrm{Eq}(f,g,h)$ is the limit of the diagram

$$
\begin{tikzcd}
(X,x_0) \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & (Y,y_0)
\end{tikzcd}
$$

   in Sets_*, being explicitly given by

$$\mathrm{Eq}(f,g,h) \cong \{a \in A \mid f(a) = g(a) = h(a)\}.$$

00AW 2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, f) \cong X.$$

00AX 3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{Eq}(f, g) \cong \mathrm{Eq}(g, f).$$

Proof. *Item 1, Associativity*: This follows from Item 1 of Proposition 2.1.5.1.2.
Item 2, Unitality: This follows from Item 4 of Proposition 2.1.5.1.2.
Item 3, Commutativity: This follows from Item 5 of Proposition 2.1.5.1.2. □

00AY 3.3 Colimits of Pointed Sets

00AZ 3.3.1 The Initial Pointed Set

00B0 **Definition 3.3.1.1.1.** The **initial pointed set** is the pair $\left((\mathrm{pt}, \star), \{\iota_X\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)}\right)$ consisting of:

- *The Limit.* The pointed set $(\mathrm{pt}, \star)$.
- *The Cone.* The collection of morphisms of pointed sets

  $$\{\iota_X : (\mathrm{pt}, \star) \to (X, x_0)\}_{(X,x_0)\in\mathrm{Obj}(\mathsf{Sets})}$$

  defined by

  $$\iota_X(\star) \stackrel{\text{def}}{=} x_0.$$

Proof. We claim that $(\mathrm{pt}, \star)$ is the initial object of Sets_*. Indeed, suppose we have a diagram of the form

$$(\mathrm{pt}, \star) \qquad (X, x_0)$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi : (\mathrm{pt}, \star) \to (X, x_0)$$

making the diagram

$$(\mathrm{pt}, \star) \overset{\phi}{\underset{\exists!}{\dashrightarrow}} (X, x_0)$$

commute, namely ι_X. □

00B1 3.3.2 Coproducts of Families of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i\in I}$ be a family of pointed sets.

00B2 **Definition 3.3.2.1.1.** The **coproduct of the family** $\{(X_i, x_0^i)\}_{i\in I}$, also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(\bigvee_{i\in I} X_i, p_0)$ consisting of:
  - *The Underlying Set.* The set $\bigvee_{i\in I} X_i$ defined by

    $$\bigvee_{i\in I} X_i \overset{\text{def}}{=} \left(\coprod_{i\in I} X_i\right)/\sim,$$

    where $\sim$ is the equivalence relation on $\coprod_{i\in I} X_i$ given by declaring

    $$\left(i, x_0^i\right) \sim \left(j, x_0^j\right)$$

    for each $i, j \in I$.
  - *The Basepoint.* The element p_0 of $\bigvee_{i\in I} X_i$ defined by

    $$\begin{aligned} p_0 &\overset{\text{def}}{=} \left[\left(i, x_0^i\right)\right] \\ &= \left[\left(j, x_0^j\right)\right] \end{aligned}$$

    for any $i, j \in I$.
- *The Cocone.* The collection

  $$\left\{\mathrm{inj}_i \colon \left(X_i, x_0^i\right) \to \left(\bigvee_{i\in I} X_i, p_0\right)\right\}_{i\in I}$$

  of morphism of pointed sets given by

  $$\mathrm{inj}_i(x) \overset{\text{def}}{=} (i, x)$$

  for each $x \in X_i$ and each $i \in I$.

Proof. We claim that $(\bigvee_{i\in I} X_i, p_0)$ is the categorical coproduct of $\{(X_i, x_0^i)\}_{i\in I}$ in Sets_*. Indeed, suppose we have, for each $i \in I$, a diagram of the form

$$\begin{tikzcd} & (C, *) \\ (X_i, x_0^i) \arrow[ur, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \left(\bigvee_{i\in I} X_i, p_0\right) \end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon \left(\bigvee_{i\in I} X_i, p_0\right) \to (C,*)$$

making the diagram

$$\begin{tikzcd}
 & (C,*) \\
(X_i,x_0^i) \arrow[ru, "\iota_i"] \arrow[r, "\mathrm{inj}_i"'] & \left(\bigvee_{i\in I} X_i, p_0\right) \arrow[u, dashed, "\phi", "\exists!"']
\end{tikzcd}$$

commute, being uniquely determined by the condition $\phi \circ \mathrm{inj}_i = \iota_i$ for each $i \in I$ via

$$\phi([(i,x)]) = \iota_i(x)$$

for each $[(i,x)] \in \bigvee_{i\in I} X_i$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\phi(p_0) &= \iota_i\Big(\Big[\Big(i, x_0^i\Big)\Big]\Big) \\ &= *,\end{aligned}$$

as ι_i is a morphism of pointed sets. □

00B3 **Proposition 3.3.2.1.2.** Let $\{(X_i,x_0^i)\}_{i\in I}$ be a family of pointed sets.

00B4 1. *Functoriality.* The assignment $\{(X_i,x_0^i)\}_{i\in I} \mapsto (\bigvee_{i\in I} X_i, p_0)$ defines a functor

$$\bigvee_{i\in I}\colon \mathsf{Fun}(I_{\mathsf{disc}}, \mathsf{Sets}_*) \to \mathsf{Sets}_*.$$

Proof. *Item 1, Functoriality*: This follows from **??** of **??**. □

00B5 3.3.3 Coproducts

Let (X,x_0) and (Y,y_0) be pointed sets.

00B6 **Definition 3.3.3.1.1.** The **coproduct of** (X,x_0) **and** (Y,y_0), also called their **wedge sum**, is the pair consisting of:

- *The Colimit.* The pointed set $(X \vee Y, p_0)$ consisting of:

– *The Underlying Set.* The set $X \vee Y$ defined by

$$
\begin{aligned}
(X \vee Y, p_0) &\stackrel{\text{def}}{=} (X, x_0) \coprod (Y, y_0) \\
&\cong \left(X \coprod\nolimits_{\text{pt}} Y, p_0\right) \\
&\cong (X \coprod Y/\sim, p_0),
\end{aligned}
\qquad
\begin{tikzcd}
X \vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\
X \arrow[u] & \text{pt}, \arrow[l, "{[x_0]}"] \arrow[u, "{[y_0]}"']
\end{tikzcd}
$$

where $\sim$ is the equivalence relation on $X \coprod Y$ obtained by declaring $(0, x_0) \sim (1, y_0)$.

– *The Basepoint.* The element p_0 of $X \vee Y$ defined by

$$
\begin{aligned}
p_0 &\stackrel{\text{def}}{=} [(0, x_0)] \\
&= [(1, y_0)].
\end{aligned}
$$

- *The Cocone.* The morphisms of pointed sets

$$
\begin{aligned}
\text{inj}_1 &\colon (X, x_0) \to (X \vee Y, p_0), \\
\text{inj}_2 &\colon (Y, y_0) \to (X \vee Y, p_0),
\end{aligned}
$$

given by

$$
\begin{aligned}
\text{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)], \\
\text{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)],
\end{aligned}
$$

for each $x \in X$ and each $y \in Y$.

Proof. We claim that $(X \vee Y, p_0)$ is the categorical coproduct of (X, x_0) and (Y, y_0) in Sets_*. Indeed, suppose we have a diagram of the form

$$
\begin{tikzcd}
 & (C, *) & \\
(X, x_0) \arrow[ur, bend left, "\iota_X"] \arrow[r, "\text{inj}_X"'] & (X \vee Y, p_0) & (Y, y_0) \arrow[ul, bend right, "\iota_Y"'] \arrow[l, "\text{inj}_Y"]
\end{tikzcd}
$$

in Sets. Then there exists a unique morphism of pointed sets

$$
\phi \colon (X \vee Y, p_0) \to (C, *)
$$

making the diagram

$$
\begin{tikzcd}
 & (C, *) & \\
(X, x_0) \arrow[ur, bend left, "\iota_X"] \arrow[r, "\text{inj}_X"'] & (X \vee Y, p_0) \arrow[u, dashed, "\phi", "\exists!"'] & (Y, y_0) \arrow[ul, bend right, "\iota_Y"'] \arrow[l, "\text{inj}_Y"]
\end{tikzcd}
$$

commute, being uniquely determined by the conditions

$$\phi \circ \mathrm{inj}_X = \iota_X,$$
$$\phi \circ \mathrm{inj}_Y = \iota_Y$$

via

$$\phi(z) = \begin{cases} \iota_X(x) & \text{if } z = [(0,x)] \text{ with } x \in X, \\ \iota_Y(y) & \text{if } z = [(1,y)] \text{ with } y \in Y \end{cases}$$

for each $z \in X \vee Y$, where we note that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned} \phi(p_0) &= \iota_X([(0,x_0)]) \\ &= \iota_Y([(1,y_0)]) \\ &= *, \end{aligned}$$

as ι_X and ι_Y are morphisms of pointed sets. □

00B7 **Proposition 3.3.3.1.2.** Let (X, x_0) and (Y, y_0) be pointed sets.

00B8 1. *Functoriality.* The assignments

$$(X, x_0), (Y, y_0), ((X, x_0), (Y, y_0)) \mapsto (X \vee Y, p_0)$$

define functors

$$\begin{aligned} X \vee - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ - \vee Y &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ -_1 \vee -_2 &\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

00B9 2. *Associativity.* We have an isomorphism of pointed sets

$$(X \vee Y) \vee Z \cong X \vee (Y \vee Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathsf{Sets}_*$.

00BA 3. *Unitality.* We have isomorphisms of pointed sets

$$(\mathrm{pt}, *) \vee (X, x_0) \cong (X, x_0),$$
$$(X, x_0) \vee (\mathrm{pt}, *) \cong (X, x_0),$$

natural in $(X, x_0) \in \mathsf{Sets}_*$.

00BB 4. *Commutativity.* We have an isomorphism of pointed sets

$$X \vee Y \cong Y \vee X,$$

natural in $(X, x_0), (Y, y_0) \in \mathsf{Sets}_*$.

00BC 5. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \vee, \mathrm{pt})$ is a symmetric monoidal category.

00BD 6. *The Fold Map.* We have a natural transformation

$$\nabla \colon \vee \circ \Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*} \Longrightarrow \mathrm{id}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[dr, "\vee"] \arrow[d, phantom, "\Downarrow\nabla"] & \\ \mathsf{Sets}_* \arrow[ur, "\Delta^{\mathsf{Cats}}_{\mathsf{Sets}_*}"] \arrow[rr, bend right, "\mathrm{id}_{\mathsf{Sets}_*}"'] & & \mathsf{Sets}_*, \end{tikzcd}$$

called the **fold map**, whose component

$$\nabla_X \colon X \vee X \to X$$

at X is given by

$$\nabla_X(p) \overset{\mathrm{def}}{=} \begin{cases} x & \text{if } p = [(0, x)], \\ x & \text{if } p = [(1, x)] \end{cases}$$

for each $p \in X \vee X$.

Proof. *Item 1, Functoriality*: This follows from **??** of **??**.
Item 2, Associativity: Clear.
Item 3, Unitality: Clear.
Item 4, Commutativity: Clear.
Item 5, Symmetric Monoidality: Omitted.
Item 6, The Fold Map: Naturality for the transformation ∇ is the statement that, given a morphism of pointed sets $f \colon (X, x_0) \to (Y, y_0)$, we have

$$\nabla_Y \circ (f \vee f) = f \circ \nabla_X, \qquad \begin{tikzcd} X \vee X \arrow[r, "\nabla_X"] \arrow[d, "f \vee f"'] & X \arrow[d, "f"] \\ Y \vee Y \arrow[r, "\nabla_Y"'] & Y. \end{tikzcd}$$

Indeed, we have

$$\begin{aligned} [\nabla_Y \circ (f \vee f)]([(i, x)]) &= \nabla_Y([(i, f(x))]) \\ &= f(x) \\ &= f(\nabla_X([(i, x)])) \\ &= [f \circ \nabla_X]([(i, x)]) \end{aligned}$$

for each $[(i, x)] \in X \vee X$, and thus ∇ is indeed a natural transformation. $\square$

00BE 3.3.4 Pushouts

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets and let $f\colon (Z, z_0) \to (X, x_0)$ and $g\colon (Z, z_0) \to (Y, y_0)$ be morphisms of pointed sets.

00BF **Definition 3.3.4.1.1.** The **pushout of** (X, x_0) **and** (Y, y_0) **over** (Z, z_0) **along** (f, g) is the pair consisting of:

- *The Colimit.* The pointed set $\left(X \coprod_{f,Z,g} Y, p_0\right)$, where:
  - The set $X \coprod_{f,Z,g} Y$ is the pushout (of unpointed sets) of X and Y over Z with respect to f and g;
  - We have $p_0 = [x_0] = [y_0]$.
- *The Cocone.* The morphisms of pointed sets

$$\begin{aligned}\mathrm{inj}_1 &\colon (X, x_0) \to (X \textstyle\coprod_Z Y, p_0),\\ \mathrm{inj}_2 &\colon (Y, y_0) \to (X \textstyle\coprod_Z Y, p_0)\end{aligned}$$

  given by

$$\begin{aligned}\mathrm{inj}_1(x) &\stackrel{\text{def}}{=} [(0, x)]\\ \mathrm{inj}_2(y) &\stackrel{\text{def}}{=} [(1, y)]\end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

Proof. Firstly, we note that indeed $[x_0] = [y_0]$, as we have

$$\begin{aligned}x_0 &= f(z_0),\\ y_0 &= g(z_0)\end{aligned}$$

since f and g are morphisms of pointed sets, with the relation $\sim$ on $X \coprod_Z Y$ then identifying $x_0 = f(z_0) \sim g(z_0) = y_0$.
We now claim that $(X \coprod_Z Y, p_0)$ is the categorical pushout of (X, x_0) and (Y, y_0) over (Z, z_0) with respect to (f, g) in Sets_*. First we need to check that the relevant pushout diagram commutes, i.e. that we have

$$\mathrm{inj}_1 \circ f = \mathrm{inj}_2 \circ g, \qquad \begin{tikzcd} (X \coprod_Z Y, p_0) & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \\ (X, x_0) \arrow[u, "\mathrm{inj}_1"] & (Z, z_0). \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Indeed, given $z \in Z$, we have

$$\begin{aligned}[\mathrm{inj}_1 \circ f](z) &= \mathrm{inj}_1(f(z))\\ &= [(0, f(z))]\\ &= [(1, g(z))]\\ &= \mathrm{inj}_2(g(z))\\ &= [\mathrm{inj}_2 \circ g](z),\end{aligned}$$

where $[(0, f(z))] = [(1, g(z))]$ by the definition of the relation $\sim$ on $X \coprod Y$ (the coproduct of unpointed sets of X and Y). Next, we prove that $X \coprod_Z Y$ satisfies the universal property of the pushout. Suppose we have a diagram of the form

$$\begin{tikzcd}
(P, *) & & \\
& (X \coprod_Z Y, p_0) \arrow[dr, phantom, "\ulcorner" very near start] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

in Sets_*. Then there exists a unique morphism of pointed sets

$$\phi\colon (X \coprod_Z Y, p_0) \to (P, *)$$

making the diagram

$$\begin{tikzcd}
(P, *) & & \\
& (X \coprod_Z Y, p_0) \arrow[ul, dashed, "\exists !"', "\phi"] \arrow[dr, phantom, "\ulcorner" very near start] & (Y, y_0) \arrow[l, "\mathrm{inj}_2"'] \arrow[ull, bend right, "\iota_2"'] \\
& (X, x_0) \arrow[u, "\mathrm{inj}_1"] \arrow[uul, bend left, "\iota_1"] & (Z, z_0) \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

commute, being uniquely determined by the conditions

$$\begin{aligned}
\phi \circ \mathrm{inj}_1 &= \iota_1, \\
\phi \circ \mathrm{inj}_2 &= \iota_2
\end{aligned}$$

via

$$\phi(p) = \begin{cases} \iota_1(x) & \text{if } x = [(0, x)], \\ \iota_2(y) & \text{if } x = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, where the well-definedness of ϕ is proven in the same way as in the proof of Definition 2.2.4.1.1. Finally, we show that ϕ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}
\phi(p_0) &= \phi([(0, x_0)]) \\
&= \iota_1(x_0) \\
&= *,
\end{aligned}$$

or alternatively

$$\begin{aligned}\phi(p_0) &= \phi([(1, y_0)]) \\ &= \iota_2(y_0) \\ &= *,\end{aligned}$$

where we use that ι_1 (resp. ι_2) is a morphism of pointed sets. $\square$

00BG **Proposition 3.3.4.1.2.** Let (X, x_0), (Y, y_0), (Z, z_0), and (A, a_0) be pointed sets.

00BH 1. *Functoriality.* The assignment $(X, Y, Z, f, g) \mapsto X \coprod_{f,Z,g} Y$ defines a functor

$$-_1 \coprod_{-_3} -_1 \colon \mathsf{Fun}(\mathcal{P}, \mathsf{Sets}) \to \mathsf{Sets}_*,$$

where $\mathcal{P}$ is the category that looks like this:

$$\begin{tikzcd} & \bullet \\ \bullet & \bullet. \arrow[l] \arrow[u] \end{tikzcd}$$

In particular, the action on morphisms of $-_1 \coprod_{-_3} -_1$ is given by sending a morphism

$$\begin{tikzcd}
X \coprod_Z Y & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
& X' \coprod_{Z'} Y' & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"'] \arrow[dr, "\chi"] & \\
& X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

in $\mathsf{Fun}(\mathcal{P}, \mathsf{Sets}_*)$ to the morphism of pointed sets

$$\xi \colon (X \coprod_Z Y, p_0) \xrightarrow{\exists!} (X' \coprod_{Z'} Y', p'_0)$$

given by

$$\xi(p) \overset{\text{def}}{=} \begin{cases} \phi(x) & \text{if } p = [(0, x)], \\ \psi(y) & \text{if } p = [(1, y)] \end{cases}$$

for each $p \in X \coprod_Z Y$, which is the unique morphism of pointed sets making the diagram

$$\begin{tikzcd}
X \coprod_Z Y \arrow[dr, dashed] & & Y \arrow[ll] \arrow[dr, "\psi"] & \\
& X' \coprod_{Z'} Y' & & Y' \arrow[ll] \\
X \arrow[uu] \arrow[dr, "\phi"'] & & Z \arrow[ll, "f"'] \arrow[uu, "g"] \arrow[dr, "\chi"] & \\
& X' \arrow[uu] & & Z' \arrow[ll, "f'"] \arrow[uu, "g'"']
\end{tikzcd}$$

commute.

00BJ 2. *Associativity.* Given a diagram

$$\begin{tikzcd}
X & & Y & & Z \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

in Sets, we have isomorphisms of pointed sets

$$(X \coprod_W Y) \coprod_V Z \cong (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) \cong X \coprod_W (Y \coprod_V Z),$$

where these pullbacks are built as in the diagrams

$$\begin{tikzcd}[column sep=small, row sep=small]
& & (X \coprod_W Y) \coprod_V Z & & \\
& X \coprod_W Y \arrow[ur] & & & \\
X \arrow[ur] & & Y \arrow[ul] & & Z, \arrow[uull] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small, row sep=small]
& & (X \coprod_W Y) \coprod_Y (Y \coprod_V Z) & & \\
& X \coprod_W Y \arrow[ur] & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[ur] & & Y \arrow[ul] \arrow[ur] & & Z, \arrow[ul] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small, row sep=small]
& & X \coprod_W (Y \coprod_V Z) & & \\
& & & Y \coprod_V Z \arrow[ul] & \\
X \arrow[uurr] & & Y \arrow[ur] & & Z. \arrow[ul] \\
& W \arrow[ul, "f"] \arrow[ur, "g"'] & & V \arrow[ul, "h"] \arrow[ur, "k"'] &
\end{tikzcd}$$

00BK 3. *Unitality.* We have isomorphisms of sets

$$\begin{tikzcd}
A \arrow[r, equal] & A \\
X \arrow[u, "f"] \arrow[r, equal] & X \arrow[u, "f"']
\end{tikzcd}
\qquad
\begin{aligned} X \coprod_X A &\cong A, \\ A \coprod_X X &\cong A, \end{aligned}
\qquad
\begin{tikzcd}
A & X \arrow[l, "f"'] \\
X \arrow[u, equal] & X. \arrow[l, "f"] \arrow[u, equal]
\end{tikzcd}$$

00BL 4. *Commutativity.* We have an isomorphism of sets

$$\begin{tikzcd}
X \coprod_Z Y & Y \arrow[l] \\
X \arrow[u] & Z, \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}
\qquad
X \coprod_Z Y \cong Y \coprod_Z X
\qquad
\begin{tikzcd}
Y \coprod_Z X & X \arrow[l] \\
Y \arrow[u] & Z. \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}$$

00BM 5. *Interaction With Coproducts.* We have

$$
X \textstyle\coprod_{\mathrm{pt}} Y \cong X \vee Y, \qquad
\begin{tikzcd}
X \vee Y \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[l] \\
X \arrow[u] & \mathrm{pt}. \arrow[l, hook', "{[x_0]}"] \arrow[u, hook', "{[y_0]}"']
\end{tikzcd}
$$

00BN 6. *Symmetric Monoidality.* The triple $(\mathsf{Sets}_*, \coprod_X, (X, x_0))$ is a symmetric monoidal category.

Proof. *Item 1, Functoriality*: This is a special case of functoriality of co/limits, **??** of **??**, with the explicit expression for ξ following from the commutativity of the cube pushout diagram.
Item 2, Associativity: This follows from Item 2 of Proposition 2.2.4.1.4.
Item 3, Unitality: This follows from Item 3 of Proposition 2.2.4.1.4.
Item 4, Commutativity: This follows from Item 4 of Proposition 2.2.4.1.4.
Item 5, Interaction With Coproducts: Clear.
Item 6, Symmetric Monoidality: Omitted. □

00BP 3.3.5 Coequalisers

Let $f, g\colon (X, x_0) \rightrightarrows (Y, y_0)$ be morphisms of pointed sets.

00BQ **Definition 3.3.5.1.1.** The **coequaliser of** (f, g) is the pointed set $(\mathrm{CoEq}(f, g), [y_0])$.

Proof. We claim that $(\mathrm{CoEq}(f, g), [y_0])$ is the categorical coequaliser of f and g in Sets_*. First we need to check that the relevant coequaliser diagram commutes, i.e. that we have

$$\mathrm{coeq}(f, g) \circ f = \mathrm{coeq}(f, g) \circ g.$$

Indeed, we have

$$
\begin{aligned}
[\mathrm{coeq}(f, g) \circ f](x) &\stackrel{\text{def}}{=} [\mathrm{coeq}(f, g)](f(x)) \\
&\stackrel{\text{def}}{=} [f(x)] \\
&= [g(x)] \\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f, g)](g(x)) \\
&\stackrel{\text{def}}{=} [\mathrm{coeq}(f, g) \circ g](x)
\end{aligned}
$$

for each $x \in X$. Next, we prove that $\mathrm{CoEq}(f, g)$ satisfies the universal property of the coequaliser. Suppose we have a diagram of the form

$$\begin{tikzcd} (X,x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,y_0) \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f,g),[y_0]) \\ & & (C,*) \end{tikzcd}$$

in Sets. Then, since $c(f(a)) = c(g(a))$ for each $a \in A$, it follows from Items 4 and 5 of Proposition 7.5.2.1.3 that there exists a unique map $\phi \colon \mathrm{CoEq}(f,g) \xrightarrow{\exists!} C$ making the diagram

$$\begin{tikzcd} (X,x_0) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,y_0) \arrow[r, hook, "\mathrm{coeq}(f,g)"] \arrow[dr, "c"'] & (\mathrm{CoEq}(f,g),[y_0]) \arrow[d, dashed, "\phi"', "\exists!"] \\ & & (C,*) \end{tikzcd}$$

commute, where we note that ϕ is indeed a morphism of pointed sets since

$$\begin{aligned} \phi([y_0]) &= [\phi \circ \mathrm{coeq}(f,g)]([y_0]) \\ &= c([y_0]) \\ &= *, \end{aligned}$$

where we have used that c is a morphism of pointed sets. □

00BR **Proposition 3.3.5.1.2.** Let (X,x_0) and (Y,y_0) be pointed sets and let $f,g,h \colon (X,x_0) \to (Y,y_0)$ be morphisms of pointed sets.

00BS 1. *Associativity.* We have isomorphisms of pointed sets

$$\underbrace{\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ f, \mathrm{coeq}(f,g)\circ h)}_{=\mathrm{CoEq}(\mathrm{coeq}(f,g)\circ g, \mathrm{coeq}(f,g)\circ h)} \cong \mathrm{CoEq}(f,g,h) \cong \underbrace{\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ g)}_{=\mathrm{CoEq}(\mathrm{coeq}(g,h)\circ f, \mathrm{coeq}(g,h)\circ h)},$$

where $\mathrm{CoEq}(f,g,h)$ is the colimit of the diagram

$$\begin{tikzcd} (X,x_0) \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & (Y,y_0) \end{tikzcd}$$

in Sets_*.

00BT 2. *Unitality.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f,f) \cong B.$$

00BU 3. *Commutativity.* We have an isomorphism of pointed sets

$$\mathrm{CoEq}(f,g) \cong \mathrm{CoEq}(g,f).$$

Proof. *Item 1*, *Associativity*: This follows from Item 1 of Proposition 2.2.5.1.4.
Item 2, *Unitality*: This follows from Item 4 of Proposition 2.2.5.1.4.
Item 3, *Commutativity*: This follows from Item 5 of Proposition 2.2.5.1.4. □

00BV 3.4 Constructions With Pointed Sets

00BW 3.4.1 Free Pointed Sets

Let X be a set.

00BX **Definition 3.4.1.1.1.** The **free pointed set on** X is the pointed set X^+ consisting of:

- *The Underlying Set.* The set X^+ defined by[11]

$$\begin{aligned} X^+ &\stackrel{\text{def}}{=} X \coprod \mathrm{pt} \\ &\stackrel{\text{def}}{=} X \coprod \{\star\}. \end{aligned}$$

- *The Basepoint.* The element $\star$ of X^+.

00BY **Proposition 3.4.1.1.2.** Let X be a set.

00BZ 1. *Functoriality.* The assignment $X \mapsto X^+$ defines a functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*,$$

where

- *Action on Objects.* For each $X \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^+\right](X) \stackrel{\text{def}}{=} X^+,$$

where X^+ is the pointed set of Definition 3.4.1.1.1;
- *Action on Morphisms.* For each morphism $f \colon X \to Y$ of Sets, the image

$$f^+ \colon X^+ \to Y^+$$

of f by $(-)^+$ is the map of pointed sets defined by

$$f^+(x) \stackrel{\text{def}}{=} \begin{cases} f(x) & \text{if } x \in X, \\ \star_Y & \text{if } x = \star_X. \end{cases}$$

[11] *Further Notation:* We sometimes write $\star_X$ for the basepoint of X^+ for clarity

00C0 2. *Adjointness.* We have an adjunction

$$\Big((-)^{+} \dashv 忘\Big) \colon \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "(-)^{+}"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_{*}, \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathsf{Sets}_{*}\Big(\Big(X^{+}, \star_{X}\Big), (Y, y_{0})\Big) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_{*})$.

00C1 3. *Symmetric Strong Monoidality With Respect to Wedge Sums.* The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\Big((-)^{+}, (-)^{+,\coprod}, (-)^{+,\coprod}_{\mathbb{1}}\Big) \colon \Big(\mathsf{Sets}, \coprod, \emptyset\Big) \to (\mathsf{Sets}_{*}, \vee, \mathrm{pt}),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\coprod}_{X,Y} \colon X^{+} \vee Y^{+} \xrightarrow{\cong} \Big(X \coprod Y\Big)^{+},$$
$$(-)^{+,\coprod}_{\mathbb{1}} \colon \mathrm{pt} \xrightarrow{\cong} \emptyset^{+},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

4. *Symmetric Strong Monoidality With Respect to Smash Products.*
00C2 The free pointed set functor of Item 1 has a symmetric strong monoidal structure

$$\Big((-)^{+}, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\Big) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \Big(\mathsf{Sets}_{*}, \wedge, S^{0}\Big),$$

being equipped with isomorphisms of pointed sets

$$(-)^{+,\times}_{X,Y} \colon X^{+} \wedge Y^{+} \xrightarrow{\cong} (X \times Y)^{+},$$
$$(-)^{+,\times}_{\mathbb{1}} \colon S^{0} \xrightarrow{\cong} \mathrm{pt}^{+},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: We claim there's an adjunction $(-)^{+} \dashv 忘$, witnessed by a bijection of sets

$$\mathsf{Sets}_{*}\Big(\Big(X^{+}, \star_{X}\Big), (Y, y_{0})\Big) \cong \mathsf{Sets}(X, Y),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets})$ and $(Y, y_0) \in \mathrm{Obj}(\mathsf{Sets}_{*})$.

- *Map I.* We define a map

$$\Phi_{X,Y} \colon \mathsf{Sets}_*\Big(\big(X^+, \star_X\big), (Y, y_0)\Big) \to \mathsf{Sets}(X, Y)$$

by sending a pointed function

$$\xi \colon \big(X^+, \star_X\big) \to (Y, y_0)$$

to the function

$$\xi^\dagger \colon X \to Y$$

given by

$$\xi^\dagger(x) \overset{\text{def}}{=} \xi(x)$$

for each $x \in X$.

- *Map II.* We define a map

$$\Psi_{X,Y} \colon \mathsf{Sets}(X, Y) \to \mathsf{Sets}_*\Big(\big(X^+, \star_X\big), (Y, y_0)\Big)$$

given by sending a function $\xi \colon X \to Y$ to the pointed function

$$\xi^\dagger \colon \big(X^+, \star_X\big) \to (Y, y_0)$$

defined by

$$\xi^\dagger(x) \overset{\text{def}}{=} \begin{cases} \xi(x) & \text{if } x \in X, \\ y_0 & \text{if } x = \star_X \end{cases}$$

for each $x \in X^+$.

- *Invertibility I.* We claim that

$$\Psi_{X,Y} \circ \Phi_{X,Y} = \mathrm{id}_{\mathsf{Sets}_*((X^+,\star_X),(Y,y_0))},$$

which is clear.

- *Invertibility II.* We claim that

$$\Phi_{X,Y} \circ \Psi_{X,Y} = \mathrm{id}_{\mathsf{Sets}(X,Y)},$$

which is clear.

- *Naturality for Φ, Part I.* We need to show that, given a pointed

when there are multiple free pointed sets involved in the current discussion.

function $g\colon (Y, y_0) \to (Y', y'_0)$, the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*\big((X^+, \star_X), (Y, y_0)\big) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "g_*"'] & \mathsf{Sets}(X, Y) \arrow[d, "g_*"] \\
\mathsf{Sets}_*\big((X^+, \star_X), (Y', y'_0)\big), \arrow[r, "\Phi_{X,Y'}"'] & \mathsf{Sets}(X, Y')
\end{tikzcd}$$

commutes. Indeed, given a pointed function

$$\xi^\dagger \colon \Big(X^+, \star_X\Big) \to (Y, y_0)$$

we have

$$\begin{aligned}
\big[\Phi_{X,Y'} \circ g_*\big](\xi) &= \Phi_{X,Y'}(g_*(\xi)) \\
&= \Phi_{X,Y'}(g \circ \xi) \\
&= g \circ \xi \\
&= g \circ \Phi_{X,Y'}(\xi) \\
&= g_*(\Phi_{X,Y'}(\xi)) \\
&= \big[g_* \circ \Phi_{X,Y'}\big](\xi).
\end{aligned}$$

- *Naturality for Φ, Part II.* We need to show that, given a pointed function $f\colon (X, x_0) \to (X', x'_0)$, the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*\Big(\Big(X'^{,+}, \star_X\Big), (Y, y_0)\Big) \arrow[r, "\Phi_{X',Y}"] \arrow[d, "f^*"'] & \mathsf{Sets}(X', Y) \arrow[d, "f^*"] \\
\mathsf{Sets}_*\big((X^+, \star_X), (Y, y_0)\big) \arrow[r, "\Phi_{X,Y}"'] & \mathsf{Sets}(X, Y)
\end{tikzcd}$$

commutes. Indeed, given a function

$$\xi \colon X' \to Y,$$

we have

$$\begin{aligned}
[\Phi_{X,Y} \circ f^*](\xi) &= \Phi_{X,Y}(f^*(\xi)) \\
&= \Phi_{X,Y}(\xi \circ f) \\
&= \xi \circ f \\
&= \Phi_{X',Y}(\xi) \circ f \\
&= f^*(\Phi_{X',Y}(\xi)) \\
&= f^*(\Phi_{X',Y}(\xi)) \\
&= \big[f^* \circ \Phi_{X',Y}\big](\xi).
\end{aligned}$$

- *Naturality for* Ψ. Since Φ is natural in each argument and Φ is a componentwise inverse to Ψ in each argument, it follows from Item 2 of Proposition 8.8.6.1.2 that Ψ is also natural in each argument.

Item 3, Symmetric Strong Monoidality With Respect to Wedge Sums: The isomorphism

$$\phi\colon X^+ \vee Y^+ \xrightarrow{\cong} (X \textstyle\coprod Y)^+$$

is given by

$$\phi(z) = \begin{cases} x & \text{if } z = [(0,x)] \text{ with } x \in X, \\ y & \text{if } z = [(1,y)] \text{ with } y \in Y, \\ \star_{X \coprod Y} & \text{if } z = [(0,\star_X)], \\ \star_{X \coprod Y} & \text{if } z = [(1,\star_Y)] \end{cases}$$

for each $z \in X^+ \vee Y^+$, with inverse

$$\phi^{-1}\colon (X \textstyle\coprod Y)^+ \xrightarrow{\cong} X^+ \vee Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} [(0,x)] & \text{if } z = [(0,x)], \\ [(0,y)] & \text{if } z = [(1,y)], \\ p_0 & \text{if } z = \star_{X \coprod Y} \end{cases}$$

for each $z \in (X \coprod Y)^+$.
Meanwhile, the isomorphism $\mathrm{pt} \cong \emptyset^+$ is given by sending $\star_X$ to $\star_\emptyset$.
That these isomorphisms satisfy the coherence conditions making the functor $(-)^+$ symmetric strong monoidal can be directly checked element by element.
Item 4, Symmetric Strong Monoidality With Respect to Smash Products: The isomorphism

$$\phi\colon X^+ \wedge Y^+ \xrightarrow{\cong} (X \times Y)^+$$

is given by

$$\phi(x \wedge y) = \begin{cases} (x,y) & \text{if } x \neq \star_X \text{ and } y \neq \star_Y \\ \star_{X \times Y} & \text{otherwise} \end{cases}$$

for each $x \wedge y \in X^+ \wedge Y^+$, with inverse

$$\phi^{-1}\colon (X \times Y)^+ \xrightarrow{\cong} X^+ \wedge Y^+$$

given by

$$\phi^{-1}(z) \stackrel{\text{def}}{=} \begin{cases} x \wedge y & \text{if } z = (x,y) \text{ with } (x,y) \in X \times Y, \\ \star_X \wedge \star_Y & \text{if } z = \star_{X \times Y}, \end{cases}$$

for each $z \in (X \coprod Y)^{+}$.
Meanwhile, the isomorphism $S^0 \cong \mathrm{pt}^{+}$ is given by sending $\star$ to $1 \in S^0 = \{0, 1\}$ and $\star_{\mathrm{pt}}$ to $0 \in S^0$.
That these isomorphisms satisfy the coherence conditions making the functor $(-)^{+}$ symmetric strong monoidal can be directly checked element by element. $\square$

Appendices

3.A Other Chapters

Chapter 4

Tensor Products of Pointed Sets

00C3 In this chapter we introduce, construct, and study tensor products of pointed sets. The most well-known among these is the *smash product of pointed sets*

$$\wedge\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

introduced in Section 4.5.1, defined via a universal property as inducing a bijection between the following data:

- Pointed maps $f\colon X \wedge Y \to Z$.
- Maps of sets $f\colon X \times Y \to Z$ satisfying

  $$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

  for each $x \in X$ and each $y \in Y$.

As it turns out, however, dropping either of the *bilinearity* conditions

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

while retaining the other leads to two other tensor products of pointed sets,

$$\begin{aligned} \lhd\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ \rhd\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*, \end{aligned}$$

called the *left* and *right tensor products of pointed sets*. In contrast to $\wedge$, which turns out to endow Sets_* with a monoidal category structure

(Proposition 4.5.9.1.1), these do not admit invertible associators and unitors, but do endow Sets_* with the structure of a skew monoidal category, however (Propositions 4.3.8.1.1 and 4.4.8.1.1).
Finally, in addition to the tensor products $\triangleleft$, $\triangleright$, and $\wedge$, we also have a “tensor product” of the form

$$\odot\colon \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*,$$

called the *tensor* of sets with pointed sets. All in all, these tensor products assemble into a family of functors of the form

$$\begin{aligned}
\otimes_{k,\ell}&\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_\ell}(\mathsf{Sets}) \to \mathsf{Mon}_{\mathbb{E}_{k+\ell}}(\mathsf{Sets}),\\
\triangleleft_{i,k}&\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \to \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}),\\
\triangleright_{i,k}&\colon \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \times \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}) \to \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Sets}),
\end{aligned}$$

where $k, \ell, i \in \mathbb{N}$ with $i \leq k-1$. Together with the Cartesian product $\times$ of Sets, the tensor products studied in this chapter form the cases:

- $(k,\ell) = (-1,-1)$ for the Cartesian product of Sets;
- $(k,\ell) = (0,-1)$ and $(-1,0)$ for the tensor of sets with pointed sets of Definition 4.2.1.1.1;
- $(i,k) = (-1,0)$ for the left and right tensor products of pointed sets of Sections 4.3 and 4.4;
- $(k,\ell) = (-1,-1)$ for the smash product of pointed sets of Section 4.5.

In this chapter, we will carefully define and study bilinearity for pointed sets, as well as all the tensor products described above. Then, in **??**, we will extend these to tensor products involving also monoids and commutative monoids, which will end up covering all cases up to $k, \ell \leq 2$, and hence *all* cases since $\mathbb{E}_k$-monoids on Sets are the same as $\mathbb{E}_2$-monoids on Sets when $k \geq 2$.

Contents

00C4 4.1 Bilinear Morphisms of Pointed Sets

00C5 4.1.1 Left Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

00C6 **Definition 4.1.1.1.1.** A **left bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon X \times Y \to Z$$

satisfying the following condition:[1,2]

(★) *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt} \times Y \arrow[ur, "\mathrm{id}_{\mathrm{pt}} \times \epsilon_Y"] \arrow[d, "{[x_0] \times \mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

00C7 **Definition 4.1.1.1.2.** The **set of left bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is left bilinear}\}.$$

[1] *Slogan:* The map f is left bilinear if it preserves basepoints in its first argument.

[2] Succinctly, f is bilinear if we have

$$f(x_0, y) = z_0$$

00C8 4.1.2 Right Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

00C9 **Definition 4.1.2.1.1.** A **right bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f \colon X \times Y \to Z$$

satisfying the following condition:[3,4]

($\star$) *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd}
& \mathrm{pt} \times \mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X \times \mathrm{pt} \arrow[ur, "\epsilon_X \times \mathrm{id}_{\mathrm{pt}}"] \arrow[d, "\mathrm{id}_X \times [y_0]"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X \times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

00CA **Definition 4.1.2.1.2.** The **set of right bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z) \stackrel{\text{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is right bilinear}\}.$$

00CB 4.1.3 Bilinear Morphisms of Pointed Sets

Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets.

00CC **Definition 4.1.3.1.1.** A **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f \colon X \times Y \to Z$$

that is both left bilinear and right bilinear.

---

for each $y \in Y$.

[3] *Slogan:* The map f is right bilinear if it preserves basepoints in its second argument.

[4] Succinctly, f is bilinear if we have

$$f(x, y_0) = z_0$$

00CD **Remark 4.1.3.1.2.** In detail, a **bilinear morphism of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is a map of sets

$$f\colon (X \times Y, (x_0, y_0)) \to (Z, z_0)$$

satisfying the following conditions:[5,6]

1. *Left Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
\mathrm{pt}\times Y \arrow[ur, "{\mathrm{id}_{\mathrm{pt}}\times\epsilon_Y}"] \arrow[d, "{[x_0]\times\mathrm{id}_Y}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X\times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

   commutes, i.e. for each $y \in Y$, we have

$$f(x_0, y) = z_0.$$

2. *Right Unital Bilinearity.* The diagram

$$\begin{tikzcd}
 & \mathrm{pt}\times\mathrm{pt} \arrow[dr, dashed, "\sim"] & \\
X\times \mathrm{pt} \arrow[ur, "{\epsilon_X\times\mathrm{id}_{\mathrm{pt}}}"] \arrow[d, "{\mathrm{id}_X\times[y_0]}"'] & & \mathrm{pt} \arrow[d, "{[z_0]}"] \\
X\times Y \arrow[rr, "f"'] & & Z
\end{tikzcd}$$

   commutes, i.e. for each $x \in X$, we have

$$f(x, y_0) = z_0.$$

00CE **Definition 4.1.3.1.3.** The **set of bilinear morphisms of pointed sets from** $(X \times Y, (x_0, y_0))$ **to** (Z, z_0) is the set $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ defined by

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \overset{\mathrm{def}}{=} \{f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \mid f \text{ is bilinear}\}.$$

for each $x \in X$.

[5] *Slogan:* The map f is bilinear if it preserves basepoints in each argument.

[6] Succinctly, f is bilinear if we have

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

00CF 4.2 Tensors and Cotensors of Pointed Sets by Sets

00CG 4.2.1 Tensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

00CH **Definition 4.2.1.1.1.** The **tensor of** (X, x_0) **by** A[7] is the pointed set[8] $A \odot (X, x_0)$ satisfying the following universal property:

(**UP**) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00CJ **Remark 4.2.1.1.2.** The universal property in Definition 4.2.1.1.1 is equivalent to the following one:

(**UP**) We have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K) \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times X, K) \,\middle|\, \begin{array}{l}\text{for each } a \in A, \text{ we}\\ \text{have } f(a, x_0) = k_0\end{array} \right\}.$$

Proof. We claim we have a bijection

$$\mathsf{Sets}(A, \mathsf{Sets}_*(X, K)) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$. Indeed, this bijection is a restriction of the bijection

$$\mathsf{Sets}(A, \mathsf{Sets}(X, K)) \cong \mathsf{Sets}(A \times X, K)$$

of Item 2 of Proposition 2.1.3.1.2:

- A map
$$\begin{aligned} \xi \colon A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a \colon X \to K), \end{aligned}$$

for each $x \in X$ and each $y \in Y$.

[7] *Further Terminology:* Also called the **copower of** (X, x_0) **by** A.

[8] *Further Notation:* Often written $A \odot X$ for simplicity.

in $\mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$ gets sent to the map

$$\xi^\dagger \colon A \times X \to K$$

defined by

$$\xi^\dagger(a, x) \stackrel{\text{def}}{=} \xi_a(x)$$

for each $(a, x) \in A \times X$, which indeed lies in $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$, as we have

$$\begin{aligned}\xi^\dagger(a, x_0) &\stackrel{\text{def}}{=} \xi_a(x_0)\\ &\stackrel{\text{def}}{=} k_0\end{aligned}$$

for each $a \in A$, where we have used that $\xi_a \in \mathsf{Sets}_*(X, K)$ is a morphism of pointed sets.

- Conversely, a map

$$\xi \colon A \times X \to K$$

in $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times X, K)$ gets sent to the map

$$\begin{aligned}\xi^\dagger \colon A &\longrightarrow \mathsf{Sets}_*(X, K),\\ a &\longmapsto \left(\xi^\dagger_a \colon X \to K\right),\end{aligned}$$

where

$$\xi^\dagger_a \colon X \to K$$

is the map defined by

$$\xi^\dagger_a(x) \stackrel{\text{def}}{=} \xi(a, x)$$

for each $x \in X$, and indeed lies in $\mathsf{Sets}_*(X, K)$, as we have

$$\begin{aligned}\xi^\dagger_a(x_0) &\stackrel{\text{def}}{=} \xi(a, x_0)\\ &\stackrel{\text{def}}{=} k_0.\end{aligned}$$

This finishes the proof. $\square$

00CK **Construction 4.2.1.1.3.** Concretely, the **tensor of** (X, x_0) **by** A is the pointed set $A \odot (X, x_0)$ consisting of:

- *The Underlying Set.* The set $A \odot X$ given by

$$A \odot X \cong \bigvee_{a \in A} (X, x_0),$$

where $\bigvee_{a \in A}(X, x_0)$ is the wedge product of the A-indexed family $((X, x_0))_{a \in A}$ of Definition 3.3.2.1.1.

- *The Basepoint.* The point $[(a, x_0)] = [(a', x_0)]$ of $\bigvee_{a\in A}(X, x_0)$.

Proof. (Proven below in a bit.) □

00CL **Notation 4.2.1.1.4.** We write $a \odot x$ for the element $[(a, x)]$ of

$$\begin{aligned} A \odot X &\cong \bigvee_{a\in A} (X, x_0) \\ &\stackrel{\text{def}}{=} \left(\coprod_{i\in I} X_i\right)/\sim. \end{aligned}$$

00CM **Remark 4.2.1.1.5.** Taking the tensor of any element of A with the basepoint x_0 of X leads to the same element in $A \odot X$, i.e. we have

$$a \odot x_0 = a' \odot x_0,$$

for each $a, a' \in A$. This is due to the equivalence relation $\sim$ on

$$\bigvee_{a\in A} (X, x_0) \stackrel{\text{def}}{=} \coprod_{a\in A} X/\sim$$

identifying (a, x_0) with (a', x_0), so that the equivalence class $a \odot x_0$ is independent from the choice of $a \in A$.

Proof. We claim we have a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

- *Map I.* We define a map

$$\Phi_K : \mathsf{Sets}_*(A \odot X, K) \to \mathsf{Sets}(A, \mathsf{Sets}_*(X, K))$$

by sending a morphism of pointed sets

$$\xi : (A \odot X, a \odot x_0) \to (K, k_0)$$

to the map of sets

$$\begin{aligned} \xi^\dagger : A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a : X \to K), \end{aligned}$$

where

$$\xi_a : (X, x_0) \to (K, k_0)$$

is the morphism of pointed sets defined by

$$\xi_a(x) \stackrel{\text{def}}{=} \xi(a \odot x)$$

for each $x \in X$. Note that we have

$$\begin{aligned}\xi_a(x_0) &\overset{\text{def}}{=} \xi(a \odot x_0) \\ &= k_0,\end{aligned}$$

so that ξ_a is indeed a morphism of pointed sets, where we have used that ξ is a morphism of pointed sets.

- *Map II.* We define a map

$$\Psi_K \colon \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)) \to \mathsf{Sets}_*(A \odot X, K)$$

given by sending a map

$$\begin{aligned}\xi \colon A &\longrightarrow \mathsf{Sets}_*(X, K), \\ a &\longmapsto (\xi_a \colon X \to K),\end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger \colon (A \odot X, a \odot x_0) \to (K, k_0)$$

defined by

$$\xi^\dagger(a \odot x) \overset{\text{def}}{=} \xi_a(x)$$

for each $a \odot x \in A \odot X$. Note that $\xi^\dagger$ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\xi^\dagger(a \odot x_0) &\overset{\text{def}}{=} \xi_a(x_0) \\ &= k_0,\end{aligned}$$

where we have used that $\xi(a) \in \mathsf{Sets}_*(X, K)$ is a morphism of pointed sets.

- *Invertibility I.* We claim that

$$\Psi_K \circ \Phi_K = \mathrm{id}_{\mathsf{Sets}_*(A \odot X, K)}.$$

Indeed, given a morphism of pointed sets

$$\xi \colon (A \odot X, a \odot x_0) \to (K, k_0),$$

we have

$$\begin{aligned}[\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi)) \\ &= \Psi_K(\llbracket a \mapsto \llbracket x \mapsto \xi(a \odot x)\rrbracket\rrbracket) \\ &= \Psi_K(\llbracket a' \mapsto \llbracket x' \mapsto \xi(a' \odot x')\rrbracket\rrbracket) \\ &= \llbracket a \odot x \mapsto \mathrm{ev}_x(\mathrm{ev}_a(\llbracket a' \mapsto \llbracket x' \mapsto \xi(a' \odot x')\rrbracket\rrbracket))\rrbracket \\ &= \llbracket a \odot x \mapsto \mathrm{ev}_x(\llbracket x' \mapsto \xi(a \odot x')\rrbracket)\rrbracket \\ &= \llbracket a \odot x \mapsto \xi(a \odot x)\rrbracket \\ &= \xi.\end{aligned}$$

- *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \mathrm{id}_{\mathsf{Sets}(A,\mathsf{Sets}_*(X,K))}.$$

Indeed, given a morphism $\xi\colon A \to \mathsf{Sets}_*(X,K)$, we have

$$\begin{aligned}
[\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\
&= \Phi_K([\![a \odot x \mapsto \xi_a(x)]\!]) \\
&= [\![a \mapsto [\![x \mapsto \xi_a(x)]\!]]\!] \\
&= [\![a \mapsto \xi(a)]\!] \\
&= \xi.
\end{aligned}$$

- *Naturality of Φ.* We need to show that, given a morphism of pointed sets

$$\phi\colon (K,k_0) \to (K',k'_0),$$

the diagram

$$\begin{tikzcd}
\mathsf{Sets}_*(A \odot X, K) \arrow[r, "\Phi_K"] \arrow[d, "\phi_*"'] & \mathsf{Sets}(A, \mathsf{Sets}_*(X,K)) \arrow[d, "(\phi_*)_*"] \\
\mathsf{Sets}_*(A \odot X, K') \arrow[r, "\Phi_{K'}"'] & \mathsf{Sets}(A, \mathsf{Sets}_*(X,K'))
\end{tikzcd}$$

commutes. Indeed, given a morphism of pointed sets

$$\xi\colon (A \odot X, a \odot x_0) \to (K,k_0),$$

we have

$$\begin{aligned}
[\Phi_{K'} \circ \phi_*](\xi) &= \Phi_{K'}(\phi_*(\xi)) \\
&= \Phi_{K'}(\phi \circ \xi) \\
&= (\phi \circ \xi)^{\dagger} \\
&= [\![a \mapsto \phi \circ \xi(a \odot -)]\!] \\
&= [\![a \mapsto \phi_*(\xi(a \odot -))]\!] \\
&= (\phi_*)_*([\![a \mapsto \xi(a \odot -]\!])) \\
&= (\phi_*)_*(\Phi_K(\xi)) \\
&= [(\phi_*)_* \circ \Phi_K](\xi).
\end{aligned}$$

- *Naturality of Ψ.* Since Φ is natural and Φ is a componentwise inverse to Ψ, it follows from Item 2 of Proposition 8.8.6.1.2 that Ψ is also natural.

This finishes the proof. □

00CN **Proposition 4.2.1.1.6.** Let (X, x_0) be a pointed set and let A be a set.

00CP 1. *Functoriality.* The assignments $A, (X, x_0), (A, (X, x_0))$ define functors

$$\begin{aligned} A \odot -\colon \mathsf{Sets}_* &\to \mathsf{Sets}_*, \\ - \odot X\colon \mathsf{Sets} &\to \mathsf{Sets}_*, \\ -_1 \odot -_2\colon \mathsf{Sets} \times \mathsf{Sets}_* &\to \mathsf{Sets}_*. \end{aligned}$$

In particular, given:

- A map of sets $f\colon A \to B$;
- A pointed map $\phi\colon (X, x_0) \to (Y, y_0)$;

the induced map

$$f \odot \phi\colon A \odot X \to B \odot Y$$

is given by

$$[f \odot \phi](a \odot x) \stackrel{\text{def}}{=} f(a) \odot \phi(x)$$

for each $a \odot x \in A \odot X$.

00CQ 2. *Adjointness I.* We have an adjunction

$$(- \odot X \dashv \mathsf{Sets}_*(X, -))\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "- \odot X"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathsf{Sets}_*(X, -)"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Sets}_*(A \odot X, K) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(X, K)),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00CR 3. *Adjointness II.* We have an adjunctions

$$(A \odot - \dashv A \pitchfork -)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "A \odot -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "A \pitchfork -"] \end{tikzcd}$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(A \odot X, Y) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, A \pitchfork Y),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00CS 4. *As a Weighted Colimit.* We have

$$A \odot X \cong \operatorname{colim}^{[A]}(X),$$

where in the right hand side we write:

- A for the functor $A\colon \mathrm{pt} \to \mathsf{Sets}$ picking $A \in \mathrm{Obj}(\mathsf{Sets})$;
- X for the functor $X\colon \mathrm{pt} \to \mathsf{Sets}_*$ picking $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00CT 5. *Iterated Tensors.* We have an isomorphism of pointed sets

$$A \odot (B \odot X) \cong (A \times B) \odot X,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$ and $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00CU 6. *Interaction With Homs.* We have a natural isomorphism

$$\mathsf{Sets}_*(A \odot X, -) \cong A \pitchfork \mathsf{Sets}_*(X, -).$$

00CV 7. *The Tensor Evaluation Map.* For each $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{ev}^{\odot}_{X,Y}\colon \mathsf{Sets}_*(X, Y) \odot X \to Y,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{ev}^{\odot}_{X,Y}(f \odot x) \stackrel{\text{def}}{=} f(x)$$

for each $f \odot x \in \mathsf{Sets}_*(X, Y) \odot X$.

00CW 8. *The Tensor Coevaluation Map.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$ and each $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{coev}^{\odot}_{A,X}\colon A \to \mathsf{Sets}_*(X, A \odot X),$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $X \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{coev}^{\odot}_{A,X}(a) \stackrel{\text{def}}{=} [\![x \mapsto a \odot x]\!]$$

for each $a \in A$.

Proof. Item 1, Functoriality: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 2, Adjointness I: This is simply a rephrasing of Definition 4.2.1.1.1.
Item 3, : Adjointness II: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 4, As a Weighted Colimit: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.

Item 5, Iterated Tensors: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 6, Interaction With Homs: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 7, The Tensor Evaluation Map: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 8, The Tensor Coevaluation Map: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$. □

00CX 4.2.2 Cotensors of Pointed Sets by Sets

Let (X, x_0) be a pointed set and let A be a set.

00CY **Definition 4.2.2.1.1.** The **cotensor of** (X, x_0) **by** A[9] is the pointed set[10] $A \pitchfork (X, x_0)$ satisfying the following universal property:

(**UP**) We have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00CZ **Remark 4.2.2.1.2.** The universal property of Definition 4.2.2.1.1 is equivalent to the following one:

(**UP**) We have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X)$ is the set defined by

$$\mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X) \stackrel{\text{def}}{=} \left\{ f \in \mathsf{Sets}(A \times K, X) \;\middle|\; \begin{array}{l} \text{for each } a \in A, \text{ we} \\ \text{have } f(a, k_0) = x_0 \end{array} \right\}.$$

Proof. This follows from the bijection

$$\mathsf{Sets}(A, \mathsf{Sets}_*(K, X)) \cong \mathsf{Sets}^{\otimes}_{\mathbb{E}_0}(A \times K, X),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ constructed in the proof of Remark 4.2.1.1.2. □

00D0 **Construction 4.2.2.1.3.** Concretely, the **cotensor of** (X, x_0) **by** A is the pointed set $A \pitchfork (X, x_0)$ consisting of:

[9] *Further Terminology:* Also called the **power of** (X, x_0) **by** A.
[10] *Further Notation:* Often written $A \pitchfork X$ for simplicity.

- *The Underlying Set.* The set $A \pitchfork X$ given by

  $$A \pitchfork X \cong \bigwedge_{a\in A}(X, x_0),$$

  where $\bigwedge_{a\in A}(X, x_0)$ is the smash product of the A-indexed family $((X, x_0))_{a\in A}$ of Definition 4.6.1.1.1.

- *The Basepoint.* The point $[(x_0)_{a\in A}] = [(x_0, x_0, x_0, \ldots)]$ of $\bigwedge_{a\in A}(X, x_0)$.

Proof. We claim we have a bijection

$$\mathsf{Sets}_*(K, A \pitchfork X) \cong \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

natural in $(K, k_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

- *Map I.* We define a map

  $$\Phi_K : \mathsf{Sets}_*(K, A \pitchfork X) \to \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)),$$

  by sending a morphism of pointed sets

  $$\xi : (K, k_0) \to (A \pitchfork X, [(x_0)_{a\in A}])$$

  to the map of sets

  $$\begin{aligned} \xi^\dagger : A &\longrightarrow \mathsf{Sets}_*(K, X), \\ a &\longmapsto (\xi_a : K \to X), \end{aligned}$$

  where

  $$\xi_a : (K, k_0) \to (X, x_0)$$

  is the morphism of pointed sets defined by

  $$\xi_a(k) = \begin{cases} x_a^k & \text{if } \xi(k) \neq [(x_0)_{a\in A}], \\ x_0 & \text{if } \xi(k) = [(x_0)_{a\in A}] \end{cases}$$

  for each $k \in K$, where x_a^k is the ath component of $\xi(k) = \left[\left(x_a^k\right)_{a\in A}\right]$. Note that:

  1. The definition of $\xi_a(k)$ is independent of the choice of equivalence class. Indeed, suppose we have

     $$\begin{aligned} \xi(k) &= \left[\left(x_a^k\right)_{a\in A}\right] \\ &= \left[\left(y_a^k\right)_{a\in A}\right] \end{aligned}$$

with $x_a^k \neq y_a^k$ for some $a \in A$. Then there exist $a_x, a_y \in A$ such that $x_{a_x}^k = y_{a_y}^k = x_0$. The equivalence relation $\sim$ on $\prod_{a\in A} X$ then forces

$$\begin{aligned}\left[\left(x_a^k\right)_{a\in A}\right] &= \left[(x_0)_{a\in A}\right],\\ \left[\left(y_a^k\right)_{a\in A}\right] &= \left[(x_0)_{a\in A}\right],\end{aligned}$$

however, and $\xi_a(k)$ is defined to be x_0 in this case.

2. The map ξ_a is indeed a morphism of pointed sets, as we have

$$\xi_a(k_0) = x_0$$

since $\xi(k_0) = \left[(x_0)_{a\in A}\right]$ as ξ is a morphism of pointed sets and $\xi_a(k_0)$, defined to be the ath component of $\left[(x_0)_{a\in A}\right]$, is equal to x_0.

- *Map II.* We define a map

$$\Psi_K \colon \mathsf{Sets}(A, \mathsf{Sets}_*(K, X)) \to \mathsf{Sets}_*(K, A \pitchfork X),$$

given by sending a map

$$\begin{aligned}\xi \colon A &\longrightarrow \mathsf{Sets}_*(K, X),\\ a &\longmapsto (\xi_a \colon K \to X),\end{aligned}$$

to the morphism of pointed sets

$$\xi^\dagger \colon (K, k_0) \to \left(A \pitchfork X, \left[(x_0)_{a\in A}\right]\right)$$

defined by

$$\xi^\dagger(k) \stackrel{\text{def}}{=} \left[(\xi_a(k))_{a\in A}\right]$$

for each $k \in K$. Note that $\xi^\dagger$ is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\xi^\dagger(k_0) &\stackrel{\text{def}}{=} \left[(\xi_a(k_0))_{a\in A}\right]\\ &= x_0,\end{aligned}$$

where we have used that $\xi_a \in \mathsf{Sets}_*(K, X)$ is a morphism of pointed sets for each $a \in A$.

- *Naturality of Ψ.* We need to show that, given a morphism of pointed sets

$$\phi \colon (K, k_0) \to (K', k_0'),$$

the diagram

$$\begin{tikzcd}
\mathsf{Sets}(A, \mathsf{Sets}_*(K', X)) \arrow[r, "\Psi_{K'}"] \arrow[d, "(\phi^*)_*"'] & \mathsf{Sets}_*(K', A \pitchfork X) \arrow[d, "\phi^*"] \\
\mathsf{Sets}(A, \mathsf{Sets}_*(K, X)) \arrow[r, "\Psi_K"'] & \mathsf{Sets}_*(K, A \pitchfork X)
\end{tikzcd}$$

commutes. Indeed, given a map of sets

$$\begin{aligned}
\xi \colon A &\longrightarrow \mathsf{Sets}_*(K', X), \\
a &\longmapsto (\xi_a \colon K' \to X),
\end{aligned}$$

we have

$$\begin{aligned}
[\Psi_K \circ (\phi^*)_*](\xi) &= \Psi_K((\phi^*)_*(\xi)) \\
&= \Psi_K((\phi^*)_*(\llbracket a \mapsto \xi_a \rrbracket)) \\
&= \Psi_K((\llbracket a \mapsto \phi^*(\xi_a) \rrbracket)) \\
&= \Psi_K((\llbracket a \mapsto \llbracket k \mapsto \xi_a(\phi(k)) \rrbracket \rrbracket)) \\
&= \llbracket k \mapsto [(\xi_a(\phi(k)))_{a \in A}] \rrbracket \\
&= \phi^*\Big(\llbracket k' \mapsto \Big[(\xi_a(k'))_{a \in A}\Big] \rrbracket\Big) \\
&= \phi^*(\Psi_{K'}(\xi)) \\
&= [\phi^* \circ \Psi_{K'}](\xi).
\end{aligned}$$

- *Naturality of* Φ. Since Ψ is natural and Ψ is a componentwise inverse to Φ, it follows from Item 2 of Proposition 8.8.6.1.2 that Φ is also natural.
- *Invertibility I.* We claim that

$$\Psi_K \circ \Phi_K = \mathrm{id}_{\mathsf{Sets}_*(K, A \pitchfork X)}.$$

Indeed, given a morphism of pointed sets

$$\xi \colon (K, k_0) \to (A \pitchfork X, [(x_0)_{a \in A}])$$

we have

$$\begin{aligned}
[\Psi_K \circ \Phi_K](\xi) &= \Psi_K(\Phi_K(\xi)) \\
&= \Psi_K(\llbracket a \mapsto \xi_a \rrbracket) \\
&= \Psi_K(\llbracket a' \mapsto \xi_{a'} \rrbracket) \\
&= \llbracket k \mapsto \Big[(\mathrm{ev}_a(\llbracket a' \mapsto \xi_{a'}(k) \rrbracket))_{a \in A}\Big] \rrbracket \\
&= \llbracket k \mapsto [(\xi_a(k))_{a \in A}] \rrbracket.
\end{aligned}$$

Now, we have two cases:

1. If $\xi(k) = [(x_0)_{a\in A}]$, we have

$$\begin{aligned}
[\Psi_K \circ \Phi_K](\xi) &= \cdots \\
&= [\![k \mapsto [(\xi_a(k))_{a\in A}]]\!] \\
&= [\![k \mapsto [(x_0)_{a\in A}]]\!] \\
&= [\![k \mapsto \xi(k)]\!] \\
&= \xi.
\end{aligned}$$

2. If $\xi(k) \neq [(x_0)_{a\in A}]$ and $\xi(k) = \left[\left(x_a^k\right)_{a\in A}\right]$ instead, we have

$$\begin{aligned}
[\Psi_K \circ \Phi_K](\xi) &= \cdots \\
&= [\![k \mapsto [(\xi_a(k))_{a\in A}]]\!] \\
&= [\![k \mapsto \left[\left(x_a^k\right)_{a\in A}\right]]\!] \\
&= [\![k \mapsto \xi(k)]\!] \\
&= \xi.
\end{aligned}$$

In both cases, we have $[\Psi_K \circ \Phi_K](\xi) = \xi$, and thus we are done.

- *Invertibility II.* We claim that

$$\Phi_K \circ \Psi_K = \mathrm{id}_{\mathsf{Sets}(A,\mathsf{Sets}_*(K,X))}.$$

Indeed, given a morphism $\xi \colon A \to \mathsf{Sets}_*(K, X)$, we have

$$\begin{aligned}
[\Phi_K \circ \Psi_K](\xi) &= \Phi_K(\Psi_K(\xi)) \\
&= \Phi_K([\![k \mapsto [(\xi_a(k))_{a\in A}]]\!]) \\
&= [\![a \mapsto [\![k \mapsto \xi_a(k)]\!]]\!] \\
&= \xi
\end{aligned}$$

This finishes the proof. □

00D1 **Proposition 4.2.2.1.4.** Let (X, x_0) be a pointed set and let A be a set.

00D2 1. *Functoriality.* The assignments $A, (X, x_0), (A, (X, x_0))$ define functors

$$\begin{aligned}
A \pitchfork - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\
- \pitchfork X &\colon \mathsf{Sets}^{\mathsf{op}} \to \mathsf{Sets}_*, \\
-_1 \pitchfork -_2 &\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.
\end{aligned}$$

In particular, given:

- A map of sets $f\colon A\to B$;
- A pointed map $\phi\colon (X,x_0)\to(Y,y_0)$;

the induced map

$$f\odot\phi\colon A\pitchfork X\to B\pitchfork Y$$

is given by

$$[f\odot\phi]([(x_a)_{a\in A}])\stackrel{\text{def}}{=}\left[\left(\phi\left(x_{f(a)}\right)\right)_{a\in A}\right]$$

for each $[(x_a)_{a\in A}]\in A\pitchfork X$.

00D3 2. *Adjointness I.* We have an adjunction

$$\left(-\pitchfork X\dashv \mathsf{Sets}_*(-,X)\right)\colon\quad \begin{tikzcd} \mathsf{Sets}^{\mathsf{op}} \arrow[r, bend left, "-\pitchfork X"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathsf{Sets}_*(-{,}X)"] \end{tikzcd}$$

witnessed by a bijection

$$\mathsf{Sets}_*^{\mathsf{op}}(A\pitchfork X,K)\cong\mathsf{Sets}(A,\mathsf{Sets}_*(K,X)),$$

i.e. by a bijection

$$\mathsf{Sets}_*(K,A\pitchfork X)\cong\mathsf{Sets}(A,\mathsf{Sets}_*(K,X)),$$

natural in $A\in\mathrm{Obj}(\mathsf{Sets})$ and $X,Y\in\mathrm{Obj}(\mathsf{Sets}_*)$.

00D4 3. *Adjointness II.* We have an adjunctions

$$(A\odot-\dashv A\pitchfork-)\colon\quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "A\odot-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "A\pitchfork-"] \end{tikzcd}$$

witnessed by a bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(A\odot X,Y)\cong\mathrm{Hom}_{\mathsf{Sets}_*}(X,A\pitchfork Y),$$

natural in $A\in\mathrm{Obj}(\mathsf{Sets})$ and $X,Y\in\mathrm{Obj}(\mathsf{Sets}_*)$.

00D5 4. *As a Weighted Limit.* We have

$$A\pitchfork X\cong\lim^{[A]}(X),$$

where in the right hand side we write:

- A for the functor $A\colon \mathrm{pt}\to\mathsf{Sets}$ picking $A\in\mathrm{Obj}(\mathsf{Sets})$;

- X for the functor $X: \mathrm{pt} \to \mathsf{Sets}_*$ picking $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00D6 5. *Iterated Cotensors.* We have an isomorphism of pointed sets

$$A \pitchfork (B \pitchfork X) \cong (A \times B) \pitchfork X,$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$ and $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00D7 6. *Commutativity With Homs.* We have natural isomorphisms

$$\begin{aligned} A \pitchfork \mathsf{Sets}_*(X, -) &\cong \mathsf{Sets}_*(A \odot X, -), \\ A \pitchfork \mathsf{Sets}_*(-, Y) &\cong \mathsf{Sets}_*(-, A \pitchfork Y). \end{aligned}$$

00D8 7. *The Cotensor Evaluation Map.* For each $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, we have a map

$$\mathrm{ev}^{\pitchfork}_{X,Y}: X \to \mathsf{Sets}_*(X, Y) \pitchfork Y,$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, and given by

$$\mathrm{ev}^{\pitchfork}_{X,Y}(x) \stackrel{\text{def}}{=} \left[(f(x))_{f \in \mathsf{Sets}_*(X,Y)}\right]$$

for each $x \in X$.

00D9 8. *The Cotensor Coevaluation Map.* For each $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have a map

$$\mathrm{coev}^{\pitchfork}_{A,X}: A \to \mathsf{Sets}_*(A \pitchfork X, X),$$

natural in $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ and $A \in \mathrm{Obj}(\mathsf{Sets})$, and given by

$$\mathrm{coev}^{\pitchfork}_{A,X}(a) \stackrel{\text{def}}{=} \llbracket [(x_b)_{b \in A}] \mapsto x_a \rrbracket$$

for each $a \in A$.

Proof. *Item 1, Functoriality*: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 2, Adjointness I: This is simply a rephrasing of Definition 4.2.2.1.1.
Item 3, : Adjointness II: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 4, As a Weighted Limit: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 5, Iterated Cotensors: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 6, Commutativity With Homs: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 7, The Cotensor Evaluation Map: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$.
Item 8, The Cotensor Coevaluation Map: This is the special case of **??** of **??** for when $\mathcal{C} = \mathsf{Sets}_*$. □

00DA 4.3 The Left Tensor Product of Pointed Sets

00DB 4.3.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

00DC **Definition 4.3.1.1.1.** The **left tensor product of pointed sets** is the functor[11]

$$\lhd \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{\mathrm{id} \times 忘} \mathsf{Sets}_* \times \mathsf{Sets} \xrightarrow{\beta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*, \mathsf{Sets}}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*,$$

where:

- $忘 \colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\beta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*, \mathsf{Sets}} \colon \mathsf{Sets}_* \times \mathsf{Sets} \xrightarrow{\cong} \mathsf{Sets} \times \mathsf{Sets}_*$ is the braiding of Cats_2, i.e. the functor witnessing the isomorphism

$$\mathsf{Sets}_* \times \mathsf{Sets} \cong \mathsf{Sets} \times \mathsf{Sets}_*.$$

- $\odot \colon \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the tensor functor of Item 1 of Proposition 4.2.1.1.6.

00DD **Remark 4.3.1.1.2.** The left tensor product of pointed sets satisfies the following natural bijection:

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathrm{Hom}^{\otimes, \mathrm{L}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

1. Pointed maps $f \colon X \lhd Y \to Z$.
2. Maps of sets $f \colon X \times Y \to Z$ satisfying $f(x_0, y) = z_0$ for each $y \in Y$.

00DE **Remark 4.3.1.1.3.** The left tensor product of pointed sets may be described as follows:

- The left tensor product of (X, x_0) and (Y, y_0) is the pair $((X \lhd Y, x_0 \lhd y_0), \iota)$ consisting of
  - A pointed set $(X \lhd Y, x_0 \lhd y_0)$;
  - A left bilinear morphism of pointed sets $\iota \colon (X \times Y, (x_0, y_0)) \to X \lhd Y$;

[11] *Further Notation:* Also written $\lhd_{\mathsf{Sets}_*}$.

satisfying the following universal property:

(UP) Given another such pair $((Z, z_0), f)$ consisting of

* $*$ A pointed set (Z, z_0);
* $*$ A left bilinear morphism of pointed sets $f \colon (X \times Y, (x_0, y_0)) \to X \lhd Y$;

there exists a unique morphism of pointed sets $X \lhd Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \lhd Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

commute.

00DF **Construction 4.3.1.1.4.** In detail, the **left tensor product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \lhd Y, [x_0])$ consisting of

- *The Underlying Set.* The set $X \lhd Y$ defined by
$$\begin{aligned} X \lhd Y &\stackrel{\text{def}}{=} |Y| \odot X \\ &\cong \bigvee_{y \in Y} (X, x_0), \end{aligned}$$
where $|Y|$ denotes the underlying set of (Y, y_0);
- *The Underlying Basepoint.* The point $[(y_0, x_0)]$ of $\bigvee_{y \in Y}(X, x_0)$, which is equal to $[(y, x_0)]$ for any $y \in Y$.

00DG **Notation 4.3.1.1.5.** We write[12] $x \lhd y$ for the element $[(y, x)]$ of
$$X \lhd Y \cong |Y| \odot X.$$

00DH **Remark 4.3.1.1.6.** Employing the notation introduced in Notation 4.3.1.1.5, we have
$$x_0 \lhd y_0 = x_0 \lhd y$$
for each $y \in Y$, and
$$x_0 \lhd y = x_0 \lhd y'$$
for each $y, y' \in Y$.

00DJ **Proposition 4.3.1.1.7.** Let (X, x_0) and (Y, y_0) be pointed sets.

[12] *Further Notation:* Also written $x \lhd_{\mathsf{Sets}_*} y$.

00DK 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \lhd Y$ define functors

$$\begin{aligned} X \lhd - &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ - \lhd Y &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ -_1 \lhd -_2 &\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f &\colon (X, x_0) \to (A, a_0), \\ g &\colon (Y, y_0) \to (B, b_0), \end{aligned}$$

the induced map

$$f \lhd g \colon X \lhd Y \to A \lhd B$$

is given by

$$[f \lhd g](x \lhd y) \stackrel{\text{def}}{=} f(x) \lhd g(y)$$

for each $x \lhd y \in X \lhd Y$.

00DL 2. *Adjointness I.* We have an adjunction

$$\left(- \lhd Y \dashv [Y, -]^{\lhd}_{\mathsf{Sets}_*}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \lhd Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[Y, -]^{\lhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\left(X, [Y, Z]^{\lhd}_{\mathsf{Sets}_*}\right)$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X, Y]^{\lhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 4.3.2.1.1.

00DM 3. *Adjointness II.* The functor

$$X \lhd - \colon \mathsf{Sets}_* \to \mathsf{Sets}_*$$

does not admit a right adjoint.

00DN 4. *Adjointness III.* We have a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|Y|, \mathsf{Sets}_*(X, Z))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness I: This follows from Item 3 of Proposition 4.2.1.1.6.
Item 3, Adjointness II: For $X \lhd -$ to admit a right adjoint would require it to preserve colimits by **??** of **??**. However, we have

$$\begin{aligned} X \lhd \mathrm{pt} &\stackrel{\text{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\ncong \mathrm{pt}, \end{aligned}$$

and thus we see that $X \lhd -$ does not have a right adjoint.
Item 4, Adjointness III: This follows from Item 2 of Proposition 4.2.1.1.6. □

00DP **Remark 4.3.1.1.8.** Here is some intuition on why $X \lhd -$ fails to be a left adjoint. Item 4 of Proposition 4.3.1.1.7 states that we have a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|Y|, \mathsf{Sets}_*(X, Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(Y, \mathbf{Sets}_*(X, Z)),$$

also holds, which would give $X \lhd - \dashv \mathbf{Sets}_*(X, -)$. However, such a bijection would require every map

$$f \colon X \lhd Y \to Z$$

to satisfy

$$f(x \lhd y_0) = z_0$$

for each $x \in X$, whereas we are imposing such a basepoint preservation condition only for elements of the form $x_0 \lhd y$. Thus $\mathbf{Sets}_*(X, -)$ can't be a right adjoint for $X \lhd -$, and as shown by Item 3 of Proposition 4.3.1.1.7, no functor can.[13]

00DQ 4.3.2 The Left Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

00DR **Definition 4.3.2.1.1.** The **left internal Hom of pointed sets** is the functor

$$[-, -]^{\lhd}_{\mathsf{Sets}_*} \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

[13]The functor $\mathbf{Sets}_*(X, -)$ is instead right adjoint to $X \wedge -$, the smash product of pointed sets of Definition 4.5.1.1.1. See Item 2 of Proposition 4.5.1.1.9.

defined as the composition

$$\mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{忘 \times \mathsf{id}} \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{\pitchfork} \mathsf{Sets}_*,$$

where:

- $忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\pitchfork\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the cotensor functor of Item 1 of Proposition 4.2.2.1.4.

Proof. For a proof that $[-,-]^{\lhd}_{\mathsf{Sets}_*}$ is indeed the left internal Hom of Sets_* with respect to the left tensor product of pointed sets, see Item 2 of Proposition 4.3.1.1.7. □

00DS **Remark 4.3.2.1.2.** The left internal Hom of pointed sets satisfies the following universal property:

$$\mathsf{Sets}_*(X \lhd Y, Z) \cong \mathsf{Sets}_*\Big(X, [Y,Z]^{\lhd}_{\mathsf{Sets}_*}\Big)$$

That is to say, the following data are in bijection:

1. Pointed maps $f\colon X \lhd Y \to Z$.
2. Pointed maps $f\colon X \to [Y,Z]^{\lhd}_{\mathsf{Sets}_*}$.

00DT **Remark 4.3.2.1.3.** In detail, the **left internal Hom of** (X, x_0) **and** (Y, y_0) is the pointed set $\Big([X,Y]^{\lhd}_{\mathsf{Sets}_*}, [(y_0)_{x\in X}]\Big)$ consisting of

- *The Underlying Set.* The set $[X,Y]^{\lhd}_{\mathsf{Sets}_*}$ defined by

$$\begin{aligned}[X,Y]^{\lhd}_{\mathsf{Sets}_*} &\overset{\mathrm{def}}{=} |X| \pitchfork Y\\ &\cong \bigwedge_{x\in X} (Y, y_0),\end{aligned}$$

  where $|X|$ denotes the underlying set of (X, x_0);

- *The Underlying Basepoint.* The point $[(y_0)_{x\in X}]$ of $\bigwedge_{x\in X}(Y, y_0)$.

00DU **Proposition 4.3.2.1.4.** Let (X, x_0) and (Y, y_0) be pointed sets.

00DV 1. *Functoriality.* The assignments $X, Y, (X,Y) \mapsto [X,Y]^{\lhd}_{\mathsf{Sets}_*}$ define functors

$$\begin{aligned}[X,-]^{\lhd}_{\mathsf{Sets}_*} &: \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ [-,Y]^{\lhd}_{\mathsf{Sets}_*} &: \mathsf{Sets}_*^{\mathsf{op}} \to \mathsf{Sets}_*,\\ [-_1,-_2]^{\lhd}_{\mathsf{Sets}_*} &: \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.\end{aligned}$$

In particular, given pointed maps

$$\begin{aligned} f &\colon (X, x_0) \to (A, a_0), \\ g &\colon (Y, y_0) \to (B, b_0), \end{aligned}$$

the induced map

$$[f, g]^{\lhd}_{\mathsf{Sets}_*} \colon [A, Y]^{\lhd}_{\mathsf{Sets}_*} \to [X, B]^{\lhd}_{\mathsf{Sets}_*}$$

is given by

$$[f, g]^{\lhd}_{\mathsf{Sets}_*}([(y_a)_{a\in A}]) \stackrel{\text{def}}{=} \left[\left(g\left(y_{f(x)}\right)\right)_{x\in X}\right]$$

for each $[(y_a)_{a\in A}] \in [A, Y]^{\lhd}_{\mathsf{Sets}_*}$.

00DW 2. *Adjointness I.* We have an adjunction

$$\left(- \lhd Y \dashv [Y, -]^{\lhd}_{\mathsf{Sets}_*}\right)\colon \quad \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "- \lhd Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_* \arrow[l, bend left, "{[Y,-]^{\lhd}_{\mathsf{Sets}_*}}"] \end{tikzcd},$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \lhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\left(X, [Y, Z]^{\lhd}_{\mathsf{Sets}_*}\right)$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00DX 3. *Adjointness II.* The functor

$$X \lhd - \colon \mathsf{Sets}_* \to \mathsf{Sets}_*$$

does not admit a right adjoint.

Proof. Item 1, Functoriality: Clear.
Item 2, Adjointness I: This is a repetition of Item 2 of Proposition 4.3.1.1.7, and is proved there.
Item 3, Adjointness II: This is a repetition of Item 3 of Proposition 4.3.1.1.7, and is proved there. □

00DY 4.3.3 The Left Skew Unit

00DZ **Definition 4.3.3.1.1.** The **left skew unit of the left tensor product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*, \lhd} \colon \mathsf{pt} \to \mathsf{Sets}_*$$

defined by

$$\mathbb{1}^{\lhd}_{\mathsf{Sets}_*} \stackrel{\text{def}}{=} S^0.$$

00E0 4.3.4 The Left Skew Associator

00E1 **Definition 4.3.4.1.1.** The **skew associator of the left tensor product of pointed sets** is the natural transformation

$$\alpha^{\mathsf{Sets}_*,\lhd}\colon \lhd\circ(\lhd\times\mathrm{id}_{\mathsf{Sets}_*})\Longrightarrow\lhd\circ(\mathrm{id}_{\mathsf{Sets}_*}\times\lhd)\circ\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

as in the diagram

$$\begin{tikzcd}
 & \mathsf{Sets}_*\times(\mathsf{Sets}_*\times\mathsf{Sets}_*) \arrow[dr, "\mathrm{id}\times\lhd"] & \\
(\mathsf{Sets}_*\times\mathsf{Sets}_*)\times\mathsf{Sets}_* \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[d, "\lhd\times\mathrm{id}"'] & & \mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[d, "\lhd"] \\
\mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[rr, "\lhd"'] \arrow[urr, phantom, "\Nearrow\alpha^{\mathsf{Sets}_*,\lhd}"] & & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}\colon (X\lhd Y)\lhd Z\to X\lhd(Y\lhd Z)$$

at $(X,x_0),(Y,y_0),(Z,z_0)\in\mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned}
(X\lhd Y)\lhd Z &\overset{\text{def}}{=} |Z|\odot(X\lhd Y)\\
&\overset{\text{def}}{=} |Z|\odot(|Y|\odot X)\\
&\cong \bigvee_{z\in Z}|Y|\odot X\\
&\cong \bigvee_{z\in Z}\left(\bigvee_{y\in Y}X\right)\\
&\to \bigvee_{[(z,y)]\in\bigvee_{z\in Z}Y}X\\
&\cong \bigvee_{[(z,y)]\in|Z|\odot Y}X\\
&\cong ||Z|\odot Y|\odot X\\
&\overset{\text{def}}{=} |Y\lhd Z|\odot X\\
&\overset{\text{def}}{=} X\lhd(Y\lhd Z),
\end{aligned}$$

where the map

$$\bigvee_{z\in Z}\left(\bigvee_{y\in Y}X\right)\to\bigvee_{(z,y)\in\bigvee_{z\in Z}Y}X$$

is given by $[(z,[(y,x)])]\mapsto[([(z,y)],x)]$.

Proof. (Proven below in a bit.) □

00E2 **Remark 4.3.4.1.2.** Unwinding the notation for elements, we have

$$\begin{aligned}[(z,[(y,x)])] &\stackrel{\text{def}}{=} [(z, x \lhd y)] \\ &\stackrel{\text{def}}{=} (x \lhd y) \lhd z\end{aligned}$$

and

$$\begin{aligned}[([(z,y)],x)] &\stackrel{\text{def}}{=} [(y \lhd z, x)] \\ &\stackrel{\text{def}}{=} x \lhd (y \lhd z).\end{aligned}$$

So, in other words, $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ acts on elements via

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x \lhd y) \lhd z) \stackrel{\text{def}}{=} x \lhd (y \lhd z)$$

for each $(x \lhd y) \lhd z \in (X \lhd Y) \lhd Z$.

00E3 **Remark 4.3.4.1.3.** Taking $y = y_0$, we see that the morphism $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x \lhd y_0) \lhd z) \stackrel{\text{def}}{=} x \lhd (y_0 \lhd z).$$

However, by the definition of $\lhd$, we have $y_0 \lhd z = y_0 \lhd z'$ for all $z, z' \in Z$, preventing $\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}$ from being non-invertible.

Proof. Firstly, note that, given $(X,x_0), (Y,y_0), (Z,z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z} \colon (X \lhd Y) \lhd Z \to X \lhd (Y \lhd Z)$$

is indeed a morphism of pointed sets, as we have

$$\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}((x_0 \lhd y_0) \lhd z_0) = x_0 \lhd (y_0 \lhd z_0).$$

Next, we claim that $\alpha^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &\colon (X,x_0) \to (X',x'_0), \\ g &\colon (Y,y_0) \to (Y',y'_0), \\ h &\colon (Z,z_0) \to (Z',z'_0) \end{aligned}$$

the diagram

$$\begin{tikzcd} (X \lhd Y) \lhd Z \arrow[r, "(f \lhd g) \lhd h"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}"'] & (X' \lhd Y') \lhd Z' \arrow[d, "\alpha^{\mathsf{Sets}_*,\lhd}_{X',Y',Z'}"] \\ X \lhd (Y \lhd Z) \arrow[r, "f \lhd (g \lhd h)"'] & X' \lhd (Y' \lhd Z') \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(x \lhd y) \lhd z \arrow[r, mapsto] \arrow[d, mapsto] & (f(x) \lhd g(y)) \lhd h(z) \arrow[d, mapsto] \\
x \lhd (y \lhd z) \arrow[r, mapsto] & f(x) \lhd (g(y) \lhd h(z))
\end{tikzcd}$$

and hence indeed commutes, showing $\alpha^{\mathsf{Sets}_*, \lhd}$ to be a natural transformation. This finishes the proof. □

00E4 4.3.5 The Left Skew Left Unitor

00E5 **Definition 4.3.5.1.1.** The **skew left unitor of the left tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*, \lhd} \colon \lhd \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*} \right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \qquad \begin{tikzcd}
\mathsf{pt} \times \mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*} \times \mathsf{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] \arrow[dr, phantom, "\Swarrow \lambda^{\mathsf{Sets}_*, \lhd}"] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\lhd"] \\
& \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*, \lhd}_X \colon S^0 \lhd X \to X$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned}
S^0 \lhd X &\cong |X| \odot S^0 \\
&\cong \bigvee_{x \in X} S^0 \\
&\to X,
\end{aligned}$$

where $\bigvee_{x \in X} S^0 \to X$ is the map given by

$$\begin{aligned}
[(x, 0)] &\mapsto x_0, \\
[(x, 1)] &\mapsto x.
\end{aligned}$$

Proof. (Proven below in a bit.) □

00E6 **Remark 4.3.5.1.2.** In other words, $\lambda^{\mathsf{Sets}_*, \lhd}_X$ acts on elements as

$$\begin{aligned}
\lambda^{\mathsf{Sets}_*, \lhd}_X (0 \lhd x) &\overset{\mathrm{def}}{=} x_0, \\
\lambda^{\mathsf{Sets}_*, \lhd}_X (1 \lhd x) &\overset{\mathrm{def}}{=} x
\end{aligned}$$

for each $1 \lhd x \in S^0 \lhd X$.

00E7 **Remark 4.3.5.1.3.** The morphism $\lambda_X^{\mathsf{Sets}_*,\lhd}$ is almost invertible, with its would-be-inverse

$$\phi_X \colon X \to S^0 \lhd X$$

given by

$$\phi_X(x) \stackrel{\text{def}}{=} 1 \lhd x$$

for each $x \in X$. Indeed, we have

$$\begin{aligned}\left[\lambda_X^{\mathsf{Sets}_*,\lhd} \circ \phi\right](x) &= \lambda_X^{\mathsf{Sets}_*,\lhd}(\phi(x)) \\ &= \lambda_X^{\mathsf{Sets}_*,\lhd}(1 \lhd x) \\ &= x \\ &= [\mathrm{id}_X](x)\end{aligned}$$

so that

$$\lambda_X^{\mathsf{Sets}_*,\lhd} \circ \phi = \mathrm{id}_X$$

and

$$\begin{aligned}\left[\phi \circ \lambda_X^{\mathsf{Sets}_*,\lhd}\right](1 \lhd x) &= \phi\Big(\lambda_X^{\mathsf{Sets}_*,\lhd}(1 \lhd x)\Big) \\ &= \phi(x) \\ &= 1 \lhd x \\ &= [\mathrm{id}_{S^0 \lhd X}](1 \lhd x),\end{aligned}$$

but

$$\begin{aligned}\left[\phi \circ \lambda_X^{\mathsf{Sets}_*,\lhd}\right](0 \lhd x) &= \phi\Big(\lambda_X^{\mathsf{Sets}_*,\lhd}(0 \lhd x)\Big) \\ &= \phi(x_0) \\ &= 1 \lhd x_0,\end{aligned}$$

where $0 \lhd x \neq 1 \lhd x_0$. Thus

$$\phi \circ \lambda_X^{\mathsf{Sets}_*,\lhd} \stackrel{?}{=} \mathrm{id}_{S^0 \lhd X}$$

holds for all elements in $S^0 \lhd X$ except one.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\lambda_X^{\mathsf{Sets}_*,\lhd} \colon S^0 \lhd X \to X$$

is indeed a morphism of pointed sets, as we have

$$\lambda_X^{\mathsf{Sets}_*,\lhd}(0 \lhd x_0) = x_0.$$

Next, we claim that $\lambda^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} S^0 \lhd X \arrow[r, "\mathrm{id}_{S^0} \lhd f"] \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_X"'] & S^0 \lhd Y \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} 0 \lhd x \arrow[d, mapsto] & \\ x_0 \arrow[r, mapsto] & f(x_0) \end{tikzcd} \qquad \begin{tikzcd} 0 \lhd x \arrow[r, mapsto] & 0 \lhd f(x) \arrow[d, mapsto] \\ & y_0 \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \lhd x \arrow[r, mapsto] \arrow[d, mapsto] & 1 \lhd f(x) \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof. □

00E8 4.3.6 The Left Skew Right Unitor

00E9 **Definition 4.3.6.1.1.** The **skew right unitor of the left tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd} \colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \lhd \circ \left(\mathsf{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right), \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathsf{pt} \arrow[r, "\mathsf{id} \times \mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] \arrow[dr, phantom, bend left=10, "\rho^{\mathsf{Sets}_*,\lhd}"] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\lhd"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets}_*,\lhd}_X \colon X \to X \lhd S^0$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} X &\to X \vee X \\ &\cong |S^0| \odot X \\ &\cong X \lhd S^0, \end{aligned}$$

where $X \to X \vee X$ is the map sending X to the second factor of X in $X \vee X$.

Proof. (Proven below in a bit.) □

00EA **Remark 4.3.6.1.2.** In other words, $\rho_X^{\mathsf{Sets}_*,\lhd}$ acts on elements as

$$\rho_X^{\mathsf{Sets}_*,\lhd}(x) \stackrel{\text{def}}{=} [(1, x)]$$

i.e. by

$$\rho_X^{\mathsf{Sets}_*,\lhd}(x) \stackrel{\text{def}}{=} x \lhd 1$$

for each $x \in X$.

00EB **Remark 4.3.6.1.3.** The morphism $\rho_X^{\mathsf{Sets}_*,\lhd}$ is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements $x \lhd 0$ of $X \lhd S^0$ with $x \neq x_0$ are outside the image of $\rho_X^{\mathsf{Sets}_*,\lhd}$, which sends x to $x \lhd 1$.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\rho_X^{\mathsf{Sets}_*,\lhd} \colon X \to X \lhd S^0$$

is indeed a morphism of pointed sets as we have

$$\begin{aligned} \rho_X^{\mathsf{Sets}_*,\lhd}(x_0) &= x_0 \lhd 1 \\ &= x_0 \lhd 0. \end{aligned}$$

Next, we claim that $\rho^{\mathsf{Sets}_*,\lhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\rho_X^{\mathsf{Sets}_*,\lhd}"'] & Y \arrow[d, "\rho_Y^{\mathsf{Sets}_*,\lhd}"] \\ X \lhd S^0 \arrow[r, "f \lhd \mathrm{id}_{S^0}"'] & Y \lhd S^0 \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x \lhd 0 \arrow[r, mapsto] & f(x) \lhd 0 \end{tikzcd}$$

and hence indeed commutes, showing $\rho^{\mathsf{Sets}_*,\lhd}$ to be a natural transformation. This finishes the proof. □

00EC 4.3.7 The Diagonal

00ED **Definition 4.3.7.1.1.** The **diagonal of the left tensor product of pointed sets** is the natural transformation

$$\Delta^{\lhd}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \lhd \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, swap, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"] & \arrow[d, phantom, "\Downarrow \Delta^{\lhd}"] & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, swap, "\lhd"] & \end{tikzcd}$$

whose component

$$\Delta^{\lhd}_X\colon (X, x_0) \to (X \lhd X, x_0 \lhd x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\Delta^{\lhd}_X(x) \stackrel{\text{def}}{=} x \lhd x$$

for each $x \in X$.

Proof. Being a Morphism of Pointed Sets: We have

$$\Delta^{\lhd}_X(x_0) \stackrel{\text{def}}{=} x_0 \lhd x_0,$$

and thus $\Delta^{\lhd}_X$ is a morphism of pointed sets.
Naturality: We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, swap, "\Delta^{\lhd}_X"] & Y \arrow[d, "\Delta^{\lhd}_Y"] \\ X \lhd X \arrow[r, swap, "f \lhd f"] & Y \lhd Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x \lhd x \arrow[r, mapsto] & f(x) \lhd f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\lhd}$ to be natural. □

4.3.8 The Left Skew Monoidal Structure on Pointed Sets Associated to $\lhd$
00EE

00EF **Proposition 4.3.8.1.1.** The category Sets_* admits a left-closed left skew monoidal category structure consisting of

- *The Underlying Category.* The category Sets_* of pointed sets;
- *The Left Skew Monoidal Product.* The left tensor product functor

$$\lhd\colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Definition 4.3.1.1.1;
- *The Left Internal Skew Hom.* The left internal Hom functor

$$[-,-]^{\lhd}_{\mathsf{Sets}_*}\colon \mathsf{Sets}^{\mathsf{op}}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Definition 4.3.2.1.1;
- *The Left Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathsf{Sets}_*,\lhd}\colon \mathsf{pt} \to \mathsf{Sets}_*$$

  of Definition 4.3.3.1.1;
- *The Left Skew Associators.* The natural transformation

$$\alpha^{\mathsf{Sets}_*,\lhd}\colon \lhd\circ(\lhd\times\mathrm{id}_{\mathsf{Sets}_*}) \Longrightarrow \lhd\circ(\mathrm{id}_{\mathsf{Sets}_*}\times\lhd)\circ\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 4.3.4.1.1;
- *The Left Skew Left Unitors.* The natural transformation

$$\lambda^{\mathsf{Sets}_*,\lhd}\colon \lhd\circ\left(\mathbb{1}^{\mathsf{Sets}_*}\times\mathrm{id}_{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

  of Definition 4.3.5.1.1;
- *The Left Skew Right Unitors.* The natural transformation

$$\rho^{\mathsf{Sets}_*,\lhd}\colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \lhd\circ\left(\mathsf{id}\times\mathbb{1}^{\mathsf{Sets}_*}\right)$$

  of Definition 4.3.6.1.1.

Proof. The Pentagon Identity: Let (W,w_0), (X,x_0), (Y,y_0) and (Z,z_0)

be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & (W \lhd (X \lhd Y)) \lhd Z \arrow[dr, "\alpha^{\mathsf{Sets}_*,\lhd}_{W,X\lhd Y,Z}"] & \\
((W \lhd X) \lhd Y) \lhd Z \arrow[ur, "\alpha^{\mathsf{Sets}_*,\lhd}_{W,X,Y} \lhd \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\lhd}_{W\lhd X,Y,Z}"'] & & W \lhd ((X \lhd Y) \lhd Z) \arrow[d, "\mathrm{id}_W \lhd \alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,Z}"] \\
(W \lhd X) \lhd (Y \lhd Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*,\lhd}_{W,X,Y\lhd Z}"'] & & W \lhd (X \lhd (Y \lhd Z))
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & (w \lhd (x \lhd y)) \lhd z \arrow[dr, mapsto] & \\
((w \lhd x) \lhd y) \lhd z \arrow[ur, mapsto] \arrow[d, mapsto] & & w \lhd ((x \lhd y) \lhd z) \arrow[d, mapsto] \\
(w \lhd x) \lhd (y \lhd z) \arrow[rr, mapsto] & & w \lhd (x \lhd (y \lhd z))
\end{tikzcd}$$

and thus we see that the pentagon identity is satisfied.
The Left Skew Left Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
(S^0 \lhd X) \lhd Y \arrow[r, "\alpha^{\mathsf{Sets}_*,\lhd}_{S^0,X,Y}"] \arrow[dr, "\lambda^{\mathsf{Sets}_*,\lhd}_X \lhd \mathrm{id}_Y"'] & S^0 \lhd (X \lhd Y) \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_{X\lhd Y}"] \\
 & X \lhd Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
(0 \lhd x) \lhd y \arrow[r, mapsto] \arrow[dr, mapsto] & 0 \lhd (x \lhd y) \arrow[d, mapsto] \\
& x_0 \lhd y = x_0 \lhd y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
(1 \lhd x) \lhd y \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \lhd (x \lhd y) \arrow[d, mapsto] \\
& x \lhd y
\end{tikzcd}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.

The Left Skew Right Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \lhd Y \arrow[d, "\rho^{\mathsf{Sets}_*,\lhd}_{X \lhd Y}"'] \arrow[dr, "\mathrm{id}_X \lhd \rho^{\mathsf{Sets}_*,\lhd}_Y"] & \\
(X \lhd Y) \lhd S^0 \arrow[r, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,Y,S^0}"'] & X \lhd (Y \lhd S^0)
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \lhd y \arrow[d, mapsto] \arrow[dr, mapsto] & \\
(x \lhd y) \lhd 1 \arrow[r, mapsto] & x \lhd (y \lhd 1)
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

The Left Skew Middle Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \lhd Y \arrow[r, equal] \arrow[d, "\rho^{\mathsf{Sets}_*,\lhd}_X \lhd \mathrm{id}_Y"'] & X \lhd Y \\
(X \lhd S^0) \lhd Y \arrow[r, "\alpha^{\mathsf{Sets}_*,\lhd}_{X,S^0,Y}"'] & X \lhd (S^0 \lhd Y) \arrow[u, "\mathrm{id}_A \lhd \lambda^{\mathsf{Sets}_*,\lhd}_Y"']
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \lhd y \arrow[r, mapsto] \arrow[d, mapsto] & x \lhd y \\
(x \lhd 1) \lhd y \arrow[r, mapsto] & x \lhd (1 \lhd y) \arrow[u, mapsto]
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

The Zig-Zag Identity: We have to show that the diagram

$$\begin{tikzcd}
S^0 \arrow[r, "\rho^{\mathsf{Sets}_*,\lhd}_{S^0}"] \arrow[dr, equal] & S^0 \lhd S^0 \arrow[d, "\lambda^{\mathsf{Sets}_*,\lhd}_{S^0}"] \\
 & S^0
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
0 \arrow[r, mapsto] \arrow[dr, mapsto] & 0 \lhd 1 \arrow[d, mapsto] \\
 & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \lhd 1 \arrow[d, mapsto] \\
 & 1
\end{tikzcd}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.

Left Skew Monoidal Left-Closedness: This follows from Item 2 of Proposition 4.3.1.1.7. □

4.3.9 Monoids With Respect to the Left Tensor Product
00EG of Pointed Sets

00EH **Proposition 4.3.9.1.1.** The category of monoids on $(\mathsf{Sets}_*, \lhd, S^0)$ is isomorphic to the category of "monoids with left zero"[14] and morphisms between them.

[14]A monoid with left zero is defined similarly as the monoids with zero of ??. Succinctly, they are monoids (A, μ_A, η_A) with a special element 0_A satisfying

$$0_A a = 0_A$$

Proof. Monoids on $(\mathsf{Sets}_*, \lhd, S^0)$: A monoid on $(\mathsf{Sets}_*, \lhd, S^0)$ consists of:

- *The Underlying Object.* A pointed set $(A, 0_A)$.
- *The Multiplication Morphism.* A morphism of pointed sets

  $$\mu_A \colon A \lhd A \to A,$$

  determining a left bilinear morphism of pointed sets

  $$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$

- *The Unit Morphism.* A morphism of pointed sets

  $$\eta_A \colon S^0 \to A$$

  picking an element 1_A of A.

satisfying the following conditions:

1. *Associativity.* The diagram

$$\begin{tikzcd} & A \lhd (A \lhd A) \arrow[dr, "\mathrm{id}_A \lhd \mu_A"] & \\ (A \lhd A) \lhd A \arrow[ur, "\alpha^{\mathsf{Sets}_*, \lhd}_{A,A,A}"] \arrow[d, "\mu_A \lhd \mathrm{id}_A"'] & & A \lhd A \arrow[d, "\mu_A"] \\ A \lhd A \arrow[rr, "\mu_A"'] & & A \end{tikzcd}$$

2. *Left Unitality.* The diagram

$$\begin{tikzcd} S^0 \lhd A \arrow[r, "\eta_A \times \mathrm{id}_A"] \arrow[dr, "\lambda^{\mathsf{Sets}_*, \lhd}_A"'] & A \lhd A \arrow[d, "\mu_A"] \\ & A \end{tikzcd}$$

   commutes.

for each $a \in A$.

3. *Right Unitality.* The diagram

$$\begin{tikzcd}
A \arrow[r, "\rho_A^{\mathsf{Sets}_*,\lhd}"] \arrow[d, equal] & A \lhd S^0 \arrow[d, "\mathrm{id}_A \times \eta_A"] \\
A & A \lhd A \arrow[l, "\mu_A"]
\end{tikzcd}$$

commutes.

Being a left-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each $a \in A$. Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

1. *Associativity.* The associativity condition acts as

$$\begin{tikzcd}
(a \lhd b) \lhd c \arrow[d, mapsto] & & & a \lhd (b \lhd c) \arrow[dr, mapsto] & \\
ab \lhd c \arrow[r, mapsto] & (ab)c & (a \lhd b) \lhd c \arrow[ur, mapsto] & & a \lhd bc \arrow[d, mapsto] \\
& & & & a(bc)
\end{tikzcd}$$

This gives

$$(ab)c = a(bc)$$

for each $a, b, c \in A$.

2. *Left Unitality.* The left unitality condition acts:

(a) On $0 \lhd a$ as

$$\begin{tikzcd}
0 \lhd a \arrow[dr, mapsto] & & 0 \lhd a \arrow[r, mapsto] \arrow[dr, mapsto] & 0_A \lhd a \arrow[d, mapsto] \\
& 0_A & & 0_A a.
\end{tikzcd}$$

(b) On $1 \lhd a$ as

$$\begin{tikzcd}
1 \lhd a \arrow[dr, mapsto] & & 1 \lhd a \arrow[r, mapsto] \arrow[dr, mapsto] & 1_A \lhd a \arrow[d, mapsto] \\
& a & & 1_A a.
\end{tikzcd}$$

This gives

$$\begin{aligned}1_A a &= a,\\ 0_A a &= 0_A\end{aligned}$$

for each $a \in A$.

3. *Right Unitality.* The right unitality condition acts as

$$\begin{tikzcd} a \arrow[d, mapsto] & & a \arrow[r, mapsto] & a \lhd 1 \arrow[d, mapsto] \\ a & & a1_A & a \lhd 1_A \arrow[l, mapsto] \end{tikzcd}$$

This gives

$$a1_A = a$$

for each $a \in A$.

Thus we see that monoids with respect to $\lhd$ are exactly monoids with left zero.

Morphisms of Monoids on $(\mathsf{Sets}_*, \lhd, S^0)$: A morphism of monoids on $(\mathsf{Sets}_*, \lhd, S^0)$ from $(A, \mu_A, \eta_A, 0_A)$ to $(B, \mu_B, \eta_B, 0_B)$ is a morphism of pointed sets

$$f\colon (A, 0_A) \to (B, 0_B)$$

satisfying the following conditions:

1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{tikzcd} A \lhd A \arrow[r, "f \lhd f"] \arrow[d, "\mu_A"'] & B \lhd B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{tikzcd} S^0 \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

commutes.

These act on elements as

$$
\begin{tikzcd}
a \lhd b \arrow[d, mapsto] & & & a \lhd b \arrow[r, mapsto] & f(a) \lhd f(b) \arrow[d, mapsto] \\
ab \arrow[r, mapsto] & f(ab) & & & f(a)f(b)
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
0 \arrow[dr, mapsto] & & 0 \arrow[r, mapsto] & 0_A \arrow[d, mapsto] \\
& 0_B & & f(0_A)
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
1 \arrow[dr, mapsto] & & 1 \arrow[r, mapsto] & 1_A \arrow[d, mapsto] \\
& 1_B & & f(1_A)
\end{tikzcd}
$$

giving

$$
\begin{aligned}
f(ab) &= f(a)f(b), \\
f(0_A) &= 0_B, \\
f(1_A) &= 1_B,
\end{aligned}
$$

for each $a, b \in A$, which is exactly a morphism of monoids with left zero. *Identities and Composition*: Similarly, the identities and composition of $\mathsf{Mon}(\mathsf{Sets}_*, \lhd, S^0)$ can be easily seen to agree with those of monoids with left zero, which finishes the proof. □

00EJ 4.4 The Right Tensor Product of Pointed Sets

00EK 4.4.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

00EL **Definition 4.4.1.1.1.** The **right tensor product of pointed sets** is the functor[15]

$$\rhd \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_* \times \mathsf{Sets}_* \xrightarrow{忘 \times \mathsf{id}} \mathsf{Sets} \times \mathsf{Sets}_* \xrightarrow{\odot} \mathsf{Sets}_*,$$

where:

[15] *Further Notation:* Also written $\rhd_{\mathsf{Sets}_*}$.

- $忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\odot\colon \mathsf{Sets} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the tensor functor of Item 1 of Proposition 4.2.1.1.6.

00EM **Remark 4.4.1.1.2.** The right tensor product of pointed sets satisfies the following natural bijection:

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathrm{Hom}^{\otimes,\mathrm{R}}_{\mathsf{Sets}_*}(X \times Y, Z).$$

That is to say, the following data are in natural bijection:

1. Pointed maps $f\colon X \rhd Y \to Z$.
2. Maps of sets $f\colon X \times Y \to Z$ satisfying $f(x, y_0) = z_0$ for each $x \in X$.

00EN **Remark 4.4.1.1.3.** The right tensor product of pointed sets may be described as follows:

- The right tensor product of (X, x_0) and (Y, y_0) is the pair $((X \rhd Y, x_0 \rhd y_0), \iota)$ consisting of
  - A pointed set $(X \rhd Y, x_0 \rhd y_0)$;
  - A right bilinear morphism of pointed sets $\iota\colon (X \times Y, (x_0, y_0)) \to X \rhd Y$;

  satisfying the following universal property:

  (UP) Given another such pair $((Z, z_0), f)$ consisting of
  - $*$ A pointed set (Z, z_0);
  - $*$ A right bilinear morphism of pointed sets $f\colon (X \times Y, (x_0, y_0)) \to X \rhd Y$;

  there exists a unique morphism of pointed sets $X \rhd Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd}
 & X \rhd Y \arrow[d, dashed, "\exists!"] \\
X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z
\end{tikzcd}$$

  commute.

00EP **Construction 4.4.1.1.4.** In detail, the **right tensor product of (X, x_0) and (Y, y_0)** is the pointed set $(X \rhd Y, [y_0])$ consisting of:

- *The Underlying Set.* The set $X \rhd Y$ defined by

$$\begin{aligned} X \rhd Y &\overset{\text{def}}{=} |X| \odot Y \\ &\cong \bigvee_{x \in X} (Y, y_0), \end{aligned}$$

  where $|X|$ denotes the underlying set of (X, x_0).

- *The Underlying Basepoint.* The point $[(x_0, y_0)]$ of $\bigvee_{x \in X}(Y, y_0)$, which is equal to $[(x, y_0)]$ for any $x \in X$.

00EQ **Notation 4.4.1.1.5.** We write[16] $x \rhd y$ for the element $[(x, y)]$ of

$$X \rhd Y \cong |X| \odot Y.$$

00ER **Remark 4.4.1.1.6.** Employing the notation introduced in Notation 4.4.1.1.5, we have

$$x_0 \rhd y_0 = x \rhd y_0$$

for each $x \in X$, and

$$x \rhd y_0 = x' \rhd y_0$$

for each $x, x' \in X$.

00ES **Proposition 4.4.1.1.7.** Let (X, x_0) and (Y, y_0) be pointed sets.

00ET 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \rhd Y$ define functors

$$\begin{aligned} X \rhd -&: \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ - \rhd Y&: \mathsf{Sets}_* \to \mathsf{Sets}_*, \\ -_1 \rhd -_2&: \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*. \end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned} f&: (X, x_0) \to (A, a_0), \\ g&: (Y, y_0) \to (B, b_0), \end{aligned}$$

   the induced map

$$f \rhd g \colon X \rhd Y \to A \rhd B$$

   is given by

$$[f \rhd g](x \rhd y) \overset{\text{def}}{=} f(x) \rhd g(y)$$

   for each $x \rhd y \in X \rhd Y$.

[16] *Further Notation:* Also written $x \rhd_{\mathsf{Sets}_*} y$.

00EU 2. *Adjointness I.* We have an adjunction

$$\left(X \rhd - \dashv [X,-]^{\rhd}_{\mathsf{Sets}_*}\right)\colon \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X\rhd -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[X,-]^{\rhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\left(Y, [X,Z]^{\rhd}_{\mathsf{Sets}_*}\right)$$

natural in $(X,x_0),(Y,y_0),(Z,z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X,Y]^{\rhd}_{\mathsf{Sets}_*}$ is the pointed set of Definition 4.4.2.1.1.

00EV 3. *Adjointness II.* The functor

$$- \rhd Y \colon \mathsf{Sets}_* \to \mathsf{Sets}_*$$

does not admit a right adjoint.

00EW 4. *Adjointness III.* We have a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|X|, \mathsf{Sets}_*(Y,Z))$$

natural in $(X,x_0),(Y,y_0),(Z,z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Adjointness I*: This follows from Item 3 of Proposition 4.2.1.1.6.
Item 3, *Adjointness II*: For $- \rhd Y$ to admit a right adjoint would require it to preserve colimits by **??** of **??**. However, we have

$$\begin{aligned} \mathrm{pt} \rhd X &\stackrel{\text{def}}{=} |\mathrm{pt}| \odot X \\ &\cong X \\ &\not\cong \mathrm{pt}, \end{aligned}$$

and thus we see that $- \rhd Y$ does not have a right adjoint.
Item 4, *Adjointness III*: This follows from Item 2 of Proposition 4.2.1.1.6. □

00EX **Remark 4.4.1.1.8.** Here is some intuition on why $- \rhd Y$ fails to be a left adjoint. Item 4 of Proposition 4.3.1.1.7 states that we have a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}}(|X|, \mathsf{Sets}_*(Y,Z)),$$

so it would be reasonable to wonder whether a natural bijection of the form

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \rhd Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y,Z)),$$

also holds, which would give $- \rhd Y \dashv \mathbf{Sets}_*(Y,-)$. However, such a bijection would require every map

$$f\colon X \rhd Y \to Z$$

to satisfy

$$f(x_0 \rhd y) = z_0$$

for each $x \in X$, whereas we are imposing such a basepoint preservation condition only for elements of the form $x \rhd y_0$. Thus $\mathbf{Sets}_*(Y,-)$ can't be a right adjoint for $- \rhd Y$, and as shown by Item 3 of Proposition 4.4.1.1.7, no functor can.[17]

00EY 4.4.2 The Right Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

00EZ **Definition 4.4.2.1.1.** The **right internal Hom of pointed sets** is the functor

$$[-,-]^{\rhd}_{\mathsf{Sets}_*}\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

defined as the composition

$$\mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{忘\times\mathrm{id}} \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \xrightarrow{\pitchfork} \mathsf{Sets}_*,$$

where:

- $忘\colon \mathsf{Sets}_* \to \mathsf{Sets}$ is the forgetful functor from pointed sets to sets.
- $\pitchfork\colon \mathsf{Sets}^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$ is the cotensor functor of Item 1 of Proposition 4.2.2.1.4.

Proof. For a proof that $[-,-]^{\rhd}_{\mathsf{Sets}_*}$ is indeed the right internal Hom of Sets_* with respect to the right tensor product of pointed sets, see Item 2 of Proposition 4.4.1.1.7. □

00F0 **Remark 4.4.2.1.2.** We have

$$[-,-]^{\lhd}_{\mathsf{Sets}_*} = [-,-]^{\rhd}_{\mathsf{Sets}_*}.$$

00F1 **Remark 4.4.2.1.3.** The right internal Hom of pointed sets satisfies the following universal property:

$$\mathsf{Sets}_*(X \rhd Y, Z) \cong \mathsf{Sets}_*\Big(Y, [X,Z]^{\rhd}_{\mathsf{Sets}_*}\Big)$$

That is to say, the following data are in bijection:

[17] The functor $\mathbf{Sets}_*(Y,-)$ is instead right adjoint to $- \wedge Y$, the smash product of

1. Pointed maps $f\colon X \rhd Y \to Z$.
2. Pointed maps $f\colon Y \to [X, Z]^{\rhd}_{\mathsf{Sets}_*}$.

00F2 **Remark 4.4.2.1.4.** In detail, the **right internal Hom of** (X, x_0) **and** (Y, y_0) is the pointed set $\left([X, Y]^{\rhd}_{\mathsf{Sets}_*}, [(y_0)_{x\in X}]\right)$ consisting of

- *The Underlying Set.* The set $[X, Y]^{\rhd}_{\mathsf{Sets}_*}$ defined by

$$\begin{aligned}[X, Y]^{\rhd}_{\mathsf{Sets}_*} &\overset{\text{def}}{=} |X| \pitchfork Y\\ &\cong \bigwedge_{x\in X} (Y, y_0),\end{aligned}$$

  where $|X|$ denotes the underlying set of (X, x_0);

- *The Underlying Basepoint.* The point $[(y_0)_{x\in X}]$ of $\bigwedge_{x\in X}(Y, y_0)$.

00F3 **Proposition 4.4.2.1.5.** Let (X, x_0) and (Y, y_0) be pointed sets.

00F4 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto [X, Y]^{\rhd}_{\mathsf{Sets}_*}$ define functors

$$\begin{aligned}[X, -]^{\rhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_* \to \mathsf{Sets}_*,\\ [-, Y]^{\rhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_*^{\mathsf{op}} \to \mathsf{Sets}_*,\\ [-_1, -_2]^{\rhd}_{\mathsf{Sets}_*} &\colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*.\end{aligned}$$

   In particular, given pointed maps

$$\begin{aligned}f &\colon (X, x_0) \to (A, a_0),\\ g &\colon (Y, y_0) \to (B, b_0),\end{aligned}$$

   the induced map

$$[f, g]^{\rhd}_{\mathsf{Sets}_*} \colon [A, Y]^{\rhd}_{\mathsf{Sets}_*} \to [X, B]^{\rhd}_{\mathsf{Sets}_*}$$

   is given by

$$[f, g]^{\rhd}_{\mathsf{Sets}_*}([(y_a)_{a\in A}]) \overset{\text{def}}{=} \left[\left(g\left(y_{f(x)}\right)\right)_{x\in X}\right]$$

   for each $[(y_a)_{a\in A}] \in [A, Y]^{\rhd}_{\mathsf{Sets}_*}$.

00F5 2. *Adjointness I.* We have an adjunction

$$\left(X \rhd - \dashv [X, -]^{\rhd}_{\mathsf{Sets}_*}\right)\colon \begin{tikzcd} \mathsf{Sets}_* \arrow[r, bend left, "X \rhd -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "{[X,-]^{\rhd}_{\mathsf{Sets}_*}}"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \triangleright Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}\left(Y, [X, Z]^{\triangleright}_{\mathsf{Sets}_*}\right)$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, where $[X, Y]^{\triangleright}_{\mathsf{Sets}_*}$ is the pointed set of Definition 4.4.2.1.1.

00F6 3. *Adjointness II.* The functor

$$- \triangleright Y \colon \mathsf{Sets}_* \to \mathsf{Sets}_*$$

does not admit a right adjoint.

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Adjointness I*: This is a repetition of Item 2 of Proposition 4.4.1.1.7, and is proved there.
Item 3, *Adjointness II*: This is a repetition of Item 3 of Proposition 4.4.1.1.7, and is proved there. □

00F7 4.4.3 The Right Skew Unit

00F8 **Definition 4.4.3.1.1.** The **right skew unit of the right tensor product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*,\triangleright} \colon \mathsf{pt} \to \mathsf{Sets}_*$$

defined by

$$\mathbb{1}^{\triangleright}_{\mathsf{Sets}_*} \stackrel{\mathrm{def}}{=} S^0.$$

00F9 4.4.4 The Right Skew Associator

00FA **Definition 4.4.4.1.1.** The **skew associator of the right tensor product of pointed sets** is the natural transformation

$$\alpha^{\mathsf{Sets}_*,\triangleright} \colon \triangleright \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \triangleright) \Longrightarrow \triangleright \circ (\triangleright \times \mathrm{id}_{\mathsf{Sets}_*}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

as in the diagram

$$\begin{tikzcd}
 & (\mathsf{Sets}_* \times \mathsf{Sets}_*) \times \mathsf{Sets}_* \arrow[dr, "\triangleright \times \mathsf{id}"] & \\
\mathsf{Sets}_* \times (\mathsf{Sets}_* \times \mathsf{Sets}_*) \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\wr"'] \arrow[d, "\mathsf{id} \times \triangleright"'] & & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\triangleright"] \\
\mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[rr, "\triangleright"'] \arrow[urr, Rightarrow, "\alpha^{\mathsf{Sets}_*,\triangleright}"] & & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}\colon X \rhd (Y \rhd Z) \to (X \rhd Y) \rhd Z$$

at $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned}
X \rhd (Y \rhd Z) &\overset{\text{def}}{=} |X| \odot (Y \rhd Z)\\
&\overset{\text{def}}{=} |X| \odot (|Y| \odot Z)\\
&\cong \bigvee_{x\in X} (|Y| \odot Z)\\
&\cong \bigvee_{x\in X}\left(\bigvee_{y\in Y} Z\right)\\
&\to \bigvee_{[(x,y)]\in\bigvee_{x\in X} Y} Z\\
&\cong \bigvee_{[(x,y)]\in|X|\odot Y} Z\\
&\cong ||X| \odot Y| \odot Z\\
&\overset{\text{def}}{=} |X \rhd Y| \odot Z\\
&\overset{\text{def}}{=} (X \rhd Y) \rhd Z,
\end{aligned}$$

where the map

$$\bigvee_{x\in X}\left(\bigvee_{y\in Y} Z\right) \to \bigvee_{[(x,y)]\in\bigvee_{x\in X} Y} Z$$

is given by $[(x, [(y, z)])] \mapsto [([(x, y)], z)]$.

Proof. (Proven below in a bit.) □

00FB **Remark 4.4.4.1.2.** Unwinding the notation for elements, we have

$$\begin{aligned}
[(x, [(y, z)])] &\overset{\text{def}}{=} [(x, y \rhd z)]\\
&\overset{\text{def}}{=} x \rhd (y \rhd z)
\end{aligned}$$

and

$$\begin{aligned}
[([(x, y)], z)] &\overset{\text{def}}{=} [(x \rhd y, z)]\\
&\overset{\text{def}}{=} (x \rhd y) \rhd z.
\end{aligned}$$

So, in other words, $\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}$ acts on elements via

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}(x \rhd (y \rhd z)) \overset{\text{def}}{=} (x \rhd y) \rhd z$$

for each $x \rhd (y \rhd z) \in X \rhd (Y \rhd Z)$.

00FC **Remark 4.4.4.1.3.** Taking $y = y_0$, we see that the morphism $\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}$ acts on elements as

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}(x \rhd (y_0 \rhd z)) \overset{\text{def}}{=} (x \rhd y_0) \rhd z.$$

However, by the definition of $\rhd$, we have $x \rhd y_0 = x' \rhd y_0$ for all $x, x' \in X$, preventing $\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}$ from being non-invertible.

Proof. Firstly, note that, given $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z} : X \rhd (Y \rhd Z) \to (X \rhd Y) \rhd Z$$

is indeed a morphism of pointed sets, as we have

$$\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}(x_0 \rhd (y_0 \rhd z_0)) = (x_0 \rhd y_0) \rhd z_0.$$

Next, we claim that $\alpha^{\mathsf{Sets}_*,\rhd}$ is a natural transformation. We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &\colon (X, x_0) \to (X', x'_0), \\ g &\colon (Y, y_0) \to (Y', y'_0), \\ h &\colon (Z, z_0) \to (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{tikzcd} X \rhd (Y \rhd Z) \arrow[r, "f \rhd (g \rhd h)"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\rhd}_{X,Y,Z}"'] & X' \rhd (Y' \rhd Z') \arrow[d, "\alpha^{\mathsf{Sets}_*,\rhd}_{X',Y',Z'}"] \\ (X \rhd Y) \rhd Z \arrow[r, "(f \rhd g) \rhd h"'] & (X' \rhd Y') \rhd Z' \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \rhd (y \rhd z) \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \rhd (g(y) \rhd h(z)) \arrow[d, mapsto] \\ (x \rhd y) \rhd z \arrow[r, mapsto] & (f(x) \rhd g(y)) \rhd h(z) \end{tikzcd}$$

and hence indeed commutes, showing $\alpha^{\mathsf{Sets}_*,\rhd}$ to be a natural transformation. This finishes the proof. □

pointed sets of Definition 4.5.1.1.1. See Item 2 of Proposition 4.5.1.1.9.

00FD 4.4.5 The Right Skew Left Unitor

00FE **Definition 4.4.5.1.1.** The **skew left unitor of the right tensor product of pointed sets** is the natural transformation

$$\lambda^{\mathsf{Sets}_*,\rhd}\colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \rhd\circ\left(\mathbb{1}^{\mathsf{Sets}_*}\times\mathrm{id}_{\mathsf{Sets}_*}\right)\qquad \begin{tikzcd} \mathsf{pt}\times\mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*}\times\mathsf{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[d, "\rhd"] \arrow[dl, phantom, "\Nearrow \lambda^{\mathsf{Sets}_*,\rhd}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*,\rhd}_X\colon X\to S^0\rhd X$$

at $(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} X &\to X\vee X\\ &\cong |S^0|\odot X\\ &\cong S^0\rhd X, \end{aligned}$$

where $X\to X\vee X$ is the map sending X to the second factor of X in $X\vee X$.

Proof. (Proven below in a bit.) □

00FF **Remark 4.4.5.1.2.** In other words, $\lambda^{\mathsf{Sets}_*,\rhd}_X$ acts on elements as

$$\lambda^{\mathsf{Sets}_*,\rhd}_X(x)\overset{\mathrm{def}}{=}[(1,x)]$$

i.e. by

$$\lambda^{\mathsf{Sets}_*,\rhd}_X(x)\overset{\mathrm{def}}{=}1\rhd x$$

for each $x\in X$.

00FG **Remark 4.4.5.1.3.** The morphism $\lambda^{\mathsf{Sets}_*,\rhd}_X$ is non-invertible, as it is non-surjective when viewed as a map of sets, since the elements $0\rhd x$ of $S^0\rhd X$ with $x\neq x_0$ are outside the image of $\lambda^{\mathsf{Sets}_*,\rhd}_X$, which sends x to $1\rhd x$.

Proof. Firstly, note that, given $(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\lambda^{\mathsf{Sets}_*,\rhd}_X\colon X\to S^0\rhd X$$

is indeed a morphism of pointed sets, as we have

$$\begin{aligned}\lambda^{\mathsf{Sets}_*,\triangleright}_X(x_0) &= 1 \triangleright x_0\\ &= 0 \triangleright x_0.\end{aligned}$$

Next, we claim that $\lambda^{\mathsf{Sets}_*,\triangleright}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f\colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\lambda^{\mathsf{Sets}_*,\triangleright}_X"'] & Y \arrow[d, "\lambda^{\mathsf{Sets}_*,\triangleright}_Y"] \\ S^0 \triangleright X \arrow[r, "\mathrm{id}_{S^0} \triangleright f"'] & S^0 \triangleright Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ 1 \triangleright x \arrow[r, mapsto] & 1 \triangleright f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*,\triangleright}$ to be a natural transformation. This finishes the proof. □

00FH 4.4.6 The Right Skew Right Unitor

00FJ **Definition 4.4.6.1.1.** The **skew right unitor of the right tensor product of pointed sets** is the natural transformation

$$\rho^{\mathsf{Sets}_*,\triangleright}\colon \triangleright \circ \left(\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathrm{pt} \arrow[r, "\mathrm{id} \times \mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] \arrow[dr, phantom, "\overset{\rho^{\mathsf{Sets}_*,\triangleright}}{\Longleftarrow}"] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\triangleright"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets}_*,\triangleright}_X\colon X \triangleright S^0 \to X$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} X \rhd S^0 &\cong |X| \odot S^0 \\ &\cong \bigvee_{x \in X} S^0 \\ &\to X, \end{aligned}$$

where $\bigvee_{x \in X} S^0 \to X$ is the map given by

$$\begin{aligned} [(x, 0)] &\mapsto x_0, \\ [(x, 1)] &\mapsto x. \end{aligned}$$

Proof. (Proven below in a bit.) □

00FK **Remark 4.4.6.1.2.** In other words, $\rho_X^{\mathsf{Sets}_*, \rhd}$ acts on elements as

$$\begin{aligned} \rho_X^{\mathsf{Sets}_*, \rhd}(x \rhd 0) &\stackrel{\text{def}}{=} x_0, \\ \rho_X^{\mathsf{Sets}_*, \rhd}(x \rhd 1) &\stackrel{\text{def}}{=} x \end{aligned}$$

for each $x \rhd 1 \in X \rhd S^0$.

00FL **Remark 4.4.6.1.3.** The morphism $\rho_X^{\mathsf{Sets}_*, \rhd}$ is almost invertible, with its would-be-inverse

$$\phi_X \colon X \to X \rhd S^0$$

given by

$$\phi_X(x) \stackrel{\text{def}}{=} x \rhd 1$$

for each $x \in X$. Indeed, we have

$$\begin{aligned} \left[\rho_X^{\mathsf{Sets}_*, \rhd} \circ \phi\right](x) &= \rho_X^{\mathsf{Sets}_*, \rhd}(\phi(x)) \\ &= \rho_X^{\mathsf{Sets}_*, \rhd}(x \rhd 1) \\ &= x \\ &= [\mathrm{id}_X](x) \end{aligned}$$

so that

$$\rho_X^{\mathsf{Sets}_*, \rhd} \circ \phi = \mathrm{id}_X$$

and

$$\begin{aligned} \left[\phi \circ \rho_X^{\mathsf{Sets}_*, \rhd}\right](x \rhd 1) &= \phi\Big(\rho_X^{\mathsf{Sets}_*, \rhd}(x \rhd 1)\Big) \\ &= \phi(x) \\ &= x \rhd 1 \\ &= [\mathrm{id}_{X \rhd S^0}](x \rhd 1), \end{aligned}$$

but

$$\begin{aligned}\left[\phi \circ \rho^{\mathsf{Sets}_*,\rhd}_X\right](x \rhd 0) &= \phi\Big(\rho^{\mathsf{Sets}_*,\rhd}_X(x \rhd 0)\Big)\\ &= \phi(x_0)\\ &= 1 \rhd x_0,\end{aligned}$$

where $x \rhd 0 \neq 1 \rhd x_0$. Thus

$$\phi \circ \rho^{\mathsf{Sets}_*,\rhd}_X \stackrel{?}{=} \mathrm{id}_{X \rhd S^0}$$

holds for all elements in $X \rhd S^0$ except one.

Proof. Firstly, note that, given $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, the map

$$\rho^{\mathsf{Sets}_*,\rhd}_X \colon X \rhd S^0 \to X$$

is indeed a morphism of pointed sets as we have

$$\rho^{\mathsf{Sets}_*,\rhd}_X(x_0 \rhd 0) = x_0.$$

Next, we claim that $\rho^{\mathsf{Sets}_*,\rhd}$ is a natural transformation. We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \rhd S^0 \arrow[r, "f \rhd \mathrm{id}_{S^0}"] \arrow[d, "\rho^{\mathsf{Sets}_*,\rhd}_X"'] & Y \rhd S^0 \arrow[d, "\rho^{\mathsf{Sets}_*,\rhd}_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \rhd 0 \arrow[d, mapsto] & \\ x_0 \arrow[r, mapsto] & f(x_0) \end{tikzcd} \qquad \begin{tikzcd} x \rhd 0 \arrow[r, mapsto] & f(x) \rhd 0 \arrow[d, mapsto] \\ & y_0 \end{tikzcd}$$

and

$$\begin{tikzcd} x \rhd 1 \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \rhd 1 \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\rho^{\mathsf{Sets}_*,\rhd}$ to be a natural transformation. This finishes the proof. $\square$

00FM 4.4.7 The Diagonal

00FN **Definition 4.4.7.1.1.** The **diagonal of the right tensor product of pointed sets** is the natural transformation

$$\Delta^{\rhd} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \rhd \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \arrow[d, phantom, "\Downarrow \Delta^{\rhd}"] & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \times \mathsf{Sets}_*, \arrow[ur, "\rhd"'] & \end{tikzcd}$$

whose component

$$\Delta^{\rhd}_X \colon (X, x_0) \to (X \rhd X, x_0 \rhd x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\Delta^{\rhd}_X(x) \stackrel{\text{def}}{=} x \rhd x$$

for each $x \in X$.

Proof. Being a Morphism of Pointed Sets: We have

$$\Delta^{\rhd}_X(x_0) \stackrel{\text{def}}{=} x_0 \rhd x_0,$$

and thus $\Delta^{\rhd}_X$ is a morphism of pointed sets.
Naturality: We need to show that, given a morphism of pointed sets

$$f \colon (X, x_0) \to (Y, y_0),$$

the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta^{\rhd}_X"'] & Y \arrow[d, "\Delta^{\rhd}_Y"] \\ X \rhd X \arrow[r, "f \rhd f"'] & Y \rhd Y \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ x \rhd x \arrow[r, mapsto] & f(x) \rhd f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\rhd}$ to be natural. $\square$

00FP 4.4.8 The Right Skew Monoidal Structure on Pointed Sets Associated to $\triangleright$

00FQ **Proposition 4.4.8.1.1.** The category Sets_* admits a right-closed right skew monoidal category structure consisting of

- *The Underlying Category.* The category Sets_* of pointed sets;
- *The Right Skew Monoidal Product.* The right tensor product functor
$$\triangleright \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$
of Definition 4.4.1.1.1;
- *The Right Internal Skew Hom.* The right internal Hom functor
$$[-,-]^{\triangleright}_{\mathsf{Sets}_*} \colon \mathsf{Sets}_*^{\mathsf{op}} \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$
of Definition 4.4.2.1.1;
- *The Right Skew Monoidal Unit.* The functor
$$\mathbb{1}^{\mathsf{Sets}_*,\triangleright} \colon \mathsf{pt} \to \mathsf{Sets}_*$$
of Definition 4.4.3.1.1;
- *The Right Skew Associators.* The natural transformation
$$\alpha^{\mathsf{Sets}_*,\triangleright} \colon \triangleright \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \triangleright) \Longrightarrow \triangleright \circ (\triangleright \times \mathrm{id}_{\mathsf{Sets}_*}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$
of Definition 4.4.4.1.1;
- *The Right Skew Left Unitors.* The natural transformation
$$\lambda^{\mathsf{Sets}_*,\triangleright} \colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*} \overset{\sim}{\Longrightarrow} \triangleright \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right)$$
of Definition 4.4.5.1.1;
- *The Right Skew Right Unitors.* The natural transformation
$$\rho^{\mathsf{Sets}_*,\triangleright} \colon \triangleright \circ \left(\mathsf{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$
of Definition 4.4.6.1.1.

Proof. The Pentagon Identity: Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0)

be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & W \triangleright ((X \triangleright Y) \triangleright Z) \arrow[dr, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W,X\triangleright Y,Z}"] & \\
W \triangleright (X \triangleright (Y \triangleright Z)) \arrow[ur, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W,X,Y} \triangleright \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W\triangleright X,Y,Z}"'] & & (W \triangleright (X \triangleright Y)) \triangleright Z \arrow[d, "\mathrm{id}_W \triangleright \alpha^{\mathsf{Sets}_*,\triangleright}_{X,Y,Z}"] \\
(W \triangleright X) \triangleright (Y \triangleright Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*,\triangleright}_{W,X,Y\triangleright Z}"'] & & ((W \triangleright X) \triangleright Y) \triangleright Z
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & w \triangleright ((x \triangleright y) \triangleright z) \arrow[dr, mapsto] & \\
w \triangleright (x \triangleright (y \triangleright z)) \arrow[ur, mapsto] \arrow[d, mapsto] & & (w \triangleright (x \triangleright y)) \triangleright z \arrow[d, mapsto] \\
(w \triangleright x) \triangleright (y \triangleright z) \arrow[rr, mapsto] & & ((w \triangleright x) \triangleright y) \triangleright z
\end{tikzcd}$$

and thus we see that the pentagon identity is satisfied.
The Right Skew Left Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed

sets. We have to show that the diagram

$$\begin{tikzcd}
X \rhd Y \arrow[d, "\lambda^{\mathsf{Sets}_*,\rhd}_{X\rhd Y}"'] \arrow[dr, "\lambda^{\mathsf{Sets}_*,\rhd}_{X} \rhd \mathrm{id}_Y"] & \\
S^0 \rhd (X \rhd Y) \arrow[r, "\alpha^{\mathsf{Sets}_*,\rhd}_{S^0,X,Y}"'] & (S^0 \rhd X) \rhd Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \rhd y \arrow[d, mapsto] \arrow[dr, mapsto] & \\
1 \rhd (x \rhd y) \arrow[r, mapsto] & (1 \rhd x) \rhd y
\end{tikzcd}$$

and hence indeed commutes. Thus the left skew triangle identity is satisfied.

The Right Skew Right Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \rhd (Y \rhd S^0) \arrow[r, "\mathrm{id}_X \rhd \rho^{\mathsf{Sets}_*,\rhd}_{Y}"] \arrow[dr, "\alpha^{\mathsf{Sets}_*,\rhd}_{S^0,X,Y}"'] & (X \rhd Y) \rhd S^0 \arrow[d, "\rho^{\mathsf{Sets}_*,\rhd}_{X\rhd Y}"] \\
& X \rhd Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \rhd (y \rhd 0) \arrow[r, mapsto] \arrow[dr, mapsto] & (x \rhd y) \rhd 0 \arrow[d, mapsto] \\
& x \rhd y_0 = x_0 \rhd y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
x \rhd (y \rhd 1) \arrow[r, mapsto] \arrow[dr, mapsto] & (x \rhd y) \rhd 1 \arrow[d, mapsto] \\
& x \rhd y
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.

The Right Skew Middle Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
X \rhd Y \arrow[rr, equal] \arrow[d, "\mathrm{id}_X \rhd \lambda^{\mathsf{Sets}_*,\rhd}_Y"'] & & X \rhd Y \\
X \rhd (S^0 \rhd Y) \arrow[rr, "\alpha^{\mathsf{Sets}_*,\rhd}_{X,S^0,Y}"'] & & (X \rhd S^0) \rhd Y \arrow[u, "\rho^{\mathsf{Sets}_*,\rhd}_X \rhd \mathrm{id}_Y"']
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \rhd y \arrow[r, mapsto] \arrow[d, mapsto] & x \rhd y \\
x \rhd (1 \rhd y) \arrow[r, mapsto] & (x \rhd 1) \rhd y \arrow[u, mapsto]
\end{tikzcd}$$

and hence indeed commutes. Thus the right skew triangle identity is satisfied.
The Zig-Zag Identity: We have to show that the diagram

$$\begin{tikzcd}
S^0 \arrow[r, "\lambda^{\mathsf{Sets}_*,\rhd}_{S^0}"] \arrow[dr, equal] & S^0 \rhd S^0 \arrow[d, "\rho^{\mathsf{Sets}_*,\rhd}_{S^0}"] \\
 & S^0
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
0 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \rhd 0 \arrow[d, mapsto] \\
 & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \rhd 1 \arrow[d, mapsto] \\
 & 1
\end{tikzcd}$$

and hence indeed commutes. Thus the zig-zag identity is satisfied.
Right Skew Monoidal Right-Closedness: This follows from Item 2 of Proposition 4.4.1.1.7. □

4.4.9 Monoids With Respect to the Right Tensor Product of Pointed Sets

00FR

00FS **Proposition 4.4.9.1.1.** The category of monoids on $(\mathsf{Sets}_*, \rhd, S^0)$ is isomorphic to the category of "monoids with right zero"[18] and morphisms between them.

Proof. Monoids on $(\mathsf{Sets}_*, \rhd, S^0)$: A monoid on $(\mathsf{Sets}_*, \rhd, S^0)$ consists of:

- *The Underlying Object.* A pointed set $(A, 0_A)$.
- *The Multiplication Morphism.* A morphism of pointed sets

  $$\mu_A \colon A \rhd A \to A,$$

  determining a right bilinear morphism of pointed sets

  $$\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto ab. \end{aligned}$$

- *The Unit Morphism.* A morphism of pointed sets

  $$\eta_A \colon S^0 \to A$$

  picking an element 1_A of A.

satisfying the following conditions:

1. *Associativity.* The diagram

$$\begin{tikzcd} & A \rhd (A \rhd A) \arrow[dr, "\mathrm{id}_A \rhd \mu_A"] & \\ (A \rhd A) \rhd A \arrow[ur, "\alpha^{\mathsf{Sets}_*, \rhd}_{A,A,A}"] \arrow[d, "\mu_A \rhd \mathrm{id}_A"'] & & A \rhd A \arrow[d, "\mu_A"] \\ A \rhd A \arrow[rr, "\mu_A"'] & & A \end{tikzcd}$$

[18] A monoid with right zero is defined similarly as the monoids with zero of **??**. Succinctly, they are monoids (A, μ_A, η_A) with a special element 0_A satisfying

$$0_A a = 0_A$$

for each $a \in A$.

2. *Left Unitality.* The diagram

$$\begin{tikzcd} A \arrow[r, "\lambda_A^{\mathsf{Sets}_*,\rhd}"] \arrow[d, equal] & S^0 \rhd A \arrow[d, "\eta_A \times \mathrm{id}_A"] \\ A & A \rhd A \arrow[l, "\mu_A"] \end{tikzcd}$$

   commutes.

3. *Right Unitality.* The diagram

$$\begin{tikzcd} A \rhd S^0 \arrow[r, "\mathrm{id}_A \times \eta_A"] \arrow[dr, "\rho_A^{\mathsf{Sets}_*,\rhd}"'] & A \rhd A \arrow[d, "\mu_A"] \\ & A \end{tikzcd}$$

   commutes.

Being a right-bilinear morphism of pointed sets, the multiplication map satisfies

$$0_A a = 0_A$$

for each $a \in A$. Now, the associativity, left unitality, and right unitality conditions act on elements as follows:

1. *Associativity.* The associativity condition acts as

$$\begin{tikzcd} (a \rhd b) \rhd c \arrow[d, mapsto] & \\ ab \rhd c \arrow[r, mapsto] & (ab)c \end{tikzcd} \qquad \begin{tikzcd} & a \rhd (b \rhd c) \arrow[dr, mapsto] & \\ (a \rhd b) \rhd c \arrow[ur, mapsto] & & a \rhd bc \arrow[d, mapsto] \\ & & a(bc) \end{tikzcd}$$

   This gives

$$(ab)c = a(bc)$$

   for each $a, b, c \in A$.

2. *Left Unitality.* The left unitality condition acts as

$$\begin{tikzcd} a \arrow[d, mapsto] \\ a \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, mapsto] & 1 \rhd a \arrow[d, mapsto] \\ 1_A a & 1_A \rhd a \arrow[l, mapsto] \end{tikzcd}$$

This gives

$$1_A a = a$$

for each $a \in A$.

3. *Right Unitality.* The right unitality condition acts:

   (a) On $1 \rhd 0$ as

$$\begin{tikzcd}
1 \rhd 0 \arrow[dr, mapsto] & & a \rhd 0 \arrow[r, mapsto] \arrow[dr, mapsto] & a \rhd 0_A \arrow[d, mapsto] \\
& 0_A & & a0_A.
\end{tikzcd}$$

   (b) On $a \rhd 1$ as

$$\begin{tikzcd}
a \rhd 1 \arrow[dr, mapsto] & & a \rhd 1 \arrow[r, mapsto] \arrow[dr, mapsto] & a \rhd 1_A \arrow[d, mapsto] \\
& a & & a1_A.
\end{tikzcd}$$

   This gives

$$\begin{aligned} a1_A &= a, \\ a0_A &= 0_A \end{aligned}$$

   for each $a \in A$.

Thus we see that monoids with respect to $\rhd$ are exactly monoids with right zero.
Morphisms of Monoids on $(\mathsf{Sets}_*, \rhd, S^0)$: A morphism of monoids on $(\mathsf{Sets}_*, \rhd, S^0)$ from $(A, \mu_A, \eta_A, 0_A)$ to $(B, \mu_B, \eta_B, 0_B)$ is a morphism of pointed sets

$$f\colon (A, 0_A) \to (B, 0_B)$$

satisfying the following conditions:

1. *Compatibility With the Multiplication Morphisms.* The diagram

$$\begin{tikzcd}
A \rhd A \arrow[r, "f \rhd f"] \arrow[d, "\mu_A"'] & B \rhd B \arrow[d, "\mu_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

   commutes.

2. *Compatibility With the Unit Morphisms.* The diagram

$$\begin{tikzcd} S^0 \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

commutes.

These act on elements as

$$\begin{tikzcd} a \rhd b \arrow[d, mapsto] & & a \rhd b \arrow[r, mapsto] & f(a) \rhd f(b) \arrow[d, mapsto] \\ ab \arrow[r, mapsto] & f(ab) & & f(a)f(b) \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[dr, mapsto] & & 0 \arrow[r, mapsto] & 0_A \arrow[d, mapsto] \\ & 0_B & & f(0_A) \end{tikzcd}$$

and

$$\begin{tikzcd} 1 \arrow[dr, mapsto] & & 1 \arrow[r, mapsto] & 1_A \arrow[d, mapsto] \\ & 1_B & & f(1_A) \end{tikzcd}$$

giving

$$\begin{aligned} f(ab) &= f(a)f(b), \\ f(0_A) &= 0_B, \\ f(1_A) &= 1_B, \end{aligned}$$

for each $a, b \in A$, which is exactly a morphism of monoids with right zero.

Identities and Composition: Similarly, the identities and composition of $\mathsf{Mon}(\mathsf{Sets}_*, \rhd, S^0)$ can be easily seen to agree with those of monoids with right zero, which finishes the proof. □

00FT 4.5 The Smash Product of Pointed Sets

00FU 4.5.1 Foundations

Let (X, x_0) and (Y, y_0) be pointed sets.

00FV **Definition 4.5.1.1.1.** The **smash product of** (X, x_0) **and** (Y, y_0)[19] is the pointed set $X \wedge Y$[20] satisfying the bijection

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z),$$

naturally in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00FW **Remark 4.5.1.1.2.** That is to say, the smash product of pointed sets is defined so as to induce a bijection between the following data:

- Pointed maps $f \colon X \wedge Y \to Z$.
- Maps of sets $f \colon X \times Y \to Z$ satisfying

$$f(x_0, y) = z_0,$$
$$f(x, y_0) = z_0$$

  for each $x \in X$ and each $y \in Y$.

00FX **Remark 4.5.1.1.3.** The smash product of pointed sets may be described as follows:

- The smash product of (X, x_0) and (Y, y_0) is the pair $((X \wedge Y, x_0 \wedge y_0), \iota)$ consisting of
  - A pointed set $(X \wedge Y, x_0 \wedge y_0)$;
  - A bilinear morphism of pointed sets $\iota \colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

  satisfying the following universal property:

  (UP) Given another such pair $((Z, z_0), f)$ consisting of
  - $*$ A pointed set (Z, z_0);
  - $*$ A bilinear morphism of pointed sets $f \colon (X \times Y, (x_0, y_0)) \to X \wedge Y$;

  there exists a unique morphism of pointed sets $X \wedge Y \xrightarrow{\exists!} Z$ making the diagram

$$\begin{tikzcd} & X \wedge Y \arrow[d, dashed, "\exists!"] \\ X \times Y \arrow[ur, "\iota"] \arrow[r, "f"'] & Z \end{tikzcd}$$

  commute.

[19] *Further Terminology:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras, the smash product $X \wedge Y$ is also called the **tensor product of** $\mathbb{F}_1$**-modules of** (X, x_0) **and** (Y, y_0) or the **tensor product of** (X, x_0) **and** (Y, y_0) **over** $\mathbb{F}_1$.

[20] *Further Notation:* In the context of monoids with zero as models for $\mathbb{F}_1$-algebras,

00FY **Construction 4.5.1.1.4.** Concretely, the **smash product of** (X, x_0) **and** (Y, y_0) is the pointed set $(X \wedge Y, x_0 \wedge y_0)$ consisting of

- *The Underlying Set.* The set $X \wedge Y$ defined by
$$X \wedge Y \cong (X \times Y)/{\sim_R},$$
where $\sim_R$ is the equivalence relation on $X \times Y$ obtained by declaring
$$\begin{aligned}(x_0, y) &\sim_R (x_0, y'),\\ (x, y_0) &\sim_R (x', y_0)\end{aligned}$$
for each $x, x' \in X$ and each $y, y' \in Y$;
- *The Basepoint.* The element $[(x_0, y_0)]$ of $X \wedge Y$ given by the equivalence class of (x_0, y_0) under the equivalence relation $\sim$ on $X \times Y$.

Proof. By Item 6 of Proposition 7.5.2.1.3, we have a natural bijection
$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z).$$
Now, by definition, $\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z)$ is the set
$$\mathrm{Hom}^R_{\mathsf{Sets}}(X \times Y, Z) \stackrel{\text{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z) \;\middle|\; \begin{array}{l}\text{for each } x, y \in X, \text{ if}\\ (x, y) \sim_R (x', y'), \text{ then}\\ f(x, y) = f(x', y')\end{array} \right\}.$$
However, the condition $(x, y) \sim_R (x', y')$ only holds when:

1. We have $x = x'$ and $y = y'$.
2. The following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

So, given $f \in \mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)$ with a corresponding $\overline{f} \colon X \wedge Y \to Z$, the latter case above implies
$$\begin{aligned}f(x_0, y) &= f(x, y_0)\\ &= f(x_0, y_0),\end{aligned}$$
and since $\overline{f} \colon X \wedge Y \to Z$ is a pointed map, we have
$$\begin{aligned}f(x_0, y_0) &= \overline{f}(x_0, y_0)\\ &= z_0.\end{aligned}$$

Thus the elements f in $\mathrm{Hom}_{\mathsf{Sets}}(X \times Y, Z)$ are precisely those functions $f \colon X \times Y \to Z$ satisfying the equalities

$$\begin{aligned} f(x_0, y) &= z_0, \\ f(x, y_0) &= z_0 \end{aligned}$$

for each $x \in X$ and each $y \in Y$, giving an equality

$$\mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z) = \mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$$

of sets, which when composed with our earlier isomorphism

$$\mathsf{Sets}_*(X \wedge Y, Z) \cong \mathrm{Hom}^{R}_{\mathsf{Sets}}(X \times Y, Z)$$

gives our desired natural bijection, finishing the proof. □

00FZ **Remark 4.5.1.1.5.** It is also somewhat common to write

$$X \wedge Y \stackrel{\text{def}}{=} \frac{X \times Y}{X \vee Y},$$

identifying $X \vee Y$ with the subspace $(\{x_0\} \times Y) \cup (X \times \{y_0\})$ of $X \times Y$, and having the quotient be defined by declaring $(x, y) \sim (x', y')$ iff we have $(x, y), (x', y') \in X \vee Y$.

00G0 **Notation 4.5.1.1.6.** We write $x \wedge y$ for the element $[(x, y)]$ of

$$X \wedge Y \cong X \times Y/{\sim}.$$

00G1 **Remark 4.5.1.1.7.** Employing the notation introduced in Notation 4.5.1.1.6, we have

$$\begin{aligned} x_0 \wedge y_0 &= x \wedge y_0, \\ &= x_0 \wedge y \end{aligned}$$

for each $x \in X$ and each $y \in Y$, and

$$\begin{aligned} x \wedge y_0 &= x' \wedge y_0, \\ x_0 \wedge y &= x_0 \wedge y' \end{aligned}$$

for each $x, x' \in X$ and each $y, y' \in Y$.

00G2 **Example 4.5.1.1.8.** Here are some examples of smash products of pointed sets.

the smash product $X \wedge Y$ is also denoted $X \otimes_{\mathbb{F}_1} Y$.

00G3 1. *Smashing With pt.* For any pointed set X, we have isomorphisms of pointed sets

$$
\begin{aligned}
\mathrm{pt} \wedge X &\cong \mathrm{pt},\\
X \wedge \mathrm{pt} &\cong \mathrm{pt}.
\end{aligned}
$$

00G4 2. *Smashing With S^0.* For any pointed set X, we have isomorphisms of pointed sets

$$
\begin{aligned}
S^0 \wedge X &\cong X,\\
X \wedge S^0 &\cong X.
\end{aligned}
$$

00G5 **Proposition 4.5.1.1.9.** Let (X, x_0) and (Y, y_0) be pointed sets.

00G6 1. *Functoriality.* The assignments $X, Y, (X, Y) \mapsto X \wedge Y$ define functors

$$
\begin{aligned}
X \wedge - &: \mathsf{Sets}_* \to \mathsf{Sets}_*,\\
- \wedge Y &: \mathsf{Sets}_* \to \mathsf{Sets}_*,\\
-_1 \wedge -_2 &: \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*.
\end{aligned}
$$

In particular, given pointed maps

$$
\begin{aligned}
f &: (X, x_0) \to (A, a_0),\\
g &: (Y, y_0) \to (B, b_0),
\end{aligned}
$$

the induced map

$$f \wedge g : X \wedge Y \to A \wedge B$$

is given by

$$[f \wedge g](x \wedge y) \overset{\mathrm{def}}{=} f(x) \wedge g(y)$$

for each $x \wedge y \in X \wedge Y$.

00G7 2. *Adjointness.* We have adjunctions

$$
(X \wedge - \dashv \mathbf{Sets}_*(X, -)) \colon \quad
\begin{tikzcd}
\mathsf{Sets}_* \arrow[r, bend left, "X \wedge -"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X, -)"]
\end{tikzcd}
$$

$$
(- \wedge Y \dashv \mathbf{Sets}_*(Y, -)) \colon \quad
\begin{tikzcd}
\mathsf{Sets}_* \arrow[r, bend left, "- \wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y, -)"]
\end{tikzcd}
$$

witnessed by bijections

$$
\begin{aligned}
\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),\\
\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) &\cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(A, Z)),
\end{aligned}
$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00G8 3. *Enriched Adjointness.* We have Sets_*-enriched adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X,-)) \colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "X\wedge-"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_* \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd},$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y,-)) \colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_* \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd},$$

witnessed by isomorphisms of pointed sets

$$\mathbf{Sets}_*(X \wedge Y, Z) \cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),$$
$$\mathbf{Sets}_*(X \wedge Y, Z) \cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$.

00G9 4. *As a Pushout.* We have an isomorphism

$$X \wedge Y \cong \mathrm{pt} \coprod_{X \vee Y} (X \times Y), \qquad \begin{tikzcd} X \wedge Y \arrow[dr, phantom, "\ulcorner", very near start] & X \times Y \arrow[l] \\ \mathrm{pt} \arrow[u] & X \vee Y, \arrow[l, "!"] \arrow[u, hook, "\iota"'] \end{tikzcd}$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$, where the pushout is taken in Sets, and the embedding $\iota \colon X \vee Y \hookrightarrow X \times Y$ is defined following Remark 4.5.1.1.5.

00GA 5. *Distributivity Over Wedge Sums.* We have isomorphisms of pointed sets

$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z),$$
$$(X \vee Y) \wedge Z \cong (X \wedge Z) \vee (Y \wedge Z),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

Proof. Item 1, Functoriality: The map $f \wedge g$ comes from Item 4 of Proposition 7.5.2.1.3 via the map

$$f \wedge g \colon X \times Y \to A \wedge B$$

sending (x, y) to $f(x) \wedge g(y)$, which we need to show satisfies

$$[f \wedge g](x, y) = [f \wedge g](x', y')$$

for each $(x, y), (x', y') \in X \times Y$ with $(x, y) \sim_R (x', y')$, where $\sim_R$ is the relation constructing $X \wedge Y$ as

$$X \wedge Y \cong (X \times Y)/{\sim_R}$$

in Construction 4.5.1.1.4. The condition defining $\sim$ is that at least one of the following conditions is satisfied:

1. We have $x = x'$ and $y = y'$;
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = y_0$.
   (b) We have $x' = x_0$ or $y' = y_0$.

We have five cases:

1. In the first case, we clearly have

$$[f \wedge g](x, y) = [f \wedge g](x', y')$$

   since $x = x'$ and $y = y'$.

2. If $x = x_0$ and $x' = x_0$, we have

$$\begin{aligned}[f \wedge g](x_0, y) &\stackrel{\text{def}}{=} f(x_0) \wedge g(y) \\ &= a_0 \wedge g(y) \\ &= a_0 \wedge g(y') \\ &= f(x_0) \wedge g(y') \\ &\stackrel{\text{def}}{=} [f \wedge g](x_0, y').\end{aligned}$$

3. If $x = x_0$ and $y' = y_0$, we have

$$\begin{aligned}[f \wedge g](x_0, y) &\stackrel{\text{def}}{=} f(x_0) \wedge g(y) \\ &= a_0 \wedge g(y) \\ &= a_0 \wedge b_0 \\ &= f(x') \wedge b_0 \\ &= f(x') \wedge g(y_0) \\ &\stackrel{\text{def}}{=} [f \wedge g](x', y_0).\end{aligned}$$

4. If $y = y_0$ and $x' = x_0$, we have

$$\begin{aligned}[f \wedge g](x, y_0) &\stackrel{\text{def}}{=} f(x) \wedge g(y_0) \\ &= f(x) \wedge b_0 \\ &= a_0 \wedge b_0 \\ &= a_0 \wedge g(y') \\ &= f(x_0) \wedge g(y') \\ &\stackrel{\text{def}}{=} [f \wedge g](x_0, y').\end{aligned}$$

5. If $y = y_0$ and $y' = y_0$, we have

$$\begin{aligned}[f \wedge g](x, y_0) &\stackrel{\text{def}}{=} f(x) \wedge g(y_0) \\ &= f(x) \wedge b_0 \\ &= f(x') \wedge b_0 \\ &= f(x) \wedge g(y_0) \\ &\stackrel{\text{def}}{=} [f \wedge g](x', y_0).\end{aligned}$$

Thus $f \wedge g$ is well-defined. Next, we claim that $\wedge$ preserves identities and composition:

- *Preservation of Identities.* We have

$$\begin{aligned}[\mathrm{id}_X \wedge \mathrm{id}_Y](x \wedge y) &\stackrel{\text{def}}{=} \mathrm{id}_X(x) \wedge \mathrm{id}_Y(y) \\ &= x \wedge y \\ &= [\mathrm{id}_{X \wedge Y}](x \wedge y)\end{aligned}$$

  for each $x \wedge y \in X \wedge Y$, and thus

$$\mathrm{id}_X \wedge \mathrm{id}_Y = \mathrm{id}_{X \wedge Y}.$$

- *Preservation of Composition.* Given pointed maps

$$\begin{aligned}f &\colon (X, x_0) \to (X', x_0'), \\ h &\colon (X', x_0') \to (X'', x_0''), \\ g &\colon (Y, y_0) \to (Y', y_0'), \\ k &\colon (Y', y_0') \to (Y'', y_0''),\end{aligned}$$

  we have

$$\begin{aligned}[(h \circ f) \wedge (k \circ g)](x \wedge y) &\stackrel{\text{def}}{=} h(f(x)) \wedge k(g(y)) \\ &\stackrel{\text{def}}{=} [h \wedge k](f(x) \wedge g(y)) \\ &\stackrel{\text{def}}{=} [h \wedge k]([f \wedge g](x \wedge y)) \\ &\stackrel{\text{def}}{=} [(h \wedge k) \circ (f \wedge g)](x \wedge y)\end{aligned}$$

  for each $x \wedge y \in X \wedge Y$, and thus

$$(h \circ f) \wedge (k \circ g) = (h \wedge k) \circ (f \wedge g).$$

This finishes the proof.
Item 2, Adjointness: We prove only the adjunction $- \wedge Y \dashv \mathsf{Sets}_*(Y, -)$, witnessed by a natural bijection

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$

as the proof of the adjunction $X \wedge - \dashv \mathsf{\textbf{Sets}}_*(X, -)$ is similar. We claim we have a bijection

$$\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathsf{\textbf{Sets}}_*(Y, Z))$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$, impliying the desired adjunction. Indeed, this bijection is a restriction of the bijection

$$\mathsf{Sets}(X \times Y, Z) \cong \mathsf{Sets}(X, \mathsf{Sets}(Y, Z))$$

of Item 2 of Proposition 2.1.3.1.2:

- A map
$$\xi \colon X \times Y \to Z$$
in $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$ gets sent to the pointed map
$$\begin{aligned} \xi^\dagger \colon (X, x_0) &\to (\mathsf{\textbf{Sets}}_*(Y, Z), \Delta_{z_0}), \\ x &\longmapsto \left(\xi^\dagger_x \colon Y \to Z\right), \end{aligned}$$
where $\xi^\dagger_x \colon Y \to Z$ is the map defined by
$$\xi^\dagger_x(y) \stackrel{\text{def}}{=} \xi(x, y)$$
for each $y \in Y$, where:
  - The map $\xi^\dagger$ is indeed pointed, as we have
$$\begin{aligned} \xi^\dagger_{x_0}(y) &\stackrel{\text{def}}{=} \xi(x_0, y) \\ &\stackrel{\text{def}}{=} z_0 \end{aligned}$$
for each $y \in Y$. Thus $\xi^\dagger_{x_0} = \Delta_{z_0}$ and $\xi^\dagger$ is pointed.
  - The map $\xi^\dagger_x$ indeed lies in $\mathsf{\textbf{Sets}}_*(Y, Z)$, as we have
$$\begin{aligned} \xi^\dagger_x(y_0) &\stackrel{\text{def}}{=} \xi(x, y_0) \\ &\stackrel{\text{def}}{=} z_0. \end{aligned}$$

- Conversely, a map
$$\begin{aligned} \xi \colon (X, x_0) &\to (\mathsf{\textbf{Sets}}_*(Y, Z), \Delta_{z_0}), \\ x &\longmapsto (\xi_x \colon Y \to Z), \end{aligned}$$
in $\mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathsf{\textbf{Sets}}_*(Y, Z))$ gets sent to the map
$$\xi^\dagger \colon X \times Y \to Z$$
defined by
$$\xi^\dagger(x, y) \stackrel{\text{def}}{=} \xi_x(y)$$
for each $(x, y) \in X \times Y$, which indeed lies in $\mathrm{Hom}^{\otimes}_{\mathsf{Sets}_*}(X \times Y, Z)$, as:

- *Left Bilinearity.* We have

$$\begin{aligned}\xi^{\dagger}(x_0, y) &\stackrel{\text{def}}{=} \xi_{x_0}(y)\\ &\stackrel{\text{def}}{=} \Delta_{z_0}(y)\\ &\stackrel{\text{def}}{=} z_0\end{aligned}$$

  for each $y \in Y$, since $\xi_{x_0} = \Delta_{z_0}$ as ξ is assumed to be a pointed map.

- *Right Bilinearity.* We have

$$\begin{aligned}\xi^{\dagger}(x, y_0) &\stackrel{\text{def}}{=} \xi_{x}(y_0)\\ &\stackrel{\text{def}}{=} z_0\end{aligned}$$

  for each $x \in X$, since $\xi_x \in \mathbf{Sets}_*(Y, Z)$ is a morphism of pointed sets.

This finishes the proof.
Item 3, Enriched Adjointness: This follows from Item 2 and **??** of **??**.
Item 4, As a Pushout: Following the description of Remark 2.2.4.1.2, we have

$$\mathrm{pt} \coprod_{X \vee Y} (X \times Y) \cong (\mathrm{pt} \times (X \times Y))/\sim,$$

where $\sim$ identifies the elemenet $\star$ in pt with all elements of the form (x_0, y) and (x, y_0) in $X \times Y$. Thus Item 4 of Proposition 7.5.2.1.3 coupled with Remark 4.5.1.1.7 then gives us a well-defined map

$$\mathrm{pt} \coprod_{X \vee Y} (X \times Y) \to X \wedge Y$$

via $[(\star, (x, y))] \mapsto x \wedge y$, with inverse

$$X \wedge Y \to \mathrm{pt} \coprod_{X \vee Y} (X \times Y)$$

given by $x \wedge y \mapsto [(\star, (x, y))]$.
Item 5, Distributivity Over Wedge Sums: This follows from Proposition 4.5.9.1.1, **??** of **??**, and the fact that $\vee$ is the coproduct in Sets_* (Definition 3.3.3.1.1). □

00GB 4.5.2 The Internal Hom of Pointed Sets

Let (X, x_0) and (Y, y_0) be pointed sets.

00GC **Definition 4.5.2.1.1.** The **internal Hom**[21] **of pointed sets from** (X, x_0) **to** (Y, y_0) is the pointed set $\mathbf{Sets}_*((X, x_0), (Y, y_0))$[22] consisting of:

[21] The pointed set $\mathbf{Sets}_*(X, Y)$ is the internal **Hom** of Sets_* with respect to the smash product of Definition 4.5.1.1.1; see Item 2 of Proposition 4.5.1.1.9.
[22] *Further Notation:* Also written $\mathbf{Hom}_{\mathsf{Sets}_*}(X, Y)$.

- *The Underlying Set.* The set $\mathsf{Sets}_*((X,x_0),(Y,y_0))$ of morphisms of pointed sets from (X,x_0) to (Y,y_0).
- *The Basepoint.* The element

$$\Delta_{y_0}\colon (X,x_0)\to(Y,y_0)$$

of $\mathsf{Sets}_*((X,x_0),(Y,y_0))$ given by

$$\Delta_{y_0}(x)\stackrel{\text{def}}{=}y_0$$

for each $x\in X$.

Proof. For a proof that $\mathbf{Sets}_*$ is indeed the internal Hom of Sets_* with respect to the smash product of pointed sets, see Item 2 of Proposition 4.5.1.1.9. □

00GD **Proposition 4.5.2.1.2.** Let (X,x_0) and (Y,y_0) be pointed sets.

00GE 1. *Functoriality.* The assignments $X,Y,(X,Y)\mapsto\mathbf{Sets}_*(X,Y)$ define functors

$$\begin{aligned}\mathbf{Sets}_*(X,-)\colon \mathsf{Sets}_*&\to\mathsf{Sets}_*,\\ \mathbf{Sets}_*(-,Y)\colon \mathsf{Sets}_*^{\mathsf{op}}&\to\mathsf{Sets}_*,\\ \mathbf{Sets}_*(-_1,-_2)\colon \mathsf{Sets}_*^{\mathsf{op}}\times\mathsf{Sets}_*&\to\mathsf{Sets}_*.\end{aligned}$$

In particular, given pointed maps

$$\begin{aligned}f\colon (X,x_0)&\to(A,a_0),\\ g\colon (Y,y_0)&\to(B,b_0),\end{aligned}$$

the induced map

$$\mathbf{Sets}_*(f,g)\colon \mathbf{Sets}_*(A,Y)\to\mathbf{Sets}_*(X,B)$$

is given by

$$[\mathbf{Sets}_*(f,g)](\phi)\stackrel{\text{def}}{=}g\circ\phi\circ f$$

for each $\phi\in\mathbf{Sets}_*(A,Y)$.

00GF 2. *Adjointness.* We have adjunctions

$$(X\wedge-\dashv\mathbf{Sets}_*(X,-))\colon\quad \begin{tikzcd}\mathsf{Sets}_* \arrow[r, bend left, "X\wedge-"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"]\end{tikzcd}$$

$$(-\wedge Y\dashv\mathbf{Sets}_*(Y,-))\colon\quad \begin{tikzcd}\mathsf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathsf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"]\end{tikzcd}$$

witnessed by bijections

$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(Y, Z)),$$
$$\mathrm{Hom}_{\mathsf{Sets}_*}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathsf{Sets}_*}(X, \mathbf{Sets}_*(A, Z)),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$.

00GG 3. *Enriched Adjointness.* We have Sets_*-enriched adjunctions

$$(X \wedge - \dashv \mathbf{Sets}_*(X, -)) \colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "X\wedge -"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(X,-)"] \end{tikzcd}$$

$$(- \wedge Y \dashv \mathbf{Sets}_*(Y, -)) \colon \quad \begin{tikzcd} \mathbf{Sets}_* \arrow[r, bend left, "-\wedge Y"] \arrow[r, phantom, "\perp"] & \mathbf{Sets}_*, \arrow[l, bend left, "\mathbf{Sets}_*(Y,-)"] \end{tikzcd}$$

witnessed by isomorphisms of pointed sets

$$\mathbf{Sets}_*(X \wedge Y, Z) \cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(Y, Z)),$$
$$\mathbf{Sets}_*(X \wedge Y, Z) \cong \mathbf{Sets}_*(X, \mathbf{Sets}_*(A, Z)),$$

natural in $(X, x_0), (Y, y_0), (Z, z_0) \in \mathrm{Obj}(\mathbf{Sets}_*)$.

Proof. Item 1, Functoriality: This follows from Item 1 of Proposition 2.3.5.1.2 and from the equalities

$$g \circ \Delta_{y_0} = \Delta_{z_0},$$
$$\Delta_{y_0} \circ f = \Delta_{y_0}$$

for morphisms $f \colon (K, k_0) \to (X, x_0)$ and $g \colon (Y, y_0) \to (Z, z_0)$, which guarantee pre- and postcomposition by morphisms of pointed sets to also be morphisms of pointed sets.

Item 2, Adjointness: This is a repetition of Item 2 of Proposition 4.5.1.1.9, and is proved there.

Item 3, Enriched Adjointness: This is a repetition of Item 3 of Proposition 4.5.1.1.9, and is proved there. □

00GH 4.5.3 The Monoidal Unit

00GJ **Definition 4.5.3.1.1.** The **monoidal unit of the smash product of pointed sets** is the functor

$$\mathbb{1}^{\mathsf{Sets}_*} \colon \mathsf{pt} \to \mathsf{Sets}_*$$

defined by

$$\mathbb{1}_{\mathsf{Sets}_*} \stackrel{\text{def}}{=} S^0.$$

00GK 4.5.4 The Associator

00GL **Definition 4.5.4.1.1.** The **associator of the smash product of pointed sets** is the natural isomorphism

$$\alpha^{\mathsf{Sets}_*}\colon \wedge\circ(\wedge\times\mathrm{id}_{\mathsf{Sets}_*}) \overset{\sim}{\Longrightarrow} \wedge\circ(\mathrm{id}_{\mathsf{Sets}_*}\times\wedge)\circ\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*},$$

as in the diagram

$$\begin{tikzcd}
 & \mathsf{Sets}_*\times(\mathsf{Sets}_*\times\mathsf{Sets}_*) \arrow[dr, "\mathsf{id}\times\wedge"] & \\
(\mathsf{Sets}_*\times\mathsf{Sets}_*)\times\mathsf{Sets}_* \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}", "\sim"'] \arrow[d, "\wedge\times\mathsf{id}"'] \arrow[rr, phantom, "\Uparrow \alpha^{\mathsf{Sets}_*}"] & & \mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[d, "\wedge"] \\
\mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[rr, "\wedge"'] & & \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}\colon (X\wedge Y)\wedge Z \xrightarrow{\cong} X\wedge(Y\wedge Z)$$

at $(X,x_0),(Y,y_0),(Z,z_0)\in\mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}((x\wedge y)\wedge z) \overset{\mathrm{def}}{=} x\wedge(y\wedge z)$$

for each $(x\wedge y)\wedge z\in(X\wedge Y)\wedge Z$.

Proof. Well-Definedness: Let $[((x,y),z)]=[((x',y'),z')]$ be an element in $(X\wedge Y)\wedge Z$. Then either:

1. We have $x=x'$, $y=y'$, and $z=z'$.
2. Both of the following conditions are satisfied:
   (a) We have $x=x_0$ or $y=y_0$ or $z=z_0$.
   (b) We have $x'=x_0$ or $y'=y_0$ or $z'=z_0$.

In the first case, $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ clearly sends both elements to the same element in $X\wedge(Y\wedge Z)$. Meanwhile, in the latter case both elements are equal to the basepoint $(x_0\wedge y_0)\wedge z_0$ of $(X\wedge Y)\wedge Z$, which gets sent to the basepoint $x_0\wedge(y_0\wedge z_0)$ of $X\wedge(Y\wedge Z)$.
Being a Morphism of Pointed Sets: As just mentioned, we have

$$\alpha^{\mathsf{Sets}_*}_{X,Y,Z}((x_0\wedge y_0)\wedge z_0) \overset{\mathrm{def}}{=} x_0\wedge(y_0\wedge z_0),$$

and thus $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ is a morphism of pointed sets.

Invertibility: Clearly, the inverse of $\alpha^{\mathsf{Sets}_*}_{X,Y,Z}$ is given by the morphism

$$\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z} \colon X \wedge (Y \wedge Z) \xrightarrow{\cong} (X \wedge Y) \wedge Z$$

defined by

$$\alpha^{\mathsf{Sets}_*,-1}_{X,Y,Z}(x \wedge (y \wedge z)) \stackrel{\text{def}}{=} (x \wedge y) \wedge z$$

for each $x \wedge (y \wedge z) \in X \wedge (Y \wedge Z)$.

Naturality: We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &\colon (X, x_0) \to (X', x'_0), \\ g &\colon (Y, y_0) \to (Y', y'_0), \\ h &\colon (Z, z_0) \to (Z', z'_0) \end{aligned}$$

the diagram

$$\begin{tikzcd} (X \wedge Y) \wedge Z \arrow[r, "(f \wedge g) \wedge h"] \arrow[d, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] & (X' \wedge Y') \wedge Z' \arrow[d, "\alpha^{\mathsf{Sets}_*}_{X',Y',Z'}"] \\ X \wedge (Y \wedge Z) \arrow[r, "f \wedge (g \wedge h)"'] & X' \wedge (Y' \wedge Z') \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} (x \wedge y) \wedge z \arrow[r, mapsto] \arrow[d, mapsto] & (f(x) \wedge g(y)) \wedge h(z) \arrow[d, mapsto] \\ x \wedge (y \wedge z) \arrow[r, mapsto] & f(x) \wedge (g(y) \wedge h(z)) \end{tikzcd}$$

and hence indeed commutes, showing α^{Sets_*} to be a natural transformation.

Being a Natural Isomorphism: Since α^{Sets_*} is natural and $\alpha^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to α^{Sets_*}, it follows from Item 2 of Proposition 8.8.6.1.2 that $\alpha^{\mathsf{Sets}_*,-1}$ is also natural. Thus α^{Sets_*} is a natural isomorphism. □

00GM 4.5.5 The Left Unitor

00GN **Definition 4.5.5.1.1.** The **left unitor of the smash product of pointed sets** is the natural isomorphism

$$\lambda^{\mathsf{Sets}_*}\colon \wedge\circ\left(\mathbb{1}^{\mathsf{Sets}_*}\times\mathrm{id}_{\mathsf{Sets}_*}\right)\overset{\sim}{\Longrightarrow}\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}\qquad
\begin{tikzcd}
\mathsf{pt}\times\mathsf{Sets}_* \arrow[r, "\mathbb{1}^{\mathsf{Sets}_*}\times\mathsf{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[d, "\wedge"] \arrow[dl, phantom, "\Leftarrow\lambda^{\mathsf{Sets}_*}"] \\
& \mathsf{Sets}_*,
\end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Sets}_*}_X\colon S^0\wedge X\xrightarrow{\cong}X$$

at $X\in\mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned}0\wedge x&\mapsto x_0,\\ 1\wedge x&\mapsto x.\end{aligned}$$

Proof. Well-Definedness: Let $[(x,y)]=[(x',y')]$ be an element in $S^0\wedge X$. Then either:

1. We have $x=x'$ and $y=y'$.
2. Both of the following conditions are satisfied:
   (a) We have $x=0$ or $y=x_0$.
   (b) We have $x'=0$ or $y'=x_0$.

In the first case, $\lambda^{\mathsf{Sets}_*}_X$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $0\wedge x_0$ of $S^0\wedge X$, which gets sent to the basepoint x_0 of X.
Being a Morphism of Pointed Sets: As just mentioned, we have

$$\lambda^{\mathsf{Sets}_*}_X(0\wedge x_0)\overset{\mathrm{def}}{=}x_0,$$

and thus $\lambda^{\mathsf{Sets}_*}_X$ is a morphism of pointed sets.
Invertibility: The inverse of $\lambda^{\mathsf{Sets}_*}_X$ is the morphism

$$\lambda^{\mathsf{Sets}_*,-1}_X\colon X\xrightarrow{\cong}S^0\wedge X$$

defined by

$$\lambda^{\mathsf{Sets}_*,-1}_X(x)\overset{\mathrm{def}}{=}1\wedge x$$

for each $x\in X$. Indeed:

- *Invertibility I.* We have
$$\begin{aligned}\left[\lambda_X^{\mathsf{Sets}_*,-1}\circ\lambda_X^{\mathsf{Sets}_*}\right](0\wedge x)&=\lambda_X^{\mathsf{Sets}_*,-1}\left(\lambda_X^{\mathsf{Sets}_*}(0\wedge x)\right)\\&=\lambda_X^{\mathsf{Sets}_*,-1}(x_0)\\&=1\wedge x_0\\&=0\wedge x,\end{aligned}$$
and
$$\begin{aligned}\left[\lambda_X^{\mathsf{Sets}_*,-1}\circ\lambda_X^{\mathsf{Sets}_*}\right](1\wedge x)&=\lambda_X^{\mathsf{Sets}_*,-1}\left(\lambda_X^{\mathsf{Sets}_*}(1\wedge x)\right)\\&=\lambda_X^{\mathsf{Sets}_*,-1}(x)\\&=1\wedge x\end{aligned}$$
for each $x\in X$, and thus we have
$$\lambda_X^{\mathsf{Sets}_*,-1}\circ\lambda_X^{\mathsf{Sets}_*}=\mathrm{id}_{S^0\wedge X}.$$

- *Invertibility II.* We have
$$\begin{aligned}\left[\lambda_X^{\mathsf{Sets}_*}\circ\lambda_X^{\mathsf{Sets}_*,-1}\right](x)&=\lambda_X^{\mathsf{Sets}_*}\left(\lambda_X^{\mathsf{Sets}_*,-1}(x)\right)\\&=\lambda_X^{\mathsf{Sets}_*,-1}(1\wedge x)\\&=x\end{aligned}$$
for each $x\in X$, and thus we have
$$\lambda_X^{\mathsf{Sets}_*}\circ\lambda_X^{\mathsf{Sets}_*,-1}=\mathrm{id}_X.$$

This shows $\lambda_X^{\mathsf{Sets}_*}$ to be invertible.
Naturality: We need to show that, given a morphism of pointed sets
$$f\colon(X,x_0)\to(Y,y_0),$$
the diagram
$$\begin{tikzcd}S^0\wedge X \arrow[r, "\mathrm{id}_{S^0}\wedge f"] \arrow[d, "\lambda_X^{\mathsf{Sets}_*}"'] & S^0\wedge Y \arrow[d, "\lambda_Y^{\mathsf{Sets}_*}"] \\ X \arrow[r, "f"'] & Y\end{tikzcd}$$
commutes. Indeed, this diagram acts on elements as
$$\begin{tikzcd}0\wedge x \arrow[d, mapsto] & \\ x_0 \arrow[r, mapsto] & f(x_0)\end{tikzcd}\qquad\begin{tikzcd}0\wedge x \arrow[r, mapsto] & 0\wedge f(x) \arrow[d, mapsto] \\ & y_0\end{tikzcd}$$

and

$$\begin{tikzcd} 1 \wedge x \arrow[r, mapsto] \arrow[d, mapsto] & 1 \wedge f(x) \arrow[d, mapsto] \\ x \arrow[r, mapsto] & f(x) \end{tikzcd}$$

and hence indeed commutes, showing $\lambda^{\mathsf{Sets}_*}$ to be a natural transformation.
Being a Natural Isomorphism: Since $\lambda^{\mathsf{Sets}_*}$ is natural and $\lambda^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to $\lambda^{\mathsf{Sets}_*}$, it follows from Item 2 of Proposition 8.8.6.1.2 that $\lambda^{\mathsf{Sets}_*,-1}$ is also natural. Thus $\lambda^{\mathsf{Sets}_*}$ is a natural isomorphism. □

00GP 4.5.6 The Right Unitor

00GQ **Definition 4.5.6.1.1.** The **right unitor of the smash product of pointed sets** is the natural isomorphism

$$\rho^{\mathsf{Sets}_*} : \wedge \circ \left(\mathsf{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \times \mathsf{pt} \arrow[r, "\mathsf{id} \times \mathbb{1}^{\mathsf{Sets}_*}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \mathsf{Sets}_* \times \mathsf{Sets}_* \arrow[d, "\wedge"] \arrow[dl, phantom, "\Swarrow \rho^{\mathsf{Sets}_*}"] \\ & \mathsf{Sets}_*, \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Sets}_*}_X : X \wedge S^0 \xrightarrow{\cong} X$$

at $X \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\begin{aligned} x \wedge 0 &\mapsto x_0, \\ x \wedge 1 &\mapsto x. \end{aligned}$$

Proof. Well-Definedness: Let $[(x, y)] = [(x', y')]$ be an element in $X \wedge S^0$. Then either:

1. We have $x = x'$ and $y = y'$.
2. Both of the following conditions are satisfied:
   (a) We have $x = x_0$ or $y = 0$.
   (b) We have $x' = x_0$ or $y' = 0$.

In the first case, $\rho_X^{\mathsf{Sets}_*}$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $x_0 \wedge 0$ of $X \wedge S^0$, which gets sent to the basepoint x_0 of X.
Being a Morphism of Pointed Sets: As just mentioned, we have

$$\rho_X^{\mathsf{Sets}_*}(x_0 \wedge 0) \stackrel{\text{def}}{=} x_0,$$

and thus $\rho_X^{\mathsf{Sets}_*}$ is a morphism of pointed sets.
Invertibility: The inverse of $\rho_X^{\mathsf{Sets}_*}$ is the morphism

$$\rho_X^{\mathsf{Sets}_*,-1} \colon X \xrightarrow{\cong} X \wedge S^0$$

defined by

$$\rho_X^{\mathsf{Sets}_*,-1}(x) \stackrel{\text{def}}{=} x \wedge 1$$

for each $x \in X$. Indeed:

- *Invertibility I.* We have

  $$\begin{aligned}\left[\rho_X^{\mathsf{Sets}_*,-1} \circ \rho_X^{\mathsf{Sets}_*}\right](x \wedge 0) &= \rho_X^{\mathsf{Sets}_*,-1}\left(\rho_X^{\mathsf{Sets}_*}(x \wedge 0)\right) \\ &= \rho_X^{\mathsf{Sets}_*,-1}(x_0) \\ &= x_0 \wedge 1 \\ &= x \wedge 0,\end{aligned}$$

  and

  $$\begin{aligned}\left[\rho_X^{\mathsf{Sets}_*,-1} \circ \rho_X^{\mathsf{Sets}_*}\right](x \wedge 1) &= \rho_X^{\mathsf{Sets}_*,-1}\left(\rho_X^{\mathsf{Sets}_*}(x \wedge 1)\right) \\ &= \rho_X^{\mathsf{Sets}_*,-1}(x) \\ &= x \wedge 1\end{aligned}$$

  for each $x \in X$, and thus we have

  $$\rho_X^{\mathsf{Sets}_*,-1} \circ \rho_X^{\mathsf{Sets}_*} = \mathrm{id}_{X \wedge S^0}.$$

- *Invertibility II.* We have

  $$\begin{aligned}\left[\rho_X^{\mathsf{Sets}_*} \circ \rho_X^{\mathsf{Sets}_*,-1}\right](x) &= \rho_X^{\mathsf{Sets}_*}\left(\rho_X^{\mathsf{Sets}_*,-1}(x)\right) \\ &= \rho_X^{\mathsf{Sets}_*,-1}(x \wedge 1) \\ &= x\end{aligned}$$

  for each $x \in X$, and thus we have

  $$\rho_X^{\mathsf{Sets}_*} \circ \rho_X^{\mathsf{Sets}_*,-1} = \mathrm{id}_X.$$

This shows $\rho_X^{\mathsf{Sets}_*}$ to be invertible.
Naturality: We need to show that, given a morphism of pointed sets

$$f\colon (X,x_0)\to (Y,y_0),$$

the diagram

$$\begin{tikzcd}
X\wedge S^0 \arrow[r, "f\wedge \mathrm{id}_{S^0}"] \arrow[d, "\rho_X^{\mathsf{Sets}_*}"'] & Y\wedge S^0 \arrow[d, "\rho_Y^{\mathsf{Sets}_*}"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x\wedge 0 \arrow[d, mapsto] & \\
x_0 \arrow[r, mapsto] & f(x_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
x\wedge 0 \arrow[r, mapsto] & f(x)\wedge 0 \arrow[d, mapsto] \\
 & y_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
x\wedge 1 \arrow[r, mapsto] \arrow[d, mapsto] & f(x)\wedge 1 \arrow[d, mapsto] \\
x \arrow[r, mapsto] & f(x)
\end{tikzcd}$$

and hence indeed commutes, showing ρ^{Sets_*} to be a natural transformation.
Being a Natural Isomorphism: Since ρ^{Sets_*} is natural and $\rho^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to ρ^{Sets_*}, it follows from Item 2 of Proposition 8.8.6.1.2 that $\rho^{\mathsf{Sets}_*,-1}$ is also natural. Thus ρ^{Sets_*} is a natural isomorphism. □

00GR 4.5.7 The Symmetry

00GS **Definition 4.5.7.1.1.** The **symmetry of the smash product of pointed sets** is the natural isomorphism

$$\sigma^{\mathsf{Sets}_*}\colon \wedge \overset{\sim}{\Longrightarrow} \wedge\circ\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*},\qquad
\begin{tikzcd}
\mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[rr, "\wedge"] \arrow[dr, "\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}"'] \arrow[rr, phantom, shift right=4ex, "\overset{\|}{\underset{\Downarrow}{\sigma^{\mathsf{Sets}_*}}}"] & & \mathsf{Sets}_*, \\
 & \mathsf{Sets}_*\times\mathsf{Sets}_* \arrow[ur, "\wedge"'] &
\end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Sets}_*}_{X,Y}\colon X\wedge Y\xrightarrow{\cong} Y\wedge X$$

at $X, Y \in \mathrm{Obj}(\mathsf{Sets}_*)$ is defined by

$$\sigma^{\mathsf{Sets}_*}_{X,Y}(x \wedge y) \stackrel{\mathrm{def}}{=} y \wedge x$$

for each $x \wedge y \in X \wedge Y$.

Proof. Well-Definedness: Let $[(x, y)] = [(x', y')]$ be an element in $X \wedge Y$. Then either:

1. We have $x = x'$ and $y = y'$.
2. Both of the following conditions are satisfied:
    (a) We have $x = x_0$ or $y = y_0$.
    (b) We have $x' = x_0$ or $y' = y_0$.

In the first case, $\sigma^{\mathsf{Sets}_*}_{X}$ clearly sends both elements to the same element in X. Meanwhile, in the latter case both elements are equal to the basepoint $x_0 \wedge y_0$ of $X \wedge Y$, which gets sent to the basepoint $y_0 \wedge x_0$ of $Y \wedge X$.
Being a Morphism of Pointed Sets: As just mentioned, we have

$$\sigma^{\mathsf{Sets}_*}_{X}(x_0 \wedge y_0) \stackrel{\mathrm{def}}{=} y_0 \wedge x_0,$$

and thus $\sigma^{\mathsf{Sets}_*}_{X}$ is a morphism of pointed sets.
Invertibility: Clearly, the inverse of $\sigma^{\mathsf{Sets}_*}_{X,Y}$ is given by the morphism

$$\sigma^{\mathsf{Sets}_*,-1}_{X,Y} \colon Y \wedge X \xrightarrow{\cong} X \wedge Y$$

defined by

$$\sigma^{\mathsf{Sets}_*,-1}_{X,Y}(y \wedge x) \stackrel{\mathrm{def}}{=} x \wedge y$$

for each $y \wedge x \in Y \wedge X$.
Naturality: We need to show that, given morphisms of pointed sets

$$\begin{aligned} f &\colon (X, x_0) \to (A, a_0), \\ g &\colon (Y, y_0) \to (B, b_0) \end{aligned}$$

the diagram

$$\begin{tikzcd} X \wedge Y \arrow[r, "f \wedge g"] \arrow[d, "\sigma^{\mathsf{Sets}_*}_{X,Y}"'] & A \wedge B \arrow[d, "\sigma^{\mathsf{Sets}_*}_{A,B}"] \\ Y \wedge X \arrow[r, "g \wedge f"'] & B \wedge A \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \wedge y \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \wedge g(y) \arrow[d, mapsto] \\ y \wedge x \arrow[r, mapsto] & g(y) \wedge f(x) \end{tikzcd}$$

and hence indeed commutes, showing σ^{Sets_*} to be a natural transformation.

Being a Natural Isomorphism: Since σ^{Sets_*} is natural and $\sigma^{\mathsf{Sets}_*,-1}$ is a componentwise inverse to σ^{Sets_*}, it follows from Item 2 of Proposition 8.8.6.1.2 that $\sigma^{\mathsf{Sets}_*,-1}$ is also natural. Thus σ^{Sets_*} is a natural isomorphism. □

00GT 4.5.8 The Diagonal

00GU **Definition 4.5.8.1.1.** The **diagonal of the smash product of pointed sets** is the natural transformation

$$\Delta^{\wedge}\colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}, \qquad \begin{tikzcd} \mathsf{Sets}_* \arrow[rr, bend left, "\mathrm{id}_{\mathsf{Sets}_*}"] \arrow[dr, "\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}"'] & \arrow[d, phantom, "\Downarrow \Delta^{\wedge}"] & \mathsf{Sets}_* \\ & \mathsf{Sets}_* \wedge \mathsf{Sets}_*, \arrow[ur, "\wedge"'] & \end{tikzcd}$$

whose component

$$\Delta^{\wedge}_X\colon (X, x_0) \to (X \wedge X, x_0 \wedge x_0)$$

at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by the composition

$$\begin{aligned} (X, x_0) &\xrightarrow{\Delta^{\wedge}_X} (X \times X, (x_0, x_0)) \\ &\twoheadrightarrow ((X \times X)/{\sim}, [(x_0, x_0)]) \\ &\stackrel{\text{def}}{=} (X \wedge X, x_0 \wedge x_0) \end{aligned}$$

in Sets_*, and thus by

$$\Delta^{\wedge}_X(x) \stackrel{\text{def}}{=} x \wedge x$$

for each $x \in X$.

Proof. Being a Morphism of Pointed Sets: We have

$$\Delta^{\wedge}_X(x_0) \stackrel{\text{def}}{=} x_0 \wedge x_0,$$

and thus $\Delta^{\wedge}_X$ is a morphism of pointed sets.

Naturality: We need to show that, given a morphism of pointed sets

$$f\colon (X,x_0)\to(Y,y_0),$$

the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Delta^{\wedge}_{X}"'] & Y \arrow[d, "\Delta^{\wedge}_{Y}"] \\
X\wedge X \arrow[r, "f\wedge f"'] & Y\wedge Y
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\
x\wedge x \arrow[r, mapsto] & f(x)\wedge f(x)
\end{tikzcd}$$

and hence indeed commutes, showing $\Delta^{\wedge}$ to be natural. □

00GV **Proposition 4.5.8.1.2.** Let $(X,x_0)\in\mathrm{Obj}(\mathsf{Sets}_*)$.

00GW 1. *Monoidality.* The diagonal

$$\Delta^{\wedge}\colon \mathrm{id}_{\mathsf{Sets}_*}\Longrightarrow \wedge\circ\Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*},$$

of the smash product of pointed sets is a monoidal natural transformation:

00GX (a) *Compatibility With Strong Monoidality Constraints.* For each $(X,x_0),(Y,y_0)\in\mathrm{Obj}(\mathsf{Sets}_*)$, the diagram

$$\begin{tikzcd}
X\wedge Y \arrow[r, "\Delta^{\wedge}_{X}\wedge\Delta^{\wedge}_{Y}"] \arrow[dr, "\Delta^{\wedge}_{X\wedge Y}"'] & (X\wedge X)\wedge(Y\wedge Y) \arrow[d, dashed, "\wr"] \\
 & (X\wedge Y)\wedge(X\wedge Y)
\end{tikzcd}$$

commutes.

00GY (b) *Compatibility With Strong Unitality Constraints.* The diagrams

$$\begin{tikzcd}
S^0 \arrow[r, "\Delta^{\wedge}_{S^0}"] \arrow[dr, equal] & S^0\wedge S^0 \arrow[d, "\lambda^{\mathsf{Sets}_*}_{S^0}"] \\
 & S^0
\end{tikzcd}
\qquad
\begin{tikzcd}
S^0 \arrow[r, "\Delta^{\wedge}_{S^0}"] \arrow[dr, equal] & S^0\wedge S^0 \arrow[d, "\rho^{\mathsf{Sets}_*}_{S^0}"] \\
 & S^0
\end{tikzcd}$$

commute, i.e. we have

$$\begin{aligned}\Delta^{\wedge}_{S^0} &= \lambda^{\mathsf{Sets}_*,-1}_{S^0}\\ &= \rho^{\mathsf{Sets}_*,-1}_{S^0},\end{aligned}$$

where we recall that the equalities

$$\begin{aligned}\lambda^{\mathsf{Sets}_*}_{S^0} &= \rho^{\mathsf{Sets}_*}_{S^0},\\ \lambda^{\mathsf{Sets}_*,-1}_{S^0} &= \rho^{\mathsf{Sets}_*,-1}_{S^0}\end{aligned}$$

are always true in any monoidal category by **??** of **??**.

00GZ 2. *The Diagonal of the Unit.* The component

$$\Delta^{\wedge}_{S^0}\colon S^0 \xrightarrow{\cong} S^0 \wedge S^0$$

of $\Delta^{\wedge}$ at S^0 is an isomorphism.

Proof. *Item 1*, *Monoidality*: We claim that $\Delta^{\wedge}$ is indeed monoidal:

1. *Item 1a: Compatibility With Strong Monoidality Constraints:* We need to show that the diagram

$$\begin{tikzcd} X \wedge Y \arrow[r, "\Delta^{\wedge}_X \wedge \Delta^{\wedge}_Y"] \arrow[dr, "\Delta^{\wedge}_{X \wedge Y}"'] & (X \wedge X) \wedge (Y \wedge Y) \arrow[d, dashed, "\wr"] \\ & (X \wedge Y) \wedge (X \wedge Y) \end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd} x \wedge y \arrow[r, mapsto] \arrow[dr, mapsto] & (x \wedge x) \wedge (y \wedge y) \arrow[d, mapsto] \\ & (x \wedge y) \wedge (x \wedge y) \end{tikzcd}$$

and hence indeed commutes.

2. *Item 1b: Compatibility With Strong Unitality Constraints:* As shown in the proof of Definition 4.5.5.1.1, the inverse of the left unitor of Sets_* with respect to to the smash product of pointed sets at $(X, x_0) \in \mathrm{Obj}(\mathsf{Sets}_*)$ is given by

$$\lambda^{\mathsf{Sets}_*,-1}_X(x) \stackrel{\text{def}}{=} 1 \wedge x$$

for each $x \in X$, so when $X = S^0$, we have

$$\lambda^{\mathsf{Sets}_*,-1}_{S^0}(0) \stackrel{\text{def}}{=} 1 \wedge 0,$$
$$\lambda^{\mathsf{Sets}_*,-1}_{S^0}(1) \stackrel{\text{def}}{=} 1 \wedge 1.$$

But since $1 \wedge 0 = 0 \wedge 0$ and

$$\Delta^{\wedge}_{S^0}(0) \stackrel{\text{def}}{=} 0 \wedge 0,$$
$$\Delta^{\wedge}_{S^0}(1) \stackrel{\text{def}}{=} 1 \wedge 1,$$

it follows that we indeed have $\Delta^{\wedge}_{S^0} = \lambda^{\mathsf{Sets}_*,-1}_{S^0}$.

This finishes the proof.

Item 2, The Diagonal of the Unit: This follows from Item 1 and the invertibility of the left/right unitor of Sets_* with respect to $\wedge$, proved in the proof of Definition 4.5.5.1.1 for the left unitor or the proof of Definition 4.5.6.1.1 for the right unitor. □

00H0
4.5.9 The Monoidal Structure on Pointed Sets Associated to $\wedge$

00H1 **Proposition 4.5.9.1.1.** The category Sets_* admits a closed monoidal category with diagonals structure consisting of

- *The Underlying Category.* The category Sets_* of pointed sets;
- *The Monoidal Product.* The smash product functor

  $$\wedge \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Item 1 of Proposition 4.5.1.1.9;
- *The Internal Hom.* The internal Hom functor

  $$\mathbf{Sets}_* \colon \mathsf{Sets}^{\mathsf{op}}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

  of Item 1 of Proposition 4.5.2.1.2;
- *The Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathsf{Sets}_*} \colon \mathsf{pt} \to \mathsf{Sets}_*$$

  of Definition 4.5.3.1.1;
- *The Associators.* The natural isomorphism

  $$\alpha^{\mathsf{Sets}_*} \colon \wedge \circ (\wedge \times \mathrm{id}_{\mathsf{Sets}_*}) \stackrel{\sim}{\Longrightarrow} \wedge \circ (\mathrm{id}_{\mathsf{Sets}_*} \times \wedge) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Sets}_*,\mathsf{Sets}_*,\mathsf{Sets}_*}$$

  of Definition 4.5.4.1.1;

- *The Left Unitors.* The natural isomorphism

$$\lambda^{\mathsf{Sets}_*} \colon \wedge \circ \left(\mathbb{1}^{\mathsf{Sets}_*} \times \mathrm{id}_{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

of Definition 4.5.5.1.1;

- *The Right Unitors.* The natural isomorphism

$$\rho^{\mathsf{Sets}_*} \colon \wedge \circ \left(\mathsf{id} \times \mathbb{1}^{\mathsf{Sets}_*}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

of Definition 4.5.6.1.1;

- *The Symmetry.* The natural isomorphism

$$\sigma^{\mathsf{Sets}_*} \colon \wedge \overset{\sim}{\Longrightarrow} \wedge \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Sets}_*,\mathsf{Sets}_*}$$

of Definition 4.5.7.1.1;

- *The Diagonals.* The monoidal natural transformation

$$\Delta^{\wedge} \colon \mathrm{id}_{\mathsf{Sets}_*} \Longrightarrow \wedge \circ \Delta^{\mathsf{Cats}_2}_{\mathsf{Sets}_*}$$

of Definition 4.5.8.1.1.

Proof. The Pentagon Identity: Let (W, w_0), (X, x_0), (Y, y_0) and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & (W \wedge (X \wedge Y)) \wedge Z \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{W, X \wedge Y, Z}"] & \\
((W \wedge X) \wedge Y) \wedge Z \arrow[ur, "\alpha^{\mathsf{Sets}_*}_{W,X,Y} \wedge \mathrm{id}_Z"] \arrow[d, "\alpha^{\mathsf{Sets}_*}_{W \wedge X, Y, Z}"'] & & W \wedge ((X \wedge Y) \wedge Z) \arrow[d, "\mathrm{id}_W \wedge \alpha^{\mathsf{Sets}_*}_{X,Y,Z}"] \\
(W \wedge X) \wedge (Y \wedge Z) \arrow[rr, "\alpha^{\mathsf{Sets}_*}_{W,X,Y \wedge Z}"'] & & W \wedge (X \wedge (Y \wedge Z))
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
 & & (w \wedge (x \wedge y)) \wedge z \arrow[ddrr, mapsto, start anchor=south east] & & \\
((w \wedge x) \wedge y) \wedge z \arrow[urr, mapsto] \arrow[dr, mapsto] & & & & w \wedge ((x \wedge y) \wedge z) \arrow[dl, mapsto] \\
 & (w \wedge x) \wedge (y \wedge z) \arrow[rr, mapsto] & & w \wedge (x \wedge (y \wedge z)) &
\end{tikzcd}
$$

and thus we see that the pentagon identity is satisfied.
The Triangle Identity: Let (X, x_0) and (Y, y_0) be pointed sets. We have to show that the diagram

$$
\begin{tikzcd}
(X \wedge S^0) \wedge Y \arrow[rr, "\alpha^{\mathsf{Sets}_*}_{X,S^0,Y}"] \arrow[dr, "\rho^{\mathsf{Sets}_*}_X \wedge \mathrm{id}_Y"'] & & X \wedge (S^0 \wedge Y) \arrow[dl, "\mathrm{id}_X \wedge \lambda^{\mathsf{Sets}_*}_Y"] \\
 & X \wedge Y &
\end{tikzcd}
$$

commutes. Indeed, this diagram acts on elements as

$$
\begin{tikzcd}
(x \wedge 0) \wedge y \arrow[dr, mapsto] & \\
 & x_0 \wedge y
\end{tikzcd}
\qquad
\begin{tikzcd}
(x \wedge 0) \wedge y \arrow[r, mapsto] & x \wedge (0 \wedge y) \arrow[dl, mapsto] \\
x \wedge y_0 &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
(x \wedge 1) \wedge y \arrow[rr, mapsto] \arrow[dr, mapsto] & & x \wedge (1 \wedge y) \arrow[dl, mapsto] \\
 & x \wedge y, &
\end{tikzcd}
$$

and thus we see that the triangle identity is satisfied.
The Left Hexagon Identity: Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed

sets. We have to show that the diagram

$$\begin{tikzcd}
 & (X \wedge Y) \wedge Z \arrow[dl, "\alpha^{\mathsf{Sets}_*}_{X,Y,Z}"'] \arrow[dr, "\beta^{\mathsf{Sets}_*}_{X,Y} \wedge \mathrm{id}_Z"] & \\
X \wedge (Y \wedge Z) \arrow[d, "\beta^{\mathsf{Sets}_*}_{X,Y \wedge Z}"'] & & (Y \wedge X) \wedge Z \arrow[d, "\alpha^{\mathsf{Sets}_*}_{Y,X,Z}"] \\
(Y \wedge Z) \wedge X \arrow[dr, "\alpha^{\mathsf{Sets}_*}_{Y,Z,X}"'] & & Y \wedge (X \wedge Z) \arrow[dl, "\mathrm{id}_Y \wedge \beta^{\mathsf{Sets}_*}_{X,Z}"] \\
 & Y \wedge (Z \wedge X) &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
 & (x \wedge y) \wedge z \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
x \wedge (y \wedge z) \arrow[d, mapsto] & & (y \wedge x) \wedge z \arrow[d, mapsto] \\
(y \wedge z) \wedge x \arrow[dr, mapsto] & & y \wedge (x \wedge z) \arrow[dl, mapsto] \\
 & y \wedge (z \wedge x) &
\end{tikzcd}$$

and thus we see that the left hexagon identity is satisfied.
The Right Hexagon Identity: Let (X, x_0), (Y, y_0), and (Z, z_0) be pointed sets. We have to show that the diagram

$$\begin{tikzcd}
 & X \wedge (Y \wedge Z) \arrow[dl, "\left(\alpha^{\mathsf{Sets}_*}_{X,Y,Z}\right)^{-1}"'] \arrow[dr, "\mathrm{id}_X \wedge \beta^{\mathsf{Sets}_*}_{Y,Z}"] & \\
(X \wedge Y) \wedge Z \arrow[d, "\beta^{\mathsf{Sets}_*}_{X \wedge Y,Z}"'] & & X \wedge (Z \wedge Y) \arrow[d, "\left(\alpha^{\mathsf{Sets}_*}_{X,Z,Y}\right)^{-1}"] \\
Z \wedge (X \wedge Y) \arrow[dr, "\left(\alpha^{\mathsf{Sets}_*}_{Z,X,Y}\right)^{-1}"'] & & (X \wedge Z) \wedge Y \arrow[dl, "\beta^{\mathsf{Sets}_*}_{X,Z} \wedge \mathrm{id}_Y"] \\
 & (Z \wedge X) \wedge Y &
\end{tikzcd}$$

commutes. Indeed, this diagram acts on elements as

$$\begin{tikzcd}
& x \wedge (y \wedge z) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
(x \wedge y) \wedge z \arrow[d, mapsto] & & x \wedge (z \wedge y) \arrow[d, mapsto] \\
z \wedge (x \wedge y) \arrow[dr, mapsto] & & (x \wedge z) \wedge y \arrow[dl, mapsto] \\
& (z \wedge x) \wedge y &
\end{tikzcd}$$

and thus we see that the right hexagon identity is satisfied.
Monoidal Closedness: This follows from Item 2 of Proposition 4.5.1.1.9.
Existence of Monoidal Diagonals: This follows from Items 1 and 2 of Proposition 4.5.8.1.2. □

00H2 4.5.10 Universal Properties of the Smash Product of Pointed Sets I

00H3 **Theorem 4.5.10.1.1.** The symmetric monoidal structure on the category Sets_* is uniquely determined by the following requirements:

1. *Two-Sided Preservation of Colimits.* The smash product

$$\wedge \colon \mathsf{Sets}_* \times \mathsf{Sets}_* \to \mathsf{Sets}_*$$

of Sets_* preserves colimits separately in each variable.

2. *The Unit Object Is S^0.* We have $\mathbb{1}_{\mathsf{Sets}_*} = S^0$.

Proof. Omitted. □

00H4 4.5.11 Universal Properties of the Smash Product of Pointed Sets II

00H5 **Theorem 4.5.11.1.1.** The symmetric monoidal structure on the category Sets_* is the unique symmetric monoidal structure on Sets_* such that the free pointed set functor

$$(-)^+ \colon \mathsf{Sets} \to \mathsf{Sets}_*$$

admits a symmetric monoidal structure.

Proof. See [GGN15, Theorem 5.1]. □

4.5.12 Monoids With Respect to the Smash Product of 00H6 Pointed Sets

00H7 **Proposition 4.5.12.1.1.** The category of monoids on $(\mathsf{Sets}_*, \wedge, S^0)$ is isomorphic to the category of monoids with zero and morphisms between them.

Proof. See **??**, in particular **??**, **??**, and **??**. □

4.5.13 Comonoids With Respect to the Smash Product of 00H8 Pointed Sets

00H9 **Proposition 4.5.13.1.1.** The symmetric monoidal functor

$$\left((-)^{+}, (-)^{+,\times}, (-)^{+,\times}_{\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to \left(\mathsf{Sets}_*, \wedge, S^0\right),$$

of Item 4 of Proposition 3.4.1.1.2 lifts to an equivalence of categories

$$\begin{aligned}\mathsf{CoMon}\left(\mathsf{Sets}_*, \wedge, S^0\right) &\overset{\text{eq.}}{\cong} \mathsf{CoMon}(\mathsf{Sets}, \times, \mathrm{pt})\\ &\cong \mathsf{Sets}.\end{aligned}$$

Proof. See [PS19, Lemma 2.4]. □

00HA 4.6 Miscellany

00HB 4.6.1 The Smash Product of a Family of Pointed Sets

Let $\{(X_i, x_0^i)\}_{i\in I}$ be a family of pointed sets.

00HC **Definition 4.6.1.1.1.** The **smash product of the family** $\{(X_i, x_0^i)\}_{i\in I}$ is the pointed set $\bigwedge_{i\in I} X_i$ consisting of:

- *The Underlying Set.* The set $\bigwedge_{i\in I} X_i$ defined by

$$\bigwedge_{i\in I} X_i \overset{\text{def}}{=} \left(\prod_{i\in I} X_i\right)/\sim,$$

  where $\sim$ is the equivalence relation on $\prod_{i\in I} X_i$ obtained by declaring

$$(x_i)_{i\in I} \sim (y_i)_{i\in I}$$

  if there exist $i_0 \in I$ such that $x_{i_0} = x_0$ and $y_{i_0} = y_0$, for each $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$.

- *The Basepoint.* The element $[(x_0)_{i\in I}]$ of $\bigwedge_{i\in I} X_i$.

Appendices

4.A Other Chapters

Part II

Relations

Chapter 5

Relations

00HD This chapter contains some material about relations. Notably, we discuss and explore:

1. The definition of relations (Section 5.1.1).
2. How relations may be viewed as decategorification of profunctors (Section 5.1.2).
3. The various kind of categories that relations form, namely:
   (a) A category (Section 5.2.1).
   (b) A monoidal category (Section 5.2.2).
   (c) A 2-category (Section 5.2.3).
   (d) A double category (Section 5.2.4).
4. The various categorical properties of the 2-category of relations, including:
   (a) The self-duality of Rel and $\mathbf{Rel}$ (Proposition 5.3.1.1.1).
   (b) Identifications of equivalences and isomorphisms in $\mathbf{Rel}$ with bijections (Proposition 5.3.2.1.1).
   (c) Identifications of adjunctions in $\mathbf{Rel}$ with functions (Proposition 5.3.3.1.1).
   (d) Identifications of monads in $\mathbf{Rel}$ with preorders (Proposition 5.3.4.1.1).
   (e) Identifications of comonads in $\mathbf{Rel}$ with subsets (Proposition 5.3.5.1.1).
   (f) A description of the monoids and comonoids in $\mathbf{Rel}$ with respect to the Cartesian product (Remark 5.3.6.1.1).
   (g) Characterisations of monomorphisms in Rel (Proposition 5.3.7.1.1).

(h) Characterisations of 2-categorical notions of monomorphisms in **Rel** (Proposition 5.3.8.1.1).

(i) Characterisations of epimorphisms in Rel (Proposition 5.3.9.1.1).

(j) Characterisations of 2-categorical notions of epimorphisms in **Rel** (Proposition 5.3.10.1.1).

(k) The partial co/completeness of Rel (Proposition 5.3.11.1.1).

(l) The existence or non-existence of Kan extensions and Kan lifts in Rel (Remark 5.3.12.1.1).

(m) The closedness of **Rel** (Proposition 5.3.13.1.1).

(n) The identification of **Rel** with the category of free algebras of the powerset monad on Sets (Proposition 5.3.14.1.1).

5. A description of two notions of "skew composition" on $\mathbf{Rel}(A, B)$, giving rise to left and right skew monoidal structures analogous to the left skew monoidal structure on $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ appearing in the definition of a relative monad (Sections 5.4 and 5.5).

Contents

00HE 5.1 Relations

00HF 5.1.1 Foundations

Let A and B be sets.

00HG **Definition 5.1.1.1.1.** A **relation** $R\colon A \mathrel{\rightarrow\mkern-14mu\mapstochar\mkern14mu} B$ **from** A **to** B[1,2] is a subset R of $A \times B$.

00HH **Notation 5.1.1.1.2.** Let $R\colon A \mathrel{\rightarrow\mkern-14mu\mapstochar\mkern14mu} B$ be a relation.

1. Given elements $a \in A$ and $b \in B$ and a relation $R\colon A \mathrel{\rightarrow\mkern-14mu\mapstochar\mkern14mu} B$, we
00HJ write $a \sim_R b$ to mean $(a, b) \in R$.

[1] *Further Terminology:* Also called a **multivalued function from** A **to** B, a **relation over** A **and** B, **relation on** A **and** B, a **binary relation over** A **and** B, or a **binary relation on** A **and** B.

[2] *Further Terminology:* When $A = B$, we also call $R \subset A \times A$ a **relation on** A.

00HK 2. Viewing R as a function

$$R\colon A \times B \to \{\mathsf{t}, \mathsf{f}\}$$

via Remark 5.1.1.1.4, we write R_a^b for the value of R at (a, b).[3]

00HL **Definition 5.1.1.1.3.** Let A and B be sets.

1. The **set of relations from A to B** is the set $\mathrm{Rel}(A, B)$ defined
00HM by

$$\mathrm{Rel}(A, B) \stackrel{\text{def}}{=} \{\text{Relations from } A \text{ to } B\}.$$

00HN 2. The **poset of relations from A to B** is the poset

$$\mathbf{Rel}(A, B) \stackrel{\text{def}}{=} (\mathrm{Rel}(A, B), \subset)$$

consisting of:

- *The Underlying Set.* The set $\mathrm{Rel}(A, B)$ of Item 1.
- *The Partial Order.* The partial order

$$\subset\colon \mathrm{Rel}(A, B) \times \mathrm{Rel}(A, B) \to \{\mathsf{true}, \mathsf{false}\}$$

on $\mathrm{Rel}(A, B)$ given by inclusion of relations.

3. The **category of relations from A to B** is the posetal cate-
00HP gory $\mathbf{Rel}(A, B)$ [4] associated to the poset $\mathbf{Rel}(A, B)$ of Item 2 via Definition 8.1.3.1.1.

00HQ **Remark 5.1.1.1.4.** A relation from A to B is equivalently:[5]

00HR 1. A subset of $A \times B$.

00HS 2. A function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$.

00HT 3. A function from A to $\mathcal{P}(B)$.

00HU 4. A function from B to $\mathcal{P}(A)$.

5. A cocontinuous morphism of posets from $(\mathcal{P}(A), \subset)$ to $(\mathcal{P}(B), \subset)$.

00HV

[3] The choice R_a^b in place of R_b^a is to keep the notation consistent with the notation we will later employ for profunctors.

[4] Here we choose to slightly abuse notation by writing $\mathbf{Rel}(A, B)$ (instead of e.g. $\mathbf{Rel}(A, B)_{\mathsf{pos}}$) for the posetal category of relations from A to B, even though the same notation is used for the poset of relations from A to B.

[5] *Intuition:* In particular, we may think of a relation $R\colon A \to \mathcal{P}(B)$ from A to B as a multivalued function from A to B (including the possibility of a given $a \in A$ having

That is: we have bijections of sets

$$\begin{aligned}\mathrm{Rel}(A,B) &\overset{\mathrm{def}}{=} \mathcal{P}(A\times B),\\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(A\times B,\{\mathsf{true},\mathsf{false}\}),\\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(A,\mathcal{P}(B)),\\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(B,\mathcal{P}(A)),\\ &\cong \mathrm{Hom}^{\mathrm{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A),\mathcal{P}(B)),\end{aligned}$$

natural in $A, B \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. We claim that Items 1 to 5 are indeed equivalent:

- *Item 1* $\iff$ *Item 2:* This is a special case of Items 1 and 2 of Proposition 2.4.3.1.6.
- *Item 2* $\iff$ *Item 3:* This follows from the bijections

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Sets}}(A\times B,\{\mathsf{true},\mathsf{false}\}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(A,\mathrm{Hom}_{\mathsf{Sets}}(B,\{\mathsf{true},\mathsf{false}\}))\\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(A,\mathcal{P}(B)),\end{aligned}$$

  where the last bijection is from Items 1 and 2 of Proposition 2.4.3.1.6.

- *Item 2* $\iff$ *Item 4:* This follows from the bijections

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Sets}}(A\times B,\{\mathsf{true},\mathsf{false}\}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(B,\mathrm{Hom}_{\mathsf{Sets}}(B,\{\mathsf{true},\mathsf{false}\}))\\ &\cong \mathrm{Hom}_{\mathsf{Sets}}(B,\mathcal{P}(A)),\end{aligned}$$

  where again the last bijection is from Items 1 and 2 of Proposition 2.4.3.1.6.

- *Item 2* $\iff$ *Item 5:* This follows from the universal property of the powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

$$\chi_X\colon X\hookrightarrow\mathcal{P}(X)$$

  of X into $\mathcal{P}(X)$, Item 2 of Proposition 2.4.3.1.8.

  In particular, the bijection

$$\mathrm{Rel}(A,B)\cong\mathrm{Hom}^{\mathrm{cocont}}_{\mathsf{Pos}}(\mathcal{P}(A),\mathcal{P}(B))$$

  is given by taking a relation $R\colon A \nrightarrow B$, passing to its associated function $f\colon A\to\mathcal{P}(B)$ from A to B and then extending f from A to all of $\mathcal{P}(A)$ by taking its left Kan extension along χ_X.

  This coincides with the direct image function $f_*\colon\mathcal{P}(A)\to\mathcal{P}(B)$ of Definition 2.4.4.1.1.

This finishes the proof. □

00HW **Proposition 5.1.1.1.5.** Let A and B be sets and let $R, S\colon A \nrightarrow B$ be relations.

00HX 1. *End Formula for the Set of Inclusions of Relations.* We have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \cong \int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(R^b_a, S^b_a\Big).$$

Proof. *Item 1, End Formula for the Set of Inclusions of Relations*: Unwinding the expression inside the end on the right hand side, we have

$$\int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(R^b_a, S^b_a\Big) \cong \begin{cases} \mathrm{pt} & \text{if, for each } a \in A \text{ and each } b \in B, \\ & \text{we have } \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(R^b_a, S^b_a\Big) \cong \mathrm{pt} \\ \emptyset & \text{otherwise.} \end{cases}$$

Since we have $\mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(R^b_a, S^b_a\Big) = \{\mathsf{true}\} \cong \mathrm{pt}$ exactly when $R^b_a = \mathsf{false}$ or $R^b_a = S^b_a = \mathsf{true}$, we get

$$\int_{a\in A}\int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\Big(R^b_a, S^b_a\Big) \cong \begin{cases} \mathrm{pt} & \text{if, for each } a \in A \text{ and each } b \in B, \\ & \text{if } a \sim_R b, \text{ then } a \sim_S b, \\ \emptyset & \text{otherwise.} \end{cases}$$

On the left hand-side, we have

$$\mathrm{Hom}_{\mathbf{Rel}(A,B)}(R,S) \cong \begin{cases} \mathrm{pt} & \text{if } R \subset S, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is then clear that the conditions for each set to evaluate to pt (up to isomorphism) are equivalent, implying that those two sets are isomorphic. □

00HY 5.1.2 Relations as Decategorifications of Profunctors

00HZ **Remark 5.1.2.1.1.** The notion of a relation is a decategorification of that of a profunctor:

1. A profunctor from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a functor

$$\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}.$$

2. A relation on sets A and B is a function

$$R\colon A \times B \to \{\mathsf{true}, \mathsf{false}\}.$$

Here we notice that:

- The opposite X^{op} of a set X is itself, as $(-)^{\mathsf{op}}\colon \mathsf{Cats} \to \mathsf{Cats}$ restricts to the identity endofunctor on Sets.
- The values that profunctors and relations take are analogous:
  - A category is enriched over the category
  $$\mathsf{Sets} \overset{\text{def}}{=} \mathsf{Cats}_0$$
  of sets, with profunctors taking values on it.
  - A set is enriched over the set
  $$\{\mathsf{true}, \mathsf{false}\} \overset{\text{def}}{=} \mathsf{Cats}_{-1}$$
  of classical truth values, with relations taking values on it.

00J0 **Remark 5.1.2.1.2.** Extending Remark 5.1.2.1.1, the equivalent definitions of relations in Remark 5.1.1.1.4 are also related to the corresponding ones for profunctors (**??**), which state that a profunctor $\mathfrak{p}\colon \mathcal{C} \nrightarrow \mathcal{D}$ is equivalently:

00J1 1. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Sets}$.

00J2 2. A functor $\mathfrak{p}\colon \mathcal{C} \to \mathsf{PSh}(\mathcal{D})$.

00J3 3. A functor $\mathfrak{p}\colon \mathcal{D}^{\mathsf{op}} \to \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$.

00J4 4. A colimit-preserving functor $\mathfrak{p}\colon \mathsf{PSh}(\mathcal{C}) \to \mathsf{PSh}(\mathcal{D})$.

Indeed:

- The equivalence between Items 1 and 2 (and also that between Items 1 and 3, which is proved analogously) is an instance of currying, both for profunctors as well as for relations, using the isomorphisms
$$\begin{aligned}
\mathsf{Sets}(A \times B, \{\mathsf{true}, \mathsf{false}\}) &\cong \mathsf{Sets}(A, \mathsf{Sets}(B, \{\mathsf{true}, \mathsf{false}\}))\\
&\cong \mathsf{Sets}(A, \mathcal{P}(B)),\\
\mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{D}, \mathsf{Sets}) &\cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}^{\mathsf{op}}, \mathsf{Sets}))\\
&\cong \mathsf{Fun}(\mathcal{C}, \mathsf{PSh}(\mathcal{D})).
\end{aligned}$$
- The equivalence between Items 1 and 3 follows from the universal properties of:

no value at all).

– The powerset $\mathcal{P}(X)$ of a set X as the free cocompletion of X via the characteristic embedding

$$\chi_{(-)} \colon X \hookrightarrow \mathcal{P}(X)$$

of X into $\mathcal{P}(X)$, as stated and proved in Item 2 of Proposition 2.4.3.1.8.

– The category $\mathsf{PSh}(\mathcal{C})$ of presheaves on a category $\mathcal{C}$ as the free cocompletion of $\mathcal{C}$ via the Yoneda embedding

$$よ \colon \mathcal{C} \hookrightarrow \mathsf{PSh}(\mathcal{C})$$

of $\mathcal{C}$ into $\mathsf{PSh}(\mathcal{C})$, as stated and proved in **??** of **??**.

00J5 5.1.3 Examples of Relations

00J6 **Example 5.1.3.1.1.** The **trivial relation on** A **and** B is the relation $\sim_{\mathrm{triv}}$ defined equivalently as follows:

1. As a subset of $A \times B$, we have

$$\sim_{\mathrm{triv}} \overset{\mathrm{def}}{=} A \times B.$$

2. As a function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\mathrm{triv}}$ is the constant function

$$\Delta_{\mathsf{true}} \colon A \times B \to \{\mathsf{true}, \mathsf{false}\}$$

from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking the value true.

3. As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\mathrm{triv}}$ is the function

$$\Delta_{\mathsf{true}} \colon A \to \mathcal{P}(B)$$

defined by

$$\Delta_{\mathsf{true}}(a) \overset{\mathrm{def}}{=} B$$

for each $a \in A$.

4. Lastly, it is the unique relation R on A and B such that we have $a \sim_R b$ for each $a \in A$ and each $b \in B$.

00J7 **Example 5.1.3.1.2.** The **cotrivial relation on** A **and** B is the relation $\sim_{\mathrm{cotriv}}$ defined equivalently as follows:

1. As a subset of $A \times B$, we have

$$\sim_{\mathrm{cotriv}} \overset{\mathrm{def}}{=} \emptyset.$$

2. As a function from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$, the relation $\sim_{\mathrm{cotriv}}$ is the constant function

$$\Delta_{\mathsf{false}} \colon A \times B \to \{\mathsf{true}, \mathsf{false}\}$$

from $A \times B$ to $\{\mathsf{true}, \mathsf{false}\}$ taking the value false.

3. As a function from A to $\mathcal{P}(B)$, the relation $\sim_{\mathrm{cotriv}}$ is the function

$$\Delta_{\mathsf{false}} \colon A \to \mathcal{P}(B)$$

defined by

$$\Delta_{\mathsf{false}}(a) \stackrel{\mathrm{def}}{=} \emptyset$$

for each $a \in A$.

4. Lastly, it is the unique relation R on A and B such that we have $a \not\sim_R b$ for each $a \in A$ and each $b \in B$.

00J8 **Example 5.1.3.1.3.** The characteristic relation

$$\chi_X(-_1, -_2) \colon X \times X \to \{\mathsf{t}, \mathsf{f}\}$$

on X of Item 3 of Definition 2.4.1.1.1, defined by

$$\chi_X(x, y) \stackrel{\mathrm{def}}{=} \begin{cases} \mathsf{true} & \text{if } x = y, \\ \mathsf{false} & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$, is another example of a relation.

00J9 **Example 5.1.3.1.4.** Square roots are examples of relations:

1. *Square Roots in* $\mathbb{R}$. The assignment $x \mapsto \sqrt{x}$ defines a relation

$$\sqrt{-} \colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$$

from $\mathbb{R}$ to itself, being explicitly given by

$$\sqrt{x} \stackrel{\mathrm{def}}{=} \begin{cases} 0 & \text{if } x = 0, \\ \left\{-\sqrt{|x|}, \sqrt{|x|}\right\} & \text{if } x \neq 0. \end{cases}$$

2. *Square Roots in* $\mathbb{Q}$. Square roots in $\mathbb{Q}$ are similar to square roots in $\mathbb{R}$, though now additionally it may also occur that $\sqrt{-} \colon \mathbb{Q} \to \mathcal{P}(\mathbb{Q})$ sends a rational number x (e.g. 2) to the empty set (since $\sqrt{2} \notin \mathbb{Q}$).

00JA **Example 5.1.3.1.5.** The complex logarithm defines a relation

$$\log\colon \mathbb{C} \to \mathcal{P}(\mathbb{C})$$

from $\mathbb{C}$ to itself, where we have

$$\log(a+bi) \overset{\text{def}}{=} \left\{\log\left(\sqrt{a^2+b^2}\right) + i\arg(a+bi) + (2\pi i)k \;\middle|\; k \in \mathbb{Z}\right\}$$

for each $a+bi \in \mathbb{C}$.

00JB **Example 5.1.3.1.6.** See [Wik24] for more examples of relations, such as antiderivation, inverse trigonometric functions, and inverse hyperbolic functions.

00JC 5.1.4 Functional Relations

Let A and B be sets.

00JD **Definition 5.1.4.1.1.** A relation $R\colon A \nrightarrow B$ is **functional** if, for each $a \in A$, the set $R(a)$ is either empty or a singleton.

00JE **Proposition 5.1.4.1.2.** Let $R\colon A \nrightarrow B$ be a relation.

00JF 1. *Characterisations.* The following conditions are equivalent:

00JG (a) The relation R is functional.

00JH (b) We have $R \diamond R^{\dagger} \subset \chi_B$.

Proof. *Item 1, Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a* $\Longrightarrow$ *Item 1b:* Let $(b,b') \in B \times B$. We need to show that

  $$\left[R \diamond R^{\dagger}\right](b,b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b,b'),$$

  i.e. that if there exists some $a \in A$ such that $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, then $b = b'$. But since $b \sim_{R^{\dagger}} a$ is the same as $a \sim_R b$, we have both $a \sim_R b$ and $a \sim_R b'$ at the same time, which implies $b = b'$ since R is functional.

- *Item 1b* $\Longrightarrow$ *Item 1a:* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

  1. Since $a \sim_R b$, we have $b \sim_{R^{\dagger}} a$.
  2. Since $R \diamond R^{\dagger} \subset \chi_B$, we have

     $$\left[R \diamond R^{\dagger}\right](b,b') \preceq_{\{\mathsf{t},\mathsf{f}\}} \chi_B(b,b'),$$

     and since $b \sim_{R^{\dagger}} a$ and $a \sim_R b'$, it follows that $\left[R \diamond R^{\dagger}\right](b,b') = \mathsf{true}$, and thus $\chi_B(b,b') = \mathsf{true}$ as well, i.e. $b = b'$.

This finishes the proof. □

00JJ 5.1.5 Total Relations

Let A and B be sets.

00JK **Definition 5.1.5.1.1.** A relation $R\colon A \nrightarrow B$ is **total** if, for each $a \in A$, we have $R(a) \neq \emptyset$.

00JL **Proposition 5.1.5.1.2.** Let $R\colon A \nrightarrow B$ be a relation.

00JM 1. *Characterisations.* The following conditions are equivalent:

00JN (a) The relation R is total.

00JP (b) We have $\chi_A \subset R^\dagger \diamond R$.

Proof. *Item 1, Characterisations*: We claim that Items 1a and 1b are indeed equivalent:

- *Item 1a $\implies$ Item 1b:* We have to show that, for each $(a, a') \in A$, we have
$$\chi_A(a, a') \preceq_{\{\mathsf{t},\mathsf{f}\}} \Big[R^\dagger \diamond R\Big](a, a'),$$
i.e. that if $a = a'$, then there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a'$ (i.e. $a \sim_R b$ again), which follows from the totality of R.

- *Item 1b $\implies$ Item 1a:* Given $a \in A$, since $\chi_A \subset R^\dagger \diamond R$, we must have
$$\{a\} \subset \Big[R^\dagger \diamond R\Big](a),$$
implying that there must exist some $b \in B$ such that $a \sim_R b$ and $b \sim_{R^\dagger} a$ (i.e. $a \sim_R b$) and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

This finishes the proof. $\square$

00JQ 5.2 Categories of Relations

00JR 5.2.1 The Category of Relations

00JS **Definition 5.2.1.1.1.** The **category of relations** is the category Rel where

- *Objects.* The objects of Rel are sets.

- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have
$$\mathsf{Rel}(A, B) \overset{\text{def}}{=} \mathrm{Rel}(A, B).$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

$$\mathbb{1}_A^{\mathsf{Rel}} \colon \mathrm{pt} \to \mathrm{Rel}(A, A)$$

of Rel at A is defined by

$$\mathrm{id}_A^{\mathsf{Rel}} \stackrel{\mathrm{def}}{=} \chi_A(-_1, -_2),$$

where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Item 3 of Definition 2.4.1.1.1.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map

$$\circ_{A,B,C}^{\mathsf{Rel}} \colon \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B) \to \mathrm{Rel}(A, C)$$

of Rel at (A, B, C) is defined by

$$S \circ_{A,B,C}^{\mathsf{Rel}} R \stackrel{\mathrm{def}}{=} S \diamond R$$

for each $(S, R) \in \mathrm{Rel}(B, C) \times \mathrm{Rel}(A, B)$, where $S \diamond R$ is the composition of S and R of Definition 6.3.12.1.1.

00JT 5.2.2 The Closed Symmetric Monoidal Category of Relations

00JU 5.2.2.1 The Monoidal Product

00JV **Definition 5.2.2.1.1.** The **monoidal product of** Rel is the functor

$$\times \colon \mathsf{Rel} \times \mathsf{Rel} \to \mathsf{Rel}$$

where

- *Action on Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Rel})$, we have

$$\times(A, B) \stackrel{\mathrm{def}}{=} A \times B,$$

where $A \times B$ is the Cartesian product of sets of Definition 2.1.3.1.1.

- *Action on Morphisms.* For each $(A, C), (B, D) \in \mathrm{Obj}(\mathsf{Rel} \times \mathsf{Rel})$, the action on morphisms

$$\times_{(A,C),(B,D)} \colon \mathrm{Rel}(A, B) \times \mathrm{Rel}(C, D) \to \mathrm{Rel}(A \times C, B \times D)$$

of $\times$ is given by sending a pair of morphisms (R, S) of the form

$$R \colon A \nrightarrow B,$$
$$S \colon C \nrightarrow D$$

to the relation

$$R \times S \colon A \times C \nrightarrow B \times D$$

of Definition 6.3.9.1.1.

00JW 5.2.2.2 The Monoidal Unit

00JX **Definition 5.2.2.2.1.** The **monoidal unit of** Rel is the functor

$$\mathbb{1}^{\mathsf{Rel}} \colon \mathrm{pt} \to \mathsf{Rel}$$

picking the set

$$\mathbb{1}_{\mathsf{Rel}} \overset{\mathrm{def}}{=} \mathrm{pt}$$

of Rel.

00JY 5.2.2.3 The Associator

00JZ **Definition 5.2.2.3.1.** The **associator of** Rel is the natural isomorphism

$$\alpha^{\mathsf{Rel}} \colon \times \circ ((\times) \times \mathsf{id}) \overset{\sim}{\Longrightarrow} \times \circ (\mathsf{id} \times (\times)) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Rel},\mathsf{Rel},\mathsf{Rel}},$$

as in the diagram

$$\begin{tikzcd}
 & \mathsf{Rel} \times (\mathsf{Rel} \times \mathsf{Rel}) \arrow[dr, "\mathsf{id} \times (\times)", "|" marking] & \\
(\mathsf{Rel} \times \mathsf{Rel}) \times \mathsf{Rel} \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathsf{Rel},\mathsf{Rel},\mathsf{Rel}}", "\sim"'] \arrow[d, "(\times) \times \mathsf{id}"', "|" marking] & & \mathsf{Rel} \times \mathsf{Rel} \arrow[d, "\times", "|" marking] \\
\mathsf{Rel} \times \mathsf{Rel} \arrow[rr, "\times"', "|" marking] \arrow[uur, phantom, "\Uparrow \alpha^{\mathsf{Rel}}"] & & \mathsf{Rel},
\end{tikzcd}$$

whose component

$$\alpha^{\mathsf{Rel}}_{A,B,C} \colon (A \times B) \times C \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} A \times (B \times C)$$

at $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$ is the relation defined by declaring

$$((a, b), c) \sim_{\alpha^{\mathsf{Rel}}_{A,B,C}} (a', (b', c'))$$

iff $a = a'$, $b = b'$, and $c = c'$.

00K0 5.2.2.4 The Left Unitor

00K1 **Definition 5.2.2.4.1.** The **left unitor of** Rel is the natural isomorphism

$$\lambda^{\mathsf{Rel}} \colon \times \circ \left(\mathbb{1}^{\mathsf{Rel}} \times \mathsf{id}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd}
\mathrm{pt} \times \mathsf{Rel} \arrow[r, "\mathbb{1}^{\mathsf{Rel}} \times \mathsf{id}"] \arrow[dr, dashed, bend right, "\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel}, \arrow[d, "\times"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathsf{Rel}}"] \\
 & \mathsf{Rel}
\end{tikzcd}$$

whose component

$$\lambda^{\mathsf{Rel}}_A \colon \mathbb{1}_{\mathsf{Rel}} \times A \nrightarrow A$$

at A is defined by declaring

$$(\star, a) \sim_{\lambda^{\mathsf{Rel}}_A} b$$

iff $a = b$.

00K2 ### 5.2.2.5 The Right Unitor

00K3 **Definition 5.2.2.5.1.** The **right unitor of** Rel is the natural isomorphism

$$\rho^{\mathsf{Rel}} \colon \times \circ \left(\mathsf{id} \times \mathbb{1}^{\mathsf{Rel}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathsf{pt} \arrow[r, "\mathsf{id} \times \mathbb{1}^{\mathsf{Rel}}"] \arrow[dr, dashed, bend right, "\boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathsf{Rel}}"'] & \mathsf{Rel} \times \mathsf{Rel}, \arrow[d, "\times"] \arrow[dl, phantom, "\Leftarrow \rho^{\mathsf{Rel}}"] \\ & \mathsf{Rel} \end{tikzcd}$$

whose component

$$\rho^{\mathsf{Rel}}_A \colon A \times \mathbb{1}_{\mathsf{Rel}} \nrightarrow A$$

at A is defined by declaring

$$(a, \star) \sim_{\rho^{\mathsf{Rel}}_A} b$$

iff $a = b$.

00K4 ### 5.2.2.6 The Symmetry

00K5 **Definition 5.2.2.6.1.** The **symmetry of** Rel is the natural isomorphism

$$\sigma^{\mathsf{Rel}} \colon \times \Longrightarrow \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}, \qquad \begin{tikzcd} \mathsf{Rel} \times \mathsf{Rel} \arrow[rr, "\times"] \arrow[dr, bend right, "\boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathsf{Rel},\mathsf{Rel}}"'] & \Downarrow \sigma^{\mathsf{Rel}} & \mathsf{Rel}, \\ & \mathsf{Rel} \times \mathsf{Rel} \arrow[ur, bend right, "\times"'] & \end{tikzcd}$$

whose component

$$\sigma^{\mathsf{Rel}}_{A,B} \colon A \times B \to B \times A$$

at (A, B) is defined by declaring

$$(a, b) \sim_{\sigma^{\mathsf{Rel}}_{A,B}} (b', a')$$

iff $a = a'$ and $b = b'$.

00K6 5.2.2.7 The Internal Hom

00K7 **Definition 5.2.2.7.1.** The **internal Hom of** Rel is the functor

$$\mathrm{Rel}\colon \mathsf{Rel}^{\mathsf{op}} \times \mathsf{Rel} \to \mathsf{Rel}$$

defined

- On objects by sending $A, B \in \mathrm{Obj}(\mathsf{Rel})$ to the set $\mathrm{Rel}(A, B)$ of Item 1 of Definition 5.1.1.1.3.
- On morphisms by pre/post-composition defined as in Definition 6.3.12.1.1.

00K8 **Proposition 5.2.2.7.2.** Let $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$.

00K9 1. *Adjointness.* We have adjunctions

$$(A \times - \dashv \mathrm{Rel}(A, -))\colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "A\times -"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathrm{Rel}(A,-)"] \end{tikzcd}$$

$$(- \times B \dashv \mathrm{Rel}(B, -))\colon \quad \begin{tikzcd} \mathsf{Rel} \arrow[r, bend left, "-\times B"] \arrow[r, phantom, "\perp"] & \mathsf{Rel}, \arrow[l, bend left, "\mathrm{Rel}(B,-)"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned} \mathrm{Rel}(A \times B, C) &\cong \mathrm{Rel}(A, \mathrm{Rel}(B, C)), \\ \mathrm{Rel}(A \times B, C) &\cong \mathrm{Rel}(B, \mathrm{Rel}(A, C)), \end{aligned}$$

natural in $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$.

Proof. *Item 1, Adjointness*: Indeed, we have

$$\begin{aligned} \mathrm{Rel}(A \times B, C) &\stackrel{\text{def}}{=} \mathsf{Sets}(A \times B \times C, \{\mathsf{true}, \mathsf{false}\}) \\ &\stackrel{\text{def}}{=} \mathrm{Rel}(A, B \times C) \\ &\stackrel{\text{def}}{=} \mathrm{Rel}(A, \mathrm{Rel}(B, C)), \end{aligned}$$

and similarly for the bijection $\mathrm{Rel}(A \times B, C) \cong \mathrm{Rel}(B, \mathrm{Rel}(A, C))$. □

00KA 5.2.2.8 The Closed Symmetric Monoidal Category of Relations

00KB **Proposition 5.2.2.8.1.** The category Rel admits a closed symmetric monoidal category structure consisting of[6]

[6] *Warning:* This is not a Cartesian monoidal structure, as the product on Rel is in fact given by the disjoint union of sets; see **??**.

- *The Underlying Category.* The category Rel of sets and relations of Definition 5.2.1.1.1.
- *The Monoidal Product.* The functor
$$\times \colon \mathrm{Rel} \times \mathrm{Rel} \to \mathrm{Rel}$$
of Definition 5.2.2.1.1.
- *The Internal Hom.* The internal Hom functor
$$\mathbf{Rel} \colon \mathrm{Rel}^{\mathsf{op}} \times \mathrm{Rel} \to \mathrm{Rel}$$
of Definition 5.2.2.7.1.
- *The Monoidal Unit.* The functor
$$\mathbb{1}^{\mathrm{Rel}} \colon \mathsf{pt} \to \mathrm{Rel}$$
of Definition 5.2.2.2.1.
- *The Associators.* The natural isomorphism
$$\alpha^{\mathrm{Rel}} \colon \times \circ (\times \times \mathrm{id}_{\mathrm{Rel}}) \overset{\sim}{\Longrightarrow} \times \circ (\mathrm{id}_{\mathrm{Rel}} \times \times) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathrm{Rel},\mathrm{Rel},\mathrm{Rel}}$$
of Definition 5.2.2.3.1.
- *The Left Unitors.* The natural isomorphism
$$\lambda^{\mathrm{Rel}} \colon \times \circ \left(\mathbb{1}^{\mathrm{Rel}} \times \mathrm{id}_{\mathrm{Rel}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathrm{Rel}}$$
of Definition 5.2.2.4.1.
- *The Right Unitors.* The natural isomorphism
$$\rho^{\mathrm{Rel}} \colon \times \circ \left(\mathsf{id} \times \mathbb{1}^{\mathrm{Rel}}\right) \overset{\sim}{\Longrightarrow} \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathrm{Rel}}$$
of Definition 5.2.2.5.1.
- *The Symmetry.* The natural isomorphism
$$\sigma^{\mathrm{Rel}} \colon \times \overset{\sim}{\Longrightarrow} \times \circ \boldsymbol{\sigma}^{\mathsf{Cats}_2}_{\mathrm{Rel},\mathrm{Rel}}$$
of Definition 5.2.2.6.1.

Proof. Omitted. □

END TEXTDBEND

00KC 5.2.3 The 2-Category of Relations

00KD **Definition 5.2.3.1.1.** The **2-category of relations** is the locally posetal 2-category Rel where

- *Objects.* The objects of Rel are sets.
- ***Hom**-Objects.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Rel}}(A,B) &\stackrel{\text{def}}{=} \mathbf{Rel}(A,B)\\ &\stackrel{\text{def}}{=} (\mathrm{Rel}(A,B), \subset).\end{aligned}$$

- *Identities.* For each $A \in \mathrm{Obj}(\mathsf{Rel})$, the unit map

$$\mathbb{1}^{\mathsf{Rel}}_A \colon \mathrm{pt} \to \mathbf{Rel}(A,A)$$

  of Rel at A is defined by

$$\mathrm{id}^{\mathsf{Rel}}_A \stackrel{\text{def}}{=} \chi_A(-_1, -_2),$$

  where $\chi_A(-_1, -_2)$ is the characteristic relation of A of Item 3 of Definition 2.4.1.1.1.

- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathsf{Rel})$, the composition map[7]

$$\circ^{\mathsf{Rel}}_{A,B,C} \colon \mathbf{Rel}(B,C) \times \mathbf{Rel}(A,B) \to \mathbf{Rel}(A,C)$$

  of Rel at (A, B, C) is defined by

$$S \circ^{\mathsf{Rel}}_{A,B,C} R \stackrel{\text{def}}{=} S \diamond R$$

  for each $(S, R) \in \mathsf{Rel}(B,C) \times \mathsf{Rel}(A,B)$, where $S \diamond R$ is the composition of S and R of Definition 6.3.12.1.1.

00KE 5.2.4 The Double Category of Relations

00KF 5.2.4.1 The Double Category of Relations

00KG **Definition 5.2.4.1.1.** The **double category of relations** is the locally posetal double category $\mathsf{Rel}^{\mathsf{dbl}}$ where

- *Objects.* The objects of $\mathsf{Rel}^{\mathsf{dbl}}$ are sets.

[7] Note that this is indeed a morphism of posets: given relations $R_1, R_2 \in \mathbf{Rel}(A,B)$ and $S_1, S_2 \in \mathbf{Rel}(B,C)$ such that

$$\begin{aligned}R_1 &\subset R_2,\\ S_1 &\subset S_2,\end{aligned}$$

- *Vertical Morphisms.* The vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are maps of sets $f\colon A \to B$.
- *Horizontal Morphisms.* The horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$ are relations $R\colon A \nrightarrow X$.
- 2-*Morphisms.* A 2-cell

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "S"', "|" marking] & Y \end{tikzcd}$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is either non-existent or an inclusion of relations of the form

$$R \subset S \circ (f \times g), \qquad \begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

- *Horizontal Identities.* The horizontal unit functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 5.2.4.2.1.
- *Vertical Identities.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}\right)$, we have

$$\mathrm{id}_A^{\mathsf{Rel}^{\mathsf{dbl}}} \overset{\text{def}}{=} \mathrm{id}_A.$$

- *Identity* 2-*Morphisms.* For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the identity 2-morphism

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "\mathrm{id}_A"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[r, "R"', "|" marking] & B \end{tikzcd}$$

of R is the identity inclusion

$$R \subset R, \qquad \begin{tikzcd} B \times A \arrow[r, "R"] \arrow[d, "\mathrm{id}_B\times\mathrm{id}_A"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times A \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

- *Horizontal Composition.* The horizontal composition functor of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor of Definition 5.2.4.3.1.
- *Vertical Composition of* 1*-Morphisms.* For each composable pair $A \xrightarrow{F} B \xrightarrow{G} C$ of vertical morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. maps of sets, we have
$$g \circ^{\mathsf{Rel}^{\mathsf{dbl}}} f \stackrel{\text{def}}{=} g \circ f.$$
- *Vertical Composition of* 2*-Morphisms.* The vertical composition of 2-morphisms in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.4.1.
- *Associators.* The associators of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.5.1.
- *Left Unitors.* The left unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.6.1.
- *Right Unitors.* The right unitors of $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as in Definition 5.2.4.7.1.

00KH 5.2.4.2 Horizontal Identities

00KJ **Definition 5.2.4.2.1.** The **horizontal unit functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor
$$\mathbb{1}^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_0 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$
of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each $A \in \mathrm{Obj}\left(\mathsf{Rel}^{\mathsf{dbl}}_0\right)$, we have
$$\mathbb{1}_A \stackrel{\text{def}}{=} \chi_A(-_1, -_2).$$
- *Action on Morphisms.* For each vertical morphism $f \colon A \to B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. each map of sets f from A to B, the identity 2-morphism
$$\begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\|}{\mathbb{1}_f \Downarrow}"] & A \arrow[d, "f"] \\ B \arrow[r, "\mathbb{1}_B"', "|"{marking}] & B \end{tikzcd}$$

we have also $S_1 \diamond R_1 \subset S_2 \diamond R_2$.

of f is the inclusion

$$\chi_B \circ (f \times f) \subset \chi_A, \qquad \begin{tikzcd} A \times A \arrow[r, "\chi_A(-_1,-_2)"] \arrow[d, "f \times f"'] \arrow[dr, phantom, "\Swarrow"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times B \arrow[r, "\chi_B(-_1,-_2)"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of Item 1 of Proposition 2.4.1.1.3.

00KK 5.2.4.3 Horizontal Composition

00KL **Definition 5.2.4.3.1.** The **horizontal composition functor** of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor

$$\odot^{\mathsf{Rel}^{\mathsf{dbl}}} \colon \mathsf{Rel}^{\mathsf{dbl}}_1 \underset{\mathsf{Rel}^{\mathsf{dbl}}_0}{\times} \mathsf{Rel}^{\mathsf{dbl}}_1 \to \mathsf{Rel}^{\mathsf{dbl}}_1$$

of $\mathsf{Rel}^{\mathsf{dbl}}$ is the functor where

- *Action on Objects.* For each composable pair $A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C$ of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, we have

  $$S \odot R \overset{\text{def}}{=} S \diamond R,$$

  where $S \diamond R$ is the composition of R and S of Definition 6.3.12.1.1.

- *Action on Morphisms.* For each horizontally composable pair

  $$\begin{tikzcd} A \arrow[r, "R", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "T"', "|"{marking}] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "|"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "h"] \\ Y \arrow[r, "U"', "|"{marking}] & Z \end{tikzcd}$$

  of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each pair

  $$\begin{tikzcd} A \times B \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\Swarrow"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Y \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times C \arrow[r, "S"] \arrow[d, "g \times h"'] \arrow[dr, phantom, "\Swarrow"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ Y \times Z \arrow[r, "U"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

  of inclusions of relations, the horizontal composition

  $$\begin{tikzcd} A \arrow[r, "S \odot R", "|"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\beta\odot\alpha"] & C \arrow[d, "h"] \\ X \arrow[r, "U \odot T"', "|"{marking}] & Z \end{tikzcd}$$

of α and β is the inclusion of relations[8]

$$(U \diamond T) \circ (f \times h) \subset (S \diamond R) \qquad \begin{tikzcd} A \times C \arrow[r, "S \diamond R"] \arrow[d, "f \times h"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ X \times Z \arrow[r, "U \diamond T"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

00KM 5.2.4.4 Vertical Composition of 2-Morphisms

00KN **Definition 5.2.4.4.1.** The **vertical composition** in $\mathsf{Rel}^{\mathsf{dbl}}$ is defined as follows: for each vertically composable pair

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[d, "g"] \\ B \arrow[r, "S"', "|" marking] & Y \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "S", "|" marking] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\beta"] & Y \arrow[d, "k"] \\ C \arrow[r, "T"', "|" marking] & Z \end{tikzcd}$$

of 2-morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, i.e. for each each pair

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f \times g"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times Y \arrow[r, "S"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd} \qquad \begin{tikzcd} B \times Y \arrow[r, "S"] \arrow[d, "h \times k"'] \arrow[dr, phantom, "\circlearrowleft"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

of inclusions of relations, we define the vertical composition

$$\begin{tikzcd} A \arrow[r, "R", "|" marking] \arrow[d, "h \circ f"'] \arrow[dr, phantom, "\Downarrow\beta\circ\alpha"] & X \arrow[d, "k \circ g"] \\ C \arrow[r, "T"', "|" marking] & Z \end{tikzcd}$$

---

[8] This is justified by noting that, given $(a, c) \in A \times C$, the statement

- We have $a \sim_{(U \diamond T) \circ (f \times h)} c$, i.e. $f(a) \sim_{U \diamond T} h(c)$, i.e. there exists some $y \in Y$ such that:
  1. We have $f(a) \sim_T y$;
  2. We have $y \sim_U h(c)$;

is implied by the statement

- We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
  1. We have $a \sim_R b$;

of α and β as the inclusion of relations

$$T \circ [(h \circ f) \times (k \circ g)] \subset R, \qquad \begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "(h\circ f)\times(k\circ g)"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\} \end{tikzcd}$$

given by the pasting of inclusions[9]

$$\begin{tikzcd} A \times X \arrow[r, "R"] \arrow[d, "f\times g"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ B \times Y \arrow[r, dashed, "S"] \arrow[d, "h\times k"'] \arrow[dr, phantom, "\subset"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ C \times Z \arrow[r, "T"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

00KP 5.2.4.5 The Associators

00KQ **Definition 5.2.4.5.1.** For each composable triple

$$A \overset{R}{\nrightarrow} B \overset{S}{\nrightarrow} C \overset{T}{\nrightarrow} D$$

of horizontal morphisms of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R} \colon (T \odot S) \odot R \overset{\sim}{\Longrightarrow} T \odot (S \odot R), \qquad \begin{tikzcd} A \arrow[r, "R", "|"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "S", "|"{marking}] \arrow[d, phantom, "\alpha^{\mathsf{Rel}^{\mathsf{dbl}}}_{T,S,R}\Downarrow"] & C \arrow[r, "T", "|"{marking}] & D \arrow[d, "\mathrm{id}_D"] \\ A \arrow[r, "R"', "|"{marking}] & B \arrow[r, "S"', "|"{marking}] & C \arrow[r, "T"', "|"{marking}] & D \end{tikzcd}$$

---

2. We have $b \sim_S c$;

since:

- If $a \sim_R b$, then $f(a) \sim_T g(b)$, as $T \circ (f \times g) \subset R$;
- If $b \sim_S c$, then $g(b) \sim_U h(c)$, as $U \circ (g \times h) \subset S$.

[9]This is justified by noting that, given $(a, x) \in A \times X$, the statement

- We have $h(f(a)) \sim_T k(g(x))$;

is implied by the statement

- We have $a \sim_R x$;

since

of the associator of $\mathsf{Rel}^{\mathsf{dbl}}$ at (R, S, T) is the identity inclusion[10]

$$(T \diamond S) \diamond R = T \diamond (S \diamond R) \qquad \begin{tikzcd} A \times B \arrow[r, "(T\diamond S)\diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\mathbin{/\!\!/}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "T\diamond(S\diamond R)"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

00KR 5.2.4.6 The Left Unitors

00KS **Definition 5.2.4.6.1.** For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_R \colon \mathbb{1}_B \odot R \overset{\sim}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "\mathbb{1}_B", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \lambda^{\mathsf{Rel}^{\mathsf{dbl}}}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "\shortmid"{marking}] & & B \end{tikzcd}$$

of the left unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[11]

$$R = \chi_B \diamond R, \qquad \begin{tikzcd} A \times B \arrow[r, "\chi_B\diamond R"] \arrow[d, equal] \arrow[dr, phantom, "\mathbin{/\!\!/}"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

00KT 5.2.4.7 The Right Unitors

00KU **Definition 5.2.4.7.1.** For each horizontal morphism $R\colon A \nrightarrow B$ of $\mathsf{Rel}^{\mathsf{dbl}}$, the component

$$\rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R \colon R \odot \mathbb{1}_A \overset{\sim}{\Longrightarrow} R, \qquad \begin{tikzcd} A \arrow[r, "\mathbb{1}_A", "\shortmid"{marking}] \arrow[d, "\mathrm{id}_A"'] & A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow \rho^{\mathsf{Rel}^{\mathsf{dbl}}}_R"] & B \arrow[d, "\mathrm{id}_B"] \\ A \arrow[rr, "R"', "\shortmid"{marking}] & & B \end{tikzcd}$$

- If $a \sim_R x$, then $f(a) \sim_S g(x)$, as $S \circ (f \times g) \subset R$;
- If $b \sim_S y$, then $h(b) \sim_T k(y)$, as $T \circ (h \times k) \subset S$, and thus, in particular:
  - If $f(a) \sim_S g(x)$, then $h(f(a)) \sim_T k(g(x))$.

[10]This is justified by Item 2 of Proposition 6.3.12.1.3.
[11]This is justified by Item 3 of Proposition 6.3.12.1.3.

of the right unitor of $\mathsf{Rel}^{\mathsf{dbl}}$ at R is the identity inclusion[12]

$$R = R \diamond \chi_A, \qquad \begin{tikzcd} A \times B \arrow[r, "R \diamond \chi_A"] \arrow[d, equal] \arrow[dr, phantom, "\sslash"] & \{\mathsf{true}, \mathsf{false}\} \arrow[d, "\mathrm{id}_{\{\mathsf{true},\mathsf{false}\}}"] \\ A \times B \arrow[r, "R"'] & \{\mathsf{true}, \mathsf{false}\}. \end{tikzcd}$$

00KV 5.3 Properties of the 2-Category of Relations

00KW 5.3.1 Self-Duality

00KX **Proposition 5.3.1.1.1.** The (2-)category of relations is self-dual:

00KY 1. *Self-Duality I.* We have an isomorphism

$$\mathrm{Rel}^{\mathsf{op}} \overset{\text{eq.}}{\cong} \mathrm{Rel}$$

of categories.

00KZ 2. *Self-Duality II.* We have a 2-isomorphism

$$\mathbf{Rel}^{\mathsf{op}} \overset{\text{eq.}}{\cong} \mathbf{Rel}$$

of 2-categories.

Proof. *Item 1, Self-Duality I*: We claim that the functor

$$F \colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Rel}$$

given by the identity on objects and by $R \mapsto R^{\dagger}$ on morphisms is an isomorphism of categories.
By Item 1 of Proposition 8.5.8.1.3, it suffices to show that F is bijective on objects (which is clear) and fully faithful. Indeed, the map

$$(-)^{\dagger} \colon \mathrm{Rel}(A, B) \to \mathrm{Rel}(B, A)$$

defined by the assignment $R \mapsto R^{\dagger}$ is a bijection by Item 5 of Proposition 6.3.11.1.3, showing F to be fully faithful.
Item 2, Self-Duality II: We claim that the 2-functor

$$F \colon \mathsf{Rel}^{\mathsf{op}} \to \mathsf{Rel}$$

given by the identity on objects, by $R \mapsto R^{\dagger}$ on morphisms, and by preserving inclusions on 2-morphisms via Item 1 of Proposition 6.3.11.1.3, is an isomorphism of categories.
By **??** of **??**, it suffices to show that F is:

[12]This is justified by Item 3 of Proposition 6.3.12.1.3.

- Bijective on objects, which is clear.
- Bijective on 1-morphisms, which was shown in Item 1.
- Bijective on 2-morphisms, which follows from Item 1 of Proposition 6.3.11.1.3.

Thus F is indeed a 2-isomorphism of categories. □

00L0 5.3.2 Isomorphisms and Equivalences in **Rel**

Let $R\colon A \mathrel{\mapstochar\rightarrow} B$ be a relation from A to B.

00L1 **Proposition 5.3.2.1.1.** The following conditions are equivalent:

00L2 1. The relation $R\colon A \mathrel{\mapstochar\rightarrow} B$ is an equivalence in **Rel**, i.e.:

   (⋆) There exists a relation $R^{-1}\colon B \mathrel{\mapstochar\rightarrow} A$ from B to A together with isomorphisms

$$R^{-1} \diamond R \cong \chi_A,$$
$$R \diamond R^{-1} \cong \chi_B.$$

00L3 2. The relation $R\colon A \mathrel{\mapstochar\rightarrow} B$ is an isomorphism in Rel, i.e.:

   (⋆) There exists a relation $R^{-1}\colon B \mathrel{\mapstochar\rightarrow} A$ from B to A such that we have

$$R^{-1} \diamond R = \chi_A,$$
$$R \diamond R^{-1} = \chi_B.$$

00L4 3. There exists a bijection $f\colon A \xrightarrow{\cong} B$ with $R = \mathrm{Gr}(f)$.

Proof. We claim that Items 1 to 3 are indeed equivalent:

- *Item 1* $\iff$ *Item 2:* This follows from the fact that **Rel** is locally posetal, so that natural isomorphisms and equalities of 1-morphisms in **Rel** coincide.
- *Item 2* $\implies$ *Item 3:* The equalities in Item 2 imply $R \dashv R^{-1}$, and thus by Proposition 5.3.3.1.1, there exists a function $f_R\colon A \to B$ associated to R, where, for each $a \in A$, the image $f_R(a)$ of a by f_R is the unique element of $R(a)$, which implies $R = \mathrm{Gr}(f_R)$ in particular. Furthermore, we have $R^{-1} = f_R^{-1}$ (as in Definition 6.3.2.1.1). The conditions from Item 2 then become the following:

$$f_R^{-1} \diamond f_R = \chi_A,$$
$$f_R \diamond f_R^{-1} = \chi_B.$$

  All that is left is to show then is that f_R is a bijection:

– *The Function f_R Is Injective.* Let $a, b \in A$ and suppose that $f_R(a) = f_R(b)$. Since $a \sim_R f_R(a)$ and $f_R(a) = f_R(b) \sim_{R^{-1}} b$, the condition $f_R^{-1} \diamond f_R = \chi_A$ implies that $a = b$, showing f_R to be injective.
– *The Function f_R Is Surjective.* Let $b \in B$. Applying the condition $f_R \diamond f_R^{-1} = \chi_B$ to (b, b), it follows that there exists some $a \in A$ such that $f_R^{-1}(b) = a$ and $f_R(a) = b$. This shows f_R to be surjective.

- *Item 3 $\Longrightarrow$ Item 2:* By Item 2 of Proposition 6.3.1.1.2, we have an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$, giving inclusions

$$\chi_A \subset f^{-1} \diamond \mathrm{Gr}(f),$$
$$\mathrm{Gr}(f) \diamond f^{-1} \subset \chi_B.$$

We claim the reverse inclusions are also true:

– *$f^{-1} \diamond Gr(f) \subset \chi_A$:* This is equivalent to the statement that if $f(a) = b$ and $f^{-1}(b) = a'$, then $a = a'$, which follows from the injectivity of f.
– *$\chi_B \subset Gr(f) \diamond f^{-1}$:* This is equivalent to the statement that given $b \in B$ there exists some $a \in A$ such that $f^{-1}(b) = a$ and $f(a) = b$, which follows from the surjectivity of f.

This finishes the proof. □

00L5 5.3.3 Adjunctions in **Rel**

Let A and B be sets.

00L6 **Proposition 5.3.3.1.1.** We have a natural bijection

$$\begin{Bmatrix} \text{Adjunctions in } \textsf{Rel} \\ \text{from } A \text{ to } B \end{Bmatrix} \cong \begin{Bmatrix} \text{Functions} \\ \text{from } A \text{ to } B \end{Bmatrix},$$

with every adjunction in **Rel** being of the form $\mathrm{Gr}(f) \dashv f^{-1}$ for some function f.

Proof. We proceed step by step:

1. *From Adjunctions in* **Rel** *to Functions.* An adjunction in **Rel** from A to B consists of a pair of relations

$$R \colon A \nrightarrow B,$$
$$S \colon B \nrightarrow A,$$

together with inclusions

$$\chi_A \subset S \diamond R,$$
$$R \diamond S \subset \chi_B.$$

We claim that these conditions imply that R is total and functional, i.e. that $R(a)$ is a singleton for each $a \in A$:

(a) *$R(a)$ Has an Element.* Given $a \in A$, since $\chi_A \subset S \diamond R$, we must have $\{a\} \subset S(R(a))$, implying that there exists some $b \in B$ such that $a \sim_R b$ and $b \sim_S a$, and thus $R(a) \neq \emptyset$, as $b \in R(a)$.

(b) *$R(a)$ Has No More Than One Element.* Suppose that we have $a \sim_R b$ and $a \sim_R b'$ for $b, b' \in B$. We claim that $b = b'$:

   i. Since $\chi_A \subset S \diamond R$, there exists some $k \in B$ such that $a \sim_R k$ and $k \sim_S a$.
   ii. Since $R \diamond S \subset \chi_B$, if $b'' \sim_S a'$ and $a' \sim_R b'''$, then $b'' = b'''$.
   iii. Applying the above to $b'' = k$, $b''' = b$, and $a' = a$, since $k \sim_S a$ and $a \sim_R b'$, we have $k = b$.
   iv. Similarly $k = b'$.
   v. Thus $b = b'$.

Together, the above two items show $R(a)$ to be a singleton, being thus given by $\mathrm{Gr}(f)$ for some function $f\colon A \to B$, which gives a map

$$\left\{\begin{matrix}\text{Adjunctions in } \mathsf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

Moreover, by uniqueness of adjoints (**??** of **??**), this implies also that $S = f^{-1}$.

2. *From Functions to Adjunctions in* **Rel**. By Item 2 of Proposition 6.3.1.1.2, every function $f\colon A \to B$ gives rise to an adjunction $\mathrm{Gr}(f) \dashv f^{-1}$ in Rel, giving a map

$$\left\{\begin{matrix}\text{Functions} \\ \text{from } A \text{ to } B\end{matrix}\right\} \to \left\{\begin{matrix}\text{Adjunctions in } \mathsf{Rel} \\ \text{from } A \text{ to } B\end{matrix}\right\}.$$

3. *Invertibility: From Functions to Adjunctions Back to Functions.* We need to show that starting with a function $f\colon A \to B$, passing to $\mathrm{Gr}(f) \dashv f^{-1}$, and then passing again to a function gives f again. This is clear however, since we have $a \sim_{\mathrm{Gr}(f)} b$ iff $f(a) = b$.

4. *Invertibility: From Adjunctions to Functions Back to Adjunctions.* We need to show that, given an adjunction $R \dashv S$ in **Rel** giving rise to a function $f_{R,S} \colon A \to B$, we have

$$\begin{aligned}\mathrm{Gr}(f_{R,S}) &= R,\\ f^{-1}_{R,S} &= S.\end{aligned}$$

We check these explicitly:

- *$Gr(f_{R,S}) = R$.* We have

$$\begin{aligned}\mathrm{Gr}(f_{R,S}) &\overset{\mathrm{def}}{=} \{(a, f_{R,S}(a)) \in A \times B \mid a \in A\}\\ &\overset{\mathrm{def}}{=} \{(a, R(a)) \in A \times B \mid a \in A\}\\ &= R.\end{aligned}$$

- *$f^{-1}_{R,S} = S$.* We first claim that, given $a \in A$ and $b \in B$, the following conditions are equivalent:
  - We have $a \sim_R b$.
  - We have $b \sim_S a$.

  Indeed:
  - *If $a \sim_R b$, then $b \sim_S a$:* Since $\chi_A \subset S \diamond R$, there exists $k \in B$ such that $a \sim_R k$ and $k \sim_S a$, but since $a \sim_R b$ and R is functional, we have $k = b$ and thus $b \sim_S a$.
  - *If $b \sim_S a$, then $a \sim_R b$:* First note that since R is total we have $a \sim_R b'$ for some $b' \in B$. Now, since $R \diamond S \subset \chi_B$, $b \sim_S a$, and $a \sim_R b'$, we have $b = b'$, and thus $a \sim_R b$.

  Having show this, we now have

$$\begin{aligned}f^{-1}_{R,S}(b) &\overset{\mathrm{def}}{=} \{a \in A \mid f_{R,S}(a) = b\}\\ &\overset{\mathrm{def}}{=} \{a \in A \mid a \sim_R b\}\\ &= \{a \in A \mid b \sim_S a\}\\ &\overset{\mathrm{def}}{=} S(b).\end{aligned}$$

  for each $b \in B$, showing $f^{-1}_{R,S} = S$.

This finishes the proof. □

00L7 5.3.4 Monads in Rel

Let A be a set.

00L8 **Proposition 5.3.4.1.1.** We have a natural identification[13]

$$\left\{\begin{matrix}\text{Monads in}\\ \textsf{Rel} \text{ on } A\end{matrix}\right\} \cong \{\text{Preorders on } A\}.$$

Proof. A monad in **Rel** on A consists of a relation $R\colon A \nrightarrow A$ together with maps

$$\begin{aligned}\mu_R &\colon R \diamond R \subset R,\\ \eta_R &\colon \chi_A \subset R\end{aligned}$$

making the diagrams

$$\begin{tikzcd}
\chi_A \diamond R \arrow[r, "\eta_R \diamond \mathrm{id}_R"] \arrow[dr, equal, "\lambda^{\mathbf{Rel}(A,B)}_R"'] & R \diamond R \arrow[d, "\mu_R"] \\
 & R
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R \diamond (R \diamond R) \arrow[dl, equal, "\alpha^{\mathbf{Rel}(A,B)}_{R,R,R}"'] \arrow[dr, "\mathrm{id}_R \diamond \mu_R"] & \\
(R \diamond R) \diamond R \arrow[d, "\mu_R \diamond \mathrm{id}_R"'] & & R \diamond R \arrow[d, "\mu_R"] \\
R \diamond R \arrow[rr, "\mu_R"'] & & R
\end{tikzcd}
\qquad
\begin{tikzcd}
R \diamond \chi_A \arrow[r, "\mathrm{id}_R \diamond \eta_R"] \arrow[dr, equal, "\rho^{\mathbf{Rel}(A,B)}_R"'] & R \diamond R \arrow[d, "\mu_R"] \\
 & R
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps μ_R and η_R, which correspond respectively to the following conditions:

1. For each $a, b, c \in A$, if $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

2. For each $a \in A$, we have $a \sim_R a$.

These are exactly the requirements for R to be a preorder (**??**). Conversely any preorder $\preceq$ gives rise to a pair of maps $\mu_{\preceq}$ and $\eta_{\preceq}$, forming a monad on A. □

00L9 5.3.5 Comonads in Rel

Let A be a set.

00LA **Proposition 5.3.5.1.1.** We have a natural identification

$$\left\{\begin{matrix}\text{Comonads in}\\ \textsf{Rel} \text{ on } A\end{matrix}\right\} \cong \{\text{Subsets of } A\}.$$

[13] See also **??** for an extension of this correspondence to “relative monads in **Rel**”.

Proof. A comonad in **Rel** on A consists of a relation $R\colon A \nrightarrow A$ together with maps

$$\begin{aligned}\Delta_R\colon R &\subset R \diamond R,\\ \epsilon_R\colon R &\subset \chi_A\end{aligned}$$

making the diagrams

$$\begin{tikzcd}
R \arrow[r, "\Delta_R", "|" marking] \arrow[dr, equal, swap, "\lambda^{\mathbf{Rel}(A,B),-1}_R"] & R \diamond R \arrow[d, "\epsilon_R \diamond \mathrm{id}_R", "|" marking] \\
& \chi_A \diamond R
\end{tikzcd}
\qquad
\begin{tikzcd}
& R \diamond R \arrow[dr, "\mathrm{id}_R \diamond \Delta_R", "|" marking] & \\
R \arrow[ur, "\Delta_R", "|" marking] \arrow[d, swap, "\Delta_R", "|" marking] & & R \diamond (R \diamond R) \\
R \diamond R \arrow[r, swap, "\Delta_R \diamond \mathrm{id}_R", "|" marking] & (R \diamond R) \diamond R \arrow[ur, equal, swap, "\alpha^{\mathbf{Rel}(A,B),-1}_{R,R,R}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\Delta_R", "|" marking] \arrow[dr, equal, swap, "\rho^{\mathbf{Rel}(A,B),-1}_R"] & R \diamond R \arrow[d, "\mathrm{id}_R \diamond \epsilon_R", "|" marking] \\
& R \diamond \chi_A
\end{tikzcd}$$

commute. However, since all morphisms involved are inclusions, the commutativity of the above diagrams is automatic, and hence all that is left is the data of the two maps Δ_R and ϵ_R, which correspond respectively to the following conditions:

1. For each $a, b \in A$, if $a \sim_R b$, then there exists some $k \in A$ such that $a \sim_R k$ and $k \sim_R b$.

2. For each $a, b \in A$, if $a \sim_R b$, then $a = b$.

Taking $k = b$ in the first condition above shows it to be trivially satisfied, while the second condition implies $R \subset \Delta_A$, i.e. R must be a subset of A. Conversely, any subset U of A satisfies $U \subset \Delta_A$, defining a comonad as above. □

00LB 5.3.6 Co/Monoids in Rel

00LC **Remark 5.3.6.1.1.** The monoids in **Rel** with respect to the Cartesian monoidal structure of Proposition 5.2.2.8.1 are called *hypermonoids*, and their theory is explored in **??**. Similarly, the comonoids in **Rel** are called *hypercomonoids*, and they are defined and studied in **??**.

00LD 5.3.7 Monomorphisms in Rel

In this section we characterise the epimorphisms in the category Rel, following **??**.

00LE **Proposition 5.3.7.1.1.** Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

00LF 1. The relation R is a monomorphism in Rel.

00LG 2. The direct image function

$$R_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

associated to R is injective.

00LH 3. The direct image with compact support function

$$R_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

associated to R is injective.

Moreover, if R is a monomorphism, then it satisfies the following condition, and the converse holds if R is total:

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that

$$\begin{aligned} a &\sim_R b, \\ a' &\sim_R b, \end{aligned}$$

then $a = a'$.

Proof. Firstly note that Items 2 and 3 are equivalent by Item 7 of Proposition 6.4.1.1.3. We then claim that Items 1 and 2 are also equivalent:

- *Item 1* $\Longrightarrow$ *Item 2:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \overset{R}{\nrightarrow} B.$$

By Remark 6.4.1.1.2, we have

$$\begin{aligned} R_*(U) &= R \diamond U, \\ R_*(V) &= R \diamond V. \end{aligned}$$

Now, if $R \diamond U = R \diamond V$, i.e. $R_*(U) = R_*(V)$, then $U = V$ since R is assumed to be a monomorphism, showing R_* to be injective.

- *Item 2* $\Longrightarrow$ *Item 1:* Conversely, suppose that R_* is injective, consider the diagram

$$X \underset{T}{\overset{S}{\rightrightarrows}} A \overset{R}{\nrightarrow} B,$$

and suppose that $R \diamond S = R \diamond T$. Note that, since R_* is injective, given a diagram of the form

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B,$$

if $R_*(U) = R \diamond U = R \diamond V = R_*(V)$, then $U = V$. In particular, for each $x \in X$, we may consider the diagram

$$\mathrm{pt} \xrightarrow{[x]} X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

for which we have $R \diamond S \diamond [x] = R \diamond T \diamond [x]$, implying that we have

$$S(x) = S \diamond [x] = T \diamond [x] = T(x)$$

for each $x \in X$, implying $S = T$, and thus R is a monomorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 1* $\Longrightarrow$ *Item 2:* Assume that R is a monomorphism.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -)\colon \mathrm{Rel} \to \mathsf{Sets}$ maps R to R_* by Remark 6.4.1.1.2.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ preserves all limits by **??** of **??**, it follows by **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ also preserves monomorphisms.
  - Since R is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to R_*, it follows that R_* is also a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R_* is injective.
- *Item 2* $\Longrightarrow$ *Item 1:* Assume that R_* is injective.
  - We first notice that the functor $\mathrm{Rel}(\mathrm{pt}, -)\colon \mathrm{Rel} \to \mathsf{Sets}$ maps R to R_* by Remark 6.4.1.1.2.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R_* is a monomorphism.
  - Since $\mathrm{Rel}(\mathrm{pt}, -)$ is faithful, it follows by **??** of **??** that $\mathrm{Rel}(\mathrm{pt}, -)$ reflects monomorphisms.
  - Since R_* is a monomorphism and $\mathrm{Rel}(\mathrm{pt}, -)$ maps R to R_*, it follows that R is also a monomorphism.

Finally, we prove the second part of the statement. Assume that R is a monomorphism, let $a, a' \in A$ such that $a \sim_R b$ and $a' \sim_R b$ for some $b \in B$, and consider the diagram

$$\mathrm{pt} \underset{[a']}{\overset{[a]}{\rightrightarrows}} A \xrightarrow{R} B.$$

Since $\star \sim_{[a]} a$ and $a \sim_R b$, we have $\star \sim_{R\diamond[a]} b$. Similarly, $\star \sim_{R\diamond[a']} b$. Thus $R\diamond[a] = R\diamond[a']$, and since R is a monomorphism, we have $[a] = [a']$, i.e. $a = a'$.
Conversely, assume the condition

($\star$) For each $a, a' \in A$, if there exists some $b \in B$ such that

$$\begin{aligned} a &\sim_R b, \\ a' &\sim_R b, \end{aligned}$$

then $a = a'$.

consider the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "S", "\mapstochar"' marking] \arrow[r, shift right, "T"', "\mapstochar" marking] & A \arrow[r, "R", "\mapstochar" marking] & B, \end{tikzcd}$$

and let $(x, a) \in S$. Since R is total and $a \in A$, there exists some $b \in B$ such that $a \sim_R b$. In this case, we have $x \sim_{R\diamond S} b$, and since $R\diamond S = R\diamond T$, we have also $x \sim_{R\diamond T} b$. Thus there must exist some $a' \in A$ such that $x \sim_T a'$ and $a' \sim_R b$. However, since $a, a' \sim_R b$, we must have $a = a'$, and thus $(x, a) \in T$ as well.
A similar argument shows that if $(x, a) \in T$, then $(x, a) \in S$, and thus $S = T$ and it follows that R is a monomorphism. $\square$

00LJ 5.3.8 2-Categorical Monomorphisms in **Rel**

In this section we characterise (for now, some of) the 2-categorical monomorphisms in **Rel**, following Section 9.1.

00LK **Proposition 5.3.8.1.1.** Let $R\colon A \mathrel{\mkern3mu\mapstochar\mkern-3mu\rightarrow} B$ be a relation.

00LL 1. *Representably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is a representably faithful morphism.

00LM 2. *Representably Full Morphisms in* **Rel**. The following conditions are equivalent:

00LN (a) The morphism $R\colon A \mathrel{\mkern3mu\mapstochar\mkern-3mu\rightarrow} B$ is a representably full morphism.

(b) For each pair of relations $S, T\colon X \rightrightarrows A$, the following condition
00LP is satisfied:

($\star$) If $R \diamond S \subset R \diamond T$, then $S \subset T$.

00LQ (c) The functor

$$R_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

is full.

00LR (d) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

00LS (e) The functor

$$R_! \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

is full.

00LT (f) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

00LU 3. *Representably Fully Faithful Morphisms in* **Rel**. Every representaly full morphism in **Rel** is a representably fully faithful morphism.

Proof. *Item 1, Representably Faithful Morphisms in* **Rel**: The relation R is a representably faithful morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

$$R_* \colon \mathbf{Rel}(X, A) \to \mathbf{Rel}(X, B)$$

is faithful, i.e. iff the morphism

$$R_{*|S,T} \colon \mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, R \diamond T)$$

is injective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(X, A))$. However, $\mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T)$ is either empty or a singleton, in either case of which the map $R_{*|S,T}$ is necessarily injective.

Item 2, Representably Full Morphisms in **Rel**: We claim Items 2a to 2f are indeed equivalent:

- *Item 2a* $\iff$ *Item 2b:* This is simply a matter of unwinding definitions: The relation R is a representably full morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

  $$R_* \colon \mathbf{Rel}(X, A) \to \mathbf{Rel}(X, B)$$

  is full, i.e. iff the morphism

  $$R_{*|S,T} \colon \mathrm{Hom}_{\mathbf{Rel}(X,A)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, R \diamond T)$$

  is surjective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(X, A))$, i.e. iff, whenever $R \diamond S \subset R \diamond T$, we also have $S \subset T$.

- *Item 2c* $\iff$ *Item 2d:* This is also simply a matter of unwinding definitions: The functor

  $$R_* \colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

  is full iff, for each $U, V \in \mathcal{P}(A)$, the morphism

  $$R_{*|U,V} \colon \mathrm{Hom}_{\mathcal{P}(A)}(U, V) \to \mathrm{Hom}_{\mathcal{P}(B)}(R_*(U), R_*(V))$$

  is surjective, i.e. iff whenever $R_*(U) \subset R_*(V)$, we also necessarily have $U \subset V$.

- *Item 2e* $\iff$ *Item 2f:* This is once again simply a matter of unwinding definitions, and proceeds exactly in the same way as in the proof of the equivalence between Items 2c and 2d given above.
- *Item 2d* $\implies$ *Item 2f:* Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  1. Suppose we have $U, V \in \mathcal{P}(A)$ with $R_!(U) \subset R_!(V)$.
  2. By Item 7 of Proposition 6.4.4.1.3, we have
  $$R_!(U) = B \setminus R_*(A \setminus U),$$
  $$R_!(V) = B \setminus R_*(A \setminus V).$$
  3. By Item 1 of Proposition 2.3.10.1.2 we have $R_*(A \setminus V) \subset R_*(A \setminus U)$.
  4. By assumption, we then have $A \setminus V \subset A \setminus U$.
  5. By Item 1 of Proposition 2.3.10.1.2 again, we have $U \subset V$.
- *Item 2f* $\implies$ *Item 2d:* Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_!(U) \subset R_!(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(A)$, if $R_*(U) \subset R_*(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  1. Suppose we have $U, V \in \mathcal{P}(A)$ with $R_*(U) \subset R_*(V)$.
  2. By Item 7 of Proposition 6.4.1.1.3, we have
  $$R_*(U) = B \setminus R_!(A \setminus U),$$
  $$R_*(V) = B \setminus R_!(A \setminus V).$$
  3. By Item 1 of Proposition 2.3.10.1.2 we have $R_!(A \setminus V) \subset R_!(A \setminus U)$.
  4. By assumption, we then have $A \setminus V \subset A \setminus U$.
  5. By Item 1 of Proposition 2.3.10.1.2 again, we have $U \subset V$.

- *Item 2b* $\implies$ *Item 2d:* Consider the diagram

$$X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

and suppose that $R \diamond S \subset R \diamond T$. Note that, by assumption, given a diagram of the form

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B,$$

if $R_*(U) = R \diamond U \subset R \diamond V = R_*(V)$, then $U \subset V$. In particular, for each $x \in X$, we may consider the diagram

$$\mathrm{pt} \xrightarrow{[x]} X \underset{T}{\overset{S}{\rightrightarrows}} A \xrightarrow{R} B,$$

for which we have $R \diamond S \diamond [x] \subset R \diamond T \diamond [x]$, implying that we have

$$S(x) = S \diamond [x] \subset T \diamond [x] = T(x)$$

for each $x \in X$, implying $S \subset T$.

- *Item 2d* $\implies$ *Item 2b:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram

$$\mathrm{pt} \underset{V}{\overset{U}{\rightrightarrows}} A \xrightarrow{R} B.$$

By Remark 6.4.1.1.2, we have

$$R_*(U) = R \diamond U,$$
$$R_*(V) = R \diamond V.$$

Now, if $R_*(U) \subset R_*(V)$, i.e. $R \diamond U \subset R \diamond V$, then $U \subset V$ by assumption.

??, *Fully Faithful Monomorphisms in* **Rel**: This follows from Items 1 and 2. □

00LV **Question 5.3.8.1.2.** Item 2 of Proposition 5.3.8.1.1 gives a characterisation of the representably full morphisms in **Rel**.
Are there other nice characterisations of these?
This question also appears as [MO 467527].

00LW 5.3.9 Epimorphisms in Rel

In this section we characterise the epimorphisms in the category Rel, following **??**.

00LX **Proposition 5.3.9.1.1.** Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

00LY 1. The relation R is an epimorphism in Rel.

00LZ 2. The weak inverse image function
$$R^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$
associated to R is injective.

00M0 3. The strong inverse image function
$$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$
associated to R is injective.

00M1 4. The function $R\colon A \to \mathcal{P}(B)$ is "surjective on singletons":
   ($\star$) For each $b \in B$, there exists some $a \in A$ such that $R(a) = \{b\}$.

Moreover, if R is total and an epimorphism, then it satisfies the following equivalent conditions:

1. For each $b \in B$, there exists some $a \in A$ such that $a \sim_R b$.
2. We have $\mathrm{Im}(R) = B$.

Proof. Firstly note that Items 2 and 3 are equivalent by Item 7 of Proposition 6.4.2.1.3. We then claim that Items 1 and 2 are also equivalent:

- *Item 1* $\implies$ *Item 2:* Let $U, V \in \mathcal{P}(A)$ and consider the diagram
$$A \overset{R}{\nrightarrow} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt}.$$
By Remark 6.4.1.1.2, we have
$$\begin{aligned} R^{-1}(U) &= U \diamond R, \\ R^{-1}(V) &= V \diamond R. \end{aligned}$$
Now, if $U \diamond R = V \diamond R$, i.e. $R^{-1}(U) = R^{-1}(V)$, then $U = V$ since R is assumed to be an epimorphism, showing R^{-1} to be injective.

- *Item 2* $\Longrightarrow$ *Item 1:* Conversely, suppose that R^{-1} is injective, consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} X,$$

and suppose that $S \diamond R = T \diamond R$. Note that, since R^{-1} is injective, given a diagram of the form

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt},$$

if $R^{-1}(U) = U \diamond R = V \diamond R = R^{-1}(V)$, then $U = V$. In particular, for each $x \in X$, we may consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} X \overset{[x]}{\longrightarrow\!\!\!\!\!\!\!+\;\;} \mathrm{pt},$$

for which we have $[x] \diamond S \diamond R = [x] \diamond T \diamond R$, implying that we have

$$S^{-1}(x) = [x] \diamond S = [x] \diamond T = T^{-1}(x)$$

for each $x \in X$, implying $S = T$, and thus R is an epimorphism.

We can also prove this in a more abstract way, following [MSE 350788]:

- *Item 1* $\Longrightarrow$ *Item 2:* Assume that R is an epimorphism.
  - We first notice that the functor $\mathrm{Rel}(-, \mathrm{pt}) \colon \mathrm{Rel}^{\mathsf{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Remark 6.4.3.1.2.
  - Since $\mathrm{Rel}(-, \mathrm{pt})$ preserves limits by **??** of **??**, it follows by **??** of **??** that $\mathrm{Rel}(-, \mathrm{pt})$ also preserves monomorphisms.
  - That is: $\mathrm{Rel}(-, \mathrm{pt})$ sends monomorphisms in $\mathrm{Rel}^{\mathsf{op}}$ to monomorphisms in Sets.
  - The monomorphisms $\mathrm{Rel}^{\mathsf{op}}$ are precisely the epimorphisms in Rel by **??** of **??**.
  - Since R is an epimorphism and $\mathrm{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R^{-1} is a monomorphism.
  - Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R^{-1} is injective.
- *Item 2* $\Longrightarrow$ *Item 1:* Assume that R^{-1} is injective.
  - We first notice that the functor $\mathrm{Rel}(-, \mathrm{pt}) \colon \mathrm{Rel}^{\mathsf{op}} \to \mathsf{Sets}$ maps R to R^{-1} by Remark 6.4.3.1.2.

– Since the monomorphisms in Sets are precisely the injections (**??** of **??**), it follows that R^{-1} is a monomorphism.
– Since $\mathrm{Rel}(-, \mathrm{pt})$ is faithful, it follows by **??** of **??** that $\mathrm{Rel}(, \mathrm{pt})$ reflects monomorphisms.
– That is: $\mathrm{Rel}(-, \mathrm{pt})$ reflects monomorphisms in Sets to monomorphisms in $\mathrm{Rel}^{\mathsf{op}}$.
– The monomorphisms $\mathrm{Rel}^{\mathsf{op}}$ are precisely the epimorphisms in Rel by **??** of **??**.
– Since R^{-1} is a monomorphism and $\mathrm{Rel}(-, \mathrm{pt})$ maps R to R^{-1}, it follows that R is an epimorphism.

We also claim that Items 2 and 4 are equivalent, following [`MO 350788`]:

- *Item 2* $\implies$ *Item 4:* Since $B \setminus \{b\} \subset B$ and R^{-1} is injective, we have $R^{-1}(B \setminus \{b\}) \subsetneq R^{-1}(B)$. So taking some $a \in R^{-1}(B) \setminus R^{-1}(B \setminus \{b\})$ we get an element of A such that $R(a) = \{b\}$.

- *Item 4* $\implies$ *Item 2:* Let $U, V \subset B$ with $U \neq V$. Without loss of generality, we can assume $U \setminus V \neq \emptyset$; otherwise just swap U and V. Let then $b \in U \setminus V$. By assumption, there exists an $a \in A$ with $R(a) = \{b\}$. Then $a \in R^{-1}(U)$ but $a \notin R^{-1}(V)$, and thus $R^{-1}(U) \neq R^{-1}(V)$, showing R^{-1} to be injective.

Finally, we prove the second part of the statement. So assume R is a total epimorphism in Rel and consider the diagram

$$A \overset{R}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} B \underset{T}{\overset{S}{\rightrightarrows}} \{0,1\},$$

where $b \sim_S 0$ for each $b \in B$ and where we have

$$b \sim_T \begin{cases} 0 & \text{if } b \in \mathrm{Im}(R), \\ 1 & \text{otherwise} \end{cases}$$

for each $b \in B$. Since R is total, we have $a \sim_{S \diamond R} 0$ and $a \sim_{T \diamond R} 0$ for all $a \in A$, and no element of A is related to 1 by $S \diamond R$ or $T \diamond R$. Thus $S \diamond R = T \diamond R$, and since R is an epimorphism, we have $S = T$. But by the definition of T, this implies $\mathrm{Im}(R) = B$. □

00M2 5.3.10 2-Categorical Epimorphisms in Rel

In this section we characterise (for now, some of) the 2-categorical epimorphisms in **Rel**, following Section 9.2.

00M3 **Proposition 5.3.10.1.1.** Let $R \colon A \nrightarrow B$ be a relation.

00M4 1. *Corepresentably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is a corepresentably faithful morphism.

00M5 2. *Corepresentably Full Morphisms in* **Rel**. The following conditions are equivalent:

   (a) The morphism $R\colon A \nrightarrow B$ is a corepresentably full morphism.

00M6

   (b) For each pair of relations $S, T\colon X \rightrightarrows A$, the following condition
00M7 is satisfied:

   (⋆) If $S \diamond R \subset T \diamond R$, then $S \subset T$.

00M8 (c) The functor

$$R^{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

   is full.

00M9 (d) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

00MA (e) The functor

$$R_{-1}\colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

   is full.

00MB (f) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

00MC 3. *Corepresentably Fully Faithful Morphisms in* **Rel**. Every corepresentably full morphism of **Rel** is a corepresentably fully faithful morphism.

Proof. *Item 1, Corepresentably Faithful Morphisms in* **Rel**: The relation R is a corepresentably faithful morphism in **Rel** iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

$$R^*\colon \mathbf{Rel}(B, X) \to \mathbf{Rel}(A, X)$$

is faithful, i.e. iff the morphism

$$R^*_{S,T}\colon \mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T \diamond R)$$

is injective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(B, X))$. However, $\mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T)$ is either empty or a singleton, in either case of which the map $R^*_{S,T}$ is necessarily injective.

Item 2, Corepresentably Full Morphisms in **Rel**: We claim Items 2a to 2f are indeed equivalent:

- *Item 2a* $\iff$ *Item 2b:* This is simply a matter of unwinding definitions: The relation R is a corepresentably full morphism in

Rel iff, for each $X \in \mathrm{Obj}(\mathbf{Rel})$, the functor

$$R^* \colon \mathbf{Rel}(B, X) \to \mathbf{Rel}(A, X)$$

is full, i.e. iff the morphism

$$R^*_{S,T} \colon \mathrm{Hom}_{\mathbf{Rel}(B,X)}(S, T) \to \mathrm{Hom}_{\mathbf{Rel}(A,X)}(S \diamond R, T \diamond R)$$

is surjective for each $S, T \in \mathrm{Obj}(\mathbf{Rel}(B, X))$, i.e. iff, whenever $S \diamond R \subset T \diamond R$, we also have $S \subset T$.

- *Item 2c* $\iff$ *Item 2d:* This is also simply a matter of unwinding definitions: The functor

$$R^{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

is full iff, for each $U, V \in \mathcal{P}(A)$, the morphism

$$R^{-1}_{U,V} \colon \mathrm{Hom}_{\mathcal{P}(B)}(U, V) \to \mathrm{Hom}_{\mathcal{P}(A)}\Big(R^{-1}(U), R^{-1}(V)\Big)$$

is surjective, i.e. iff whenever $R^{-1}(U) \subset R^{-1}(V)$, we also necessarily have $U \subset V$.

- *Item 2e* $\iff$ *Item 2f:* This is once again simply a matter of unwinding definitions, and proceeds exactly in the same way as in the proof of the equivalence between Items 2c and 2d given above.

- *Item 2d* $\implies$ *Item 2f:* Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  1. Suppose we have $U, V \in \mathcal{P}(B)$ with $R_{-1}(U) \subset R_{-1}(V)$.
  2. By Item 7 of Proposition 6.4.2.1.3, we have

  $$\begin{aligned} R_{-1}(U) &= B \setminus R^{-1}(A \setminus U), \\ R_{-1}(V) &= B \setminus R^{-1}(A \setminus V). \end{aligned}$$

  3. By Item 1 of Proposition 2.3.10.1.2 we have $R^{-1}(A \setminus V) \subset R^{-1}(A \setminus U)$.
  4. By assumption, we then have $A \setminus V \subset A \setminus U$.
  5. By Item 1 of Proposition 2.3.10.1.2 again, we have $U \subset V$.

- *Item 2f* $\Longrightarrow$ *Item 2d:* Suppose that the following condition is true:

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R_{-1}(U) \subset R_{-1}(V)$, then $U \subset V$.

  We need to show that the condition

  ($\star$) For each $U, V \in \mathcal{P}(B)$, if $R^{-1}(U) \subset R^{-1}(V)$, then $U \subset V$.

  is also true. We proceed step by step:

  1. Suppose we have $U, V \in \mathcal{P}(B)$ with $R^{-1}(U) \subset R^{-1}(V)$.
  2. By Item 7 of Proposition 6.4.3.1.3, we have
  $$\begin{aligned} R^{-1}(U) &= B \setminus R_{-1}(A \setminus U), \\ R^{-1}(V) &= B \setminus R_{-1}(A \setminus V). \end{aligned}$$
  3. By Item 1 of Proposition 2.3.10.1.2 we have $R_{-1}(A \setminus V) \subset R_{-1}(A \setminus U)$.
  4. By assumption, we then have $A \setminus V \subset A \setminus U$.
  5. By Item 1 of Proposition 2.3.10.1.2 again, we have $U \subset V$.

- *Item 2b* $\Longrightarrow$ *Item 2d:* Consider the diagram

  $$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] & B \arrow[r, shift left, "S"] \arrow[r, shift right, "T"'] & X, \end{tikzcd}$$

  and suppose that $S \diamond R \subset T \diamond R$. Note that, by assumption, given a diagram of the form

  $$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] & B \arrow[r, shift left, "U"] \arrow[r, shift right, "V"'] & \mathrm{pt}, \end{tikzcd}$$

  if $R^{-1}(U) = R \diamond U \subset R \diamond V = R^{-1}(V)$, then $U \subset V$. In particular, for each $x \in X$, we may consider the diagram

  $$\begin{tikzcd} A \arrow[r, "R", "\shortmid"{marking}] & B \arrow[r, shift left, "S"] \arrow[r, shift right, "T"'] & X \arrow[r, "{[x]}", "\shortmid"{marking}] & \mathrm{pt}, \end{tikzcd}$$

  for which we have $[x] \diamond S \diamond R \subset [x] \diamond T \diamond R$, implying that we have

  $$S^{-1}(x) = [x] \diamond S \subset [x] \diamond T = T^{-1}(x)$$

  for each $x \in X$, implying $S \subset T$.

- *Item 2d* $\implies$ *Item 2b:* Let $U, V \in \mathcal{P}(B)$ and consider the diagram

$$A \overset{R}{\nrightarrow} B \underset{V}{\overset{U}{\rightrightarrows}} \mathrm{pt}.$$

By Remark 6.4.1.1.2, we have

$$R^{-1}(U) = U \diamond R,$$
$$R^{-1}(V) = V \diamond R.$$

Now, if $R^{-1}(U) \subset R^{-1}(V)$, i.e. $U \diamond R \subset V \diamond R$, then $U \subset V$ by assumption.

Item 3, Corepresentably Fully Faithful Morphisms in **Rel**: This follows from Items 1 and 2. □

00MD **Question 5.3.10.1.2.** Item 2 of Proposition 5.3.10.1.1 gives a characterisation of the corepresentably full morphisms in **Rel**.
Are there other nice characterisations of these?
This question also appears as [MO 467527].

00ME 5.3.11 Co/Limits in Rel

00MF **Proposition 5.3.11.1.1.** This will be properly written later on.

Proof. Omitted. □

00MG 5.3.12 Kan Extensions and Kan Lifts in Rel

00MH **Remark 5.3.12.1.1.** The 2-category **Rel** admits all right Kan extensions and right Kan lifts, though not all left Kan extensions and neither does it admit all left Kan lifts. See Section 6.2 for a detailed discussion of this.

00MJ 5.3.13 Closedness of Rel

00MK **Proposition 5.3.13.1.1.** The 2-category **Rel** is a closed bicategory, there being, for each $R\colon A \nrightarrow B$ and set X, a pair of adjunctions

$$(R^* \dashv \mathrm{Ran}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(B,X) \arrow[r, bend left, "R^*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(A,X), \arrow[l, bend left, "\mathrm{Ran}_R"] \end{tikzcd}$$

$$(R_* \dashv \mathrm{Rift}_R)\colon \quad \begin{tikzcd} \mathrm{Rel}(X,A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}(X,B), \arrow[l, bend left, "\mathrm{Rift}_R"] \end{tikzcd}$$

witnessed by bijections

$$\begin{aligned}\mathbf{Rel}(S \diamond R, T) &\cong \mathbf{Rel}(S, \mathrm{Ran}_R(T)),\\ \mathbf{Rel}(R \diamond U, V) &\cong \mathbf{Rel}(U, \mathrm{Rift}_R(V)),\end{aligned}$$

natural in $S \in \mathrm{Rel}(B, X)$, $T \in \mathrm{Rel}(A, X)$, $U \in \mathrm{Rel}(X, A)$, and $V \in \mathrm{Rel}(X, B)$.

Proof. This follows from Propositions 6.2.3.1.1 and 6.2.4.1.1. □

00ML 5.3.14 Rel as a Category of Free Algebras

00MM **Proposition 5.3.14.1.1.** We have an isomorphism of categories

$$\mathrm{Rel} \cong \mathsf{FreeAlg}_{\mathcal{P}_*}(\mathsf{Sets}),$$

where $\mathcal{P}_*$ is the powerset monad of **??**.

Proof. Omitted. □

00MN 5.4 The Left Skew Monoidal Structure on Rel(A, B)

00MP 5.4.1 The Left Skew Monoidal Product

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

00MQ **Definition 5.4.1.1.1.** The **left J-skew monoidal product of Rel(A, B)** is the functor

$$\lhd_J\colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \lhd_J R \overset{\mathrm{def}}{=} S \diamond \mathrm{Rift}_J(R), \qquad \begin{tikzcd} & A \arrow[r, "S", "|" marking] \arrow[d, "J", "|" marking] \arrow[dl, phantom, "\Swarrow"] & B. \\ A \arrow[r, "R"', "|" marking] \arrow[ur, dashed, "\mathrm{Rift}_J(R)", "|" marking] & B & \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\lhd_J)_{(G,F),(G',F')}\colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}\left(S, S'\right) \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}\left(R, R'\right) \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}\left(S \lhd_J R, S' \lhd_J R'\right)$$

of $\lhd_J$ at $((R, S), (R', S'))$ is defined by[14]

$$\beta \lhd_J \alpha \overset{\text{def}}{=} \beta \diamond \mathrm{Rift}_J(\alpha), \qquad \begin{tikzcd} & A \arrow[r, bend left, "S"] \arrow[r, bend right, "S'"'] \arrow[r, phantom, "\Downarrow\beta"] \arrow[d, "J"] & B \\ A \arrow[ur, dashed, bend left=40, "\mathrm{Rift}_J(R)"] \arrow[ur, dashed, "\mathrm{Rift}_J(R')"'] \arrow[ur, phantom, bend left=20, "\Downarrow \mathrm{Rift}_J(\alpha)"] \arrow[r, "R"] \arrow[r, bend right=40, "R'"'] \arrow[r, phantom, bend right=20, "\Downarrow\alpha"] & B \end{tikzcd}$$

for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

00MR 5.4.2 The Left Skew Monoidal Unit

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

00MS **Definition 5.4.2.1.1.** The **left J-skew monoidal unit of $\mathbf{Rel}(A, B)$** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J} \colon \mathsf{pt} \to \mathbf{Rel}(A, B)$$

picking the object

$$\mathbb{1}^{\lhd_J}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of $\mathbf{Rel}(A, B)$.

00MT 5.4.3 The Left Skew Associators

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

00MU **Definition 5.4.3.1.1.** The **left J-skew associator of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\lhd_J} \colon \lhd_J \circ (\lhd_J \times \mathrm{id}) \Longrightarrow \lhd_J \circ (\mathrm{id} \times \lhd_J) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)},$$

[14]Since $\mathbf{Rel}(A, B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then $S \lhd_J R \subset S' \lhd_J R'$.

as in the diagram

$$
\begin{tikzcd}
 & \mathbf{Rel}(A,B)\times(\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B)) \arrow[dr, "\mathsf{id}\times\triangleleft_J"] & \\
(\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B))\times\mathbf{Rel}(A,B) \arrow[ur, dashed, "\alpha^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}", "\sim"'] \arrow[d, "\triangleleft_J\times\mathsf{id}"'] \arrow[r, phantom, "\Uparrow\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}"] & & \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \arrow[d, "\triangleleft_J"] \\
\mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B) \arrow[rr, "\triangleleft_J"'] & & \mathbf{Rel}(A,B),
\end{tikzcd}
$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}_{T,S,R}\colon \underbrace{(T\triangleleft_J S)\triangleleft_J R}_{\stackrel{\text{def}}{=}T\diamond\mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R)} \hookrightarrow \underbrace{T\triangleleft_J(S\triangleleft_J R)}_{\stackrel{\text{def}}{=}T\diamond\mathrm{Rift}_J(S\diamond\mathrm{Rift}_J(R))}$$

at (T,S,R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleleft_J}_{T,S,R} \stackrel{\text{def}}{=} \mathrm{id}_T\diamond\gamma,$$

where

$$\gamma\colon \mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R)\hookrightarrow\mathrm{Rift}_J(S\diamond\mathrm{Rift}_J(R))$$

is the inclusion adjunct to the inclusion

$$\epsilon_S\star\mathrm{id}_{\mathrm{Rift}_J(R)}\colon \underbrace{J\diamond\mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R)}_{\stackrel{\text{def}}{=}J_*(\mathrm{Rift}_J(S)\diamond\mathrm{Rift}_J(R))}\hookrightarrow S\diamond\mathrm{Rift}_J(R)$$

under the adjunction $J_*\dashv\mathrm{Rift}_J$, where $\epsilon\colon J\diamond\mathrm{Rift}_J\Longrightarrow\mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_*\dashv\mathrm{Rift}_J$.

00MV 5.4.4 The Left Skew Left Unitors

Let A and B be sets and let $J\colon A\nrightarrow B$ be a relation.

00MW **Definition 5.4.4.1.1.** The **left J-skew left unitor of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \triangleleft_J\circ\left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J}\times\mathsf{id}\right)\Longrightarrow\boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

as in the diagram

$$\begin{tikzcd}
\mathsf{pt} \times \mathbf{Rel}(A, B) \arrow[r, "\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J} \times \mathsf{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \arrow[d, "\triangleleft_J"] \arrow[dl, phantom, "\Swarrow \lambda^{\mathbf{Rel}(A,B),\triangleleft_J}"] \\
& \mathbf{Rel}(A, B),
\end{tikzcd}$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}_R \colon \underbrace{J \triangleleft_J R}_{\overset{\text{def}}{=} J \diamond \mathrm{Rift}_J(R)} \hookrightarrow R$$

at R is given by

$$\lambda^{\mathbf{Rel}(A,B),\triangleleft_J}_R \overset{\text{def}}{=} \epsilon_R,$$

where $\epsilon \colon J_* \diamond \mathrm{Rift}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

00MX 5.4.5 The Left Skew Right Unitors

Let A and B be sets and let $J \colon A \nrightarrow B$ be a relation.

00MY **Definition 5.4.5.1.1.** The **left J-skew right unitor of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleleft_J} \colon \rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleleft_J \circ \left(\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J}\right)$$

as in the diagram

$$\begin{tikzcd}
\mathbf{Rel}(A, B) \times \mathsf{pt} \arrow[r, "\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleleft_J}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B), \arrow[d, "\triangleleft_J"] \arrow[dl, phantom, "\Nearrow \rho^{\mathbf{Rel}(A,B),\triangleleft_J}"] \\
& \mathbf{Rel}(A, B)
\end{tikzcd}$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\triangleleft_J}_R \colon R \hookrightarrow \underbrace{R \triangleleft_J J}_{\overset{\text{def}}{=} R \diamond \mathrm{Rift}_J(J)}$$

at R is given by the composition

$$\begin{aligned} R &\overset{\sim}{\Longrightarrow} R \diamond \chi_A \\ &\overset{\mathrm{id}_R \diamond \eta_{\chi_A}}{\Longrightarrow} R \diamond \mathrm{Rift}_J(J_*(\chi_A)) \\ &\overset{\mathrm{def}}{=} R \diamond \mathrm{Rift}_J(J \diamond \chi_A) \\ &\overset{\sim}{\Longrightarrow} R \diamond \mathrm{Rift}_J(J) \\ &\overset{\mathrm{def}}{=} R \lhd_J J, \end{aligned}$$

where $\eta\colon \mathrm{id}_{\mathbf{Rel}(A,A)} \Longrightarrow \mathrm{Rift}_J \circ J_*$ is the unit of the adjunction $J_* \dashv \mathrm{Rift}_J$.

00MZ 5.4.6 The Left Skew Monoidal Structure on Rel(A, B)

00N0 **Proposition 5.4.6.1.1.** The category **Rel**(A, B) admits a left skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset **Rel**(A, B) of relations from A to B of Item 2 of Definition 5.1.1.1.3.
- *The Left Skew Monoidal Product.* The left J-skew monoidal product

  $$\lhd_J\colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

  of Definition 5.4.1.1.1.
- *The Left Skew Monoidal Unit.* The functor

  $$\mathbb{1}^{\mathbf{Rel}(A,B),\lhd_J}\colon \mathsf{pt} \to \mathbf{Rel}(A, B)$$

  of Definition 5.4.2.1.1.
- *The Left Skew Associators.* The natural transformation

  $$\alpha^{\mathbf{Rel}(A,B),\lhd_J}\colon \lhd_J \circ (\lhd_J \times \mathsf{id}) \Longrightarrow \lhd_J \circ (\mathsf{id} \times \lhd_J) \circ \boldsymbol{\alpha}^{\mathsf{Cats}}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}$$

  of Definition 5.4.3.1.1.
- *The Left Skew Left Unitors.* The natural transformation

  $$\lambda^{\mathbf{Rel}(A,B),\lhd_J}\colon \lhd_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J} \times \mathsf{id}\right) \Longrightarrow \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

  of Definition 5.4.4.1.1.
- *The Left Skew Right Unitors.* The natural transformation

  $$\rho^{\mathbf{Rel}(A,B),\lhd_J}\colon \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \lhd_J \circ \left(\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\lhd_J}\right)$$

  of Definition 5.4.5.1.1.

Proof. Since $\mathbf{Rel}(A, B)$ is posetal, the commutativity of the pentagon identity, the left skew left triangle identity, the left skew right triangle identity, the left skew middle triangle identity, and the zigzag identity is automatic, and thus $\mathbf{Rel}(A, B)$ together with the data in the statement forms a left skew monoidal category. □

00N1 5.5 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

Let A and B be sets and let $J\colon A \nrightarrow B$ be a relation.

00N2 5.5.1 The Right Skew Monoidal Product

00N3 **Definition 5.5.1.1.1.** The **right J-skew monoidal product of $\mathbf{Rel}(A, B)$** is the functor

$$\triangleright_J\colon \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \to \mathbf{Rel}(A, B)$$

where

- *Action on Objects.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

$$S \triangleright_J R \overset{\mathrm{def}}{=} \mathrm{Ran}_J(S) \diamond R, \qquad \begin{tikzcd} A \arrow[r, "R", "\mid" marking] & B \arrow[r, dashed, "\mathrm{Ran}_J(S)", "\mid" marking] & B. \\ & A \arrow[u, "J", "\mid" marking] \arrow[ur, "S"', "\mid" marking] \arrow[u, phantom, shift left=4, "\Leftarrow"] & \end{tikzcd}$$

- *Action on Morphisms.* For each $R, S, R', S' \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

$$(\triangleright_J)_{(S,R),(S',R')}\colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S') \times \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R') \to \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S \triangleright_J R, S' \triangleright_J R')$$

of $\triangleright_J$ at $((S, R), (S', R'))$ is defined by[15]

$$\beta \triangleright_J \alpha \overset{\mathrm{def}}{=} \mathrm{Ran}_J(\beta) \diamond \alpha, \qquad \begin{tikzcd} A \arrow[r, bend left, "R'", "\mid" marking] \arrow[r, bend right, "R"', "\mid" marking] \arrow[r, phantom, "\alpha\Uparrow"] & B \arrow[rr, bend left, dashed, "\mathrm{Ran}_J(S')", "\mid" marking] \arrow[rr, dashed, "\mathrm{Ran}_J(S)"', "\mid" marking] \arrow[rr, phantom, bend left=15, "\mathrm{Ran}_J(\beta)\Uparrow"] & & B \\ & & S' & \\ & A \arrow[uu, "J", "\mid" marking] \arrow[uurr, "S'", "\mid" marking] \arrow[uurr, bend right, "S"'] \arrow[ur, phantom, "\Leftarrow\beta"] & & \end{tikzcd}$$

for each $\beta \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(S, S')$ and each $\alpha \in \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, R')$.

[15] Since $\mathbf{Rel}(A, B)$ is posetal, this is to say that if $S \subset S'$ and $R \subset R'$, then

00N4 5.5.2 The Right Skew Monoidal Unit

00N5 **Definition 5.5.2.1.1.** The **right J-skew monoidal unit of $\mathbf{Rel}(A,B)$** is the functor

$$\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J} \colon \mathsf{pt} \to \mathbf{Rel}(A,B)$$

picking the object

$$\mathbb{1}^{\triangleright_J}_{\mathbf{Rel}(A,B)} \overset{\text{def}}{=} J$$

of $\mathbf{Rel}(A,B)$.

00N6 5.5.3 The Right Skew Associators

00N7 **Definition 5.5.3.1.1.** The **right J-skew associator of $\mathbf{Rel}(A,B)$** is the natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J} \colon \triangleright_J \circ (\mathsf{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \mathsf{id}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)},$$

as in the diagram

$$\begin{tikzcd}
& (\mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B)) \times \mathbf{Rel}(A,B) \arrow[dr, "\triangleright_J \times \mathsf{id}"] & \\
\mathbf{Rel}(A,B) \times (\mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B)) \arrow[ur, dashed, "\boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}", "\sim"'] \arrow[d, "\mathsf{id} \times \triangleright_J"'] \arrow[rr, phantom, "\Uparrow \alpha^{\mathbf{Rel}(A,B),\triangleright_J}"] & & \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[d, "\triangleright_J"] \\
\mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \arrow[rr, "\triangleright_J"'] & & \mathbf{Rel}(A,B),
\end{tikzcd}$$

whose component

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J}_{T,S,R} \colon \underbrace{T \triangleright_J (S \triangleright_J R)}_{\overset{\text{def}}{=} \mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \diamond R} \hookrightarrow \underbrace{(T \triangleright_J S) \triangleright_J R}_{\overset{\text{def}}{=} \mathrm{Ran}_J(\mathrm{Ran}_J(T) \diamond S) \diamond R}$$

at (T,S,R) is given by

$$\alpha^{\mathbf{Rel}(A,B),\triangleright}_{T,S,R} \overset{\text{def}}{=} \gamma \diamond \mathrm{id}_R,$$

where

$$\gamma \colon \mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \hookrightarrow \mathrm{Ran}_J(\mathrm{Ran}_J(T) \diamond S)$$

is the inclusion adjunct to the inclusion

$$\mathrm{id}_{\mathrm{Ran}_J(T)} \diamond \epsilon_S \colon \underbrace{\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S) \diamond J}_{\overset{\text{def}}{=} J^*(\mathrm{Ran}_J(T) \diamond \mathrm{Ran}_J(S))} \hookrightarrow \mathrm{Ran}_J(T) \diamond S$$

under the adjunction $J^* \dashv \mathrm{Ran}_J$, where $\epsilon \colon \mathrm{Ran}_J \diamond J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

00N8 5.5.4 The Right Skew Left Unitors

00N9 **Definition 5.5.4.1.1.** The **right J-skew left unitor of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J}\colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleright_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \times \mathsf{id}\right),$$

as in the diagram

$$\begin{tikzcd}
\mathsf{pt} \times \mathbf{Rel}(A, B) \arrow[r, "\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J} \times \mathsf{id}"] \arrow[dr, dashed, bend right, "\lambda^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] \arrow[r, phantom, shift right=6ex, "\Uparrow \lambda^{\mathbf{Rel}(A,B),\triangleright_J}"] & \mathbf{Rel}(A, B) \times \mathbf{Rel}(A, B) \arrow[d, "\triangleright_J"] \\
 & \mathbf{Rel}(A, B),
\end{tikzcd}$$

whose component

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J}_{R}\colon R \hookrightarrow \underbrace{J \triangleright_J R}_{\overset{\text{def}}{=} \mathrm{Ran}_J(J) \diamond R}$$

at R is given by the composition

$$\begin{aligned}
R &\overset{\sim}{\Longrightarrow} \chi_B \diamond R \\
&\xRightarrow{\eta_{\chi_B} \diamond \mathrm{id}_R} \mathrm{Ran}_J(J^*(\chi_A)) \diamond R \\
&\overset{\text{def}}{=} \mathrm{Ran}_J(J^* \diamond \chi_A) \diamond R \\
&\overset{\sim}{\Longrightarrow} \mathrm{Ran}_J(J) \diamond R \\
&\overset{\text{def}}{=} R \triangleright_J J,
\end{aligned}$$

where $\eta\colon \mathrm{id}_{\mathbf{Rel}(B,B)} \Longrightarrow \mathrm{Ran}_J \circ J^*$ is the unit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

00NA 5.5.5 The Right Skew Right Unitors

00NB **Definition 5.5.5.1.1.** The **right J-skew right unitor of $\mathbf{Rel}(A, B)$** is the natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J}\colon \triangleright_J \circ \left(\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}\right) \Longrightarrow \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)},$$

$S \triangleright_J R \subset S' \triangleright_J R'$.

as in the diagram

$$\begin{tikzcd}
\mathbf{Rel}(A,B)\times\mathsf{pt} \arrow[r, "\mathsf{id}\times\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright_J}"] \arrow[dr, dashed, bend right, "\rho^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}"'] & \mathbf{Rel}(A,B)\times\mathbf{Rel}(A,B), \arrow[d, "\triangleright_J"] \arrow[dl, phantom, "\Swarrow \rho^{\mathbf{Rel}(A,B),\triangleright_J}"] \\
 & \mathbf{Rel}(A,B)
\end{tikzcd}$$

whose component

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J}_S\colon \underbrace{S \triangleright_J J}_{\overset{\mathrm{def}}{=}\mathrm{Ran}_J(S)\diamond J} \hookrightarrow S$$

at S is given by

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J}_S \overset{\mathrm{def}}{=} \epsilon_R,$$

where $\epsilon\colon J^* \circ \mathrm{Ran}_J \Longrightarrow \mathrm{id}_{\mathbf{Rel}(A,B)}$ is the counit of the adjunction $J^* \dashv \mathrm{Ran}_J$.

00NC 5.5.6 The Right Skew Monoidal Structure on $\mathbf{Rel}(A, B)$

00ND **Proposition 5.5.6.1.1.** The category $\mathbf{Rel}(A, B)$ admits a right skew monoidal category structure consisting of

- *The Underlying Category.* The posetal category associated to the poset $\mathbf{Rel}(A, B)$ of relations from A to B of Item 2 of Definition 5.1.1.1.3.
- *The Right Skew Monoidal Product.* The right J-skew monoidal product

$$\triangleleft_J\colon \mathbf{Rel}(A,B) \times \mathbf{Rel}(A,B) \to \mathbf{Rel}(A,B)$$

  of Definition 5.5.1.1.1.
- *The Right Skew Monoidal Unit.* The functor

$$\mathbb{1}^{\mathbf{Rel}(A,B),\triangleleft_J}\colon \mathsf{pt} \to \mathbf{Rel}(A,B)$$

  of Definition 5.5.2.1.1.
- *The Right Skew Associators.* The natural transformation

$$\alpha^{\mathbf{Rel}(A,B),\triangleright_J}\colon \triangleright_J \circ (\mathsf{id} \times \triangleright_J) \Longrightarrow \triangleright_J \circ (\triangleright_J \times \mathsf{id}) \circ \boldsymbol{\alpha}^{\mathsf{Cats},-1}_{\mathbf{Rel}(A,B),\mathbf{Rel}(A,B),\mathbf{Rel}(A,B)}$$

  of Definition 5.5.3.1.1.

- *The Right Skew Left Unitors.* The natural transformation

$$\lambda^{\mathbf{Rel}(A,B),\triangleright_J} \colon \boldsymbol{\lambda}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)} \Longrightarrow \triangleright_J \circ \left(\mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright} \times \mathsf{id}\right)$$

  of Definition 5.5.4.1.1.

- *The Right Skew Right Unitors.* The natural transformation

$$\rho^{\mathbf{Rel}(A,B),\triangleright_J} \colon \triangleright_J \circ \left(\mathsf{id} \times \mathbb{1}^{\mathbf{Rel}(A,B)}_{\triangleright}\right) \Longrightarrow \boldsymbol{\rho}^{\mathsf{Cats}_2}_{\mathbf{Rel}(A,B)}$$

  of Definition 5.5.5.1.1.

Proof. Since $\mathbf{Rel}(A, B)$ is posetal, the commutativity of the pentagon identity, the right skew left triangle identity, the right skew right triangle identity, the right skew middle triangle identity, and the zigzag identity is automatic, and thus $\mathbf{Rel}(A, B)$ together with the data in the statement forms a right skew monoidal category. □

Appendices

5.A Other Chapters

Chapter 6

Constructions With Relations

00NE This chapter contains some material about constructions with relations. Notably, we discuss and explore:

1. The existence or non-existence of Kan extensions and Kan lifts in the 2-category **Rel** (Section 6.2).
2. The various kinds of constructions involving relations, such as graphs, domains, ranges, unions, intersections, products, inverse relations, composition of relations, and collages (Section 6.3).
3. The adjoint pairs

$$R_* \dashv R_{-1} \colon \mathcal{P}(A) \rightleftarrows \mathcal{P}(B),$$
$$R^{-1} \dashv R_! \colon \mathcal{P}(B) \rightleftarrows \mathcal{P}(A)$$

of functors (morphisms of posets) between $\mathcal{P}(A)$ and $\mathcal{P}(B)$ induced by a relation $R \colon A \nrightarrow B$, as well as the properties of R_*, R_{-1}, R^{-1}, and $R_!$ (Section 6.4).

Of particular note are the following points:

(a) These two pairs of adjoint functors are the counterpart for relations of the adjoint triple $f_* \dashv f^{-1} \dashv f_!$ induced by a function $f \colon A \to B$ studied in Section 2.4.

(b) We have $R_{-1} = R^{-1}$ iff R is total and functional (Item 8 of Proposition 6.4.2.1.3).

(c) As a consequence of the previous item, when R comes from a function f, the pair of adjunctions

$$R_* \dashv R_{-1} = R^{-1} \dashv R_!$$

reduces to the triple adjunction

$$f_* \dashv f^{-1} \dashv f_!$$

from Section 2.4.

(d) The pairs $R_* \dashv R_{-1}$ and $R^{-1} \dashv R_!$ turn out to be rather important later on, as they appear in the definition and study of continuous, open, and closed relations between topological spaces (**??**).

Contents

00NF 6.1 Co/Limits in the Category of Relations

This section is currently just a stub, and will be properly developed later on.

00NG 6.2 Kan Extensions and Kan Lifts in the 2-Category of Relations

00NH 6.2.1 Left Kan Extensions in **Rel**

00NJ **Proposition 6.2.1.1.1.** Let $R\colon A \relbar\joinrel\mapstochar\joinrel\rightarrow B$ be a relation.

00NK 1. *Non-Existence of All Left Kan Extensions in* **Rel**. Not all relations in **Rel** admit left Kan extensions.

00NL 2. *Characterisation of Relations Admitting Left Kan Extensions Along Them.* The following conditions are equivalent:

   (a) The left Kan extension

$$\mathrm{Lan}_R\colon \mathbf{Rel}(A,X) \to \mathbf{Rel}(B,X)$$

   along R exists.

   (b) The relation R admits a left adjoint in **Rel**.

   (c) The relation R is of the form f^{-1} (as in Definition 6.3.2.1.1) for some function f.

Proof. *Item 1, Non-Existence of All Left Kan Extensions in* **Rel**: Omitted, but will eventually follow Fosco Loregian's comment on [MO 460656].

Item 2, Characterisation of Relations Admitting Left Kan Extensions Along Them: Omitted, but will eventually follow Tim Campion's answer to to [MO 460656]. □

00NM **Question 6.2.1.1.2.** Given relations $S\colon A \relbar\joinrel\mapstochar\joinrel\rightarrow X$ and $R\colon A \relbar\joinrel\mapstochar\joinrel\rightarrow B$, is there a characterisation of when the left Kan extension

$$\mathrm{Lan}_S(R)\colon B \relbar\joinrel\mapstochar\joinrel\rightarrow X$$

exists in terms of properties of R and S?

This question also appears as [MO 461592].

00NN **Question 6.2.1.1.3.** As shown in Item 2 of Proposition 6.2.1.1.1, the left Kan extension

$$\mathrm{Lan}_R\colon \mathbf{Rel}(A,X) \to \mathbf{Rel}(B,X)$$

along a relation of the form $R = f^{-1}$ exists. Is there a explicit description of it, similarly to the explicit description of right Kan extensions given in Proposition 6.2.3.1.1?

This question also appears as [MO 461592].

00NP 6.2.2 Left Kan Lifts in **Rel**

00NQ **Proposition 6.2.2.1.1.** Let $R\colon A \nrightarrow B$ be a relation.

00NR 1. *Non-Existence of All Left Kan Lifts in* **Rel**. Not all relations in **Rel** admit left Kan lifts.

2. *Characterisation of Relations Admitting Left Kan Lifts Along Them.*
00NS The following conditions are equivalent:

   (a) The left Kan lift

   $$\mathrm{Lift}_R\colon \mathbf{Rel}(X,B) \to \mathbf{Rel}(X,A)$$

   along R exists.

   (b) The relation R admits a right adjoint in **Rel**.

   (c) The relation R is of the form $\mathrm{Gr}(f)$ (as in Definition 6.3.1.1.1) for some function f.

Proof. *Item 1, Non-Existence of All Left Kan Lifts in* **Rel**: Omitted, but will eventually follow (the dual of) Fosco Loregian's comment on [MO 460656].
Item 2, Characterisation of Relations Admitting Left Kan Lifts Along Them: Omitted, but will eventually follow Tim Campion's answer to to [MO 460656]. □

00NT **Question 6.2.2.1.2.** Given relations $S\colon A \nrightarrow X$ and $R\colon A \nrightarrow B$, is there a characterisation of when the left Kan lift

$$\mathrm{Lift}_S(R)\colon X \nrightarrow A$$

exists in terms of properties of R and S?
This question also appears as [MO 461592].

00NU **Question 6.2.2.1.3.** As shown in Item 2 of Proposition 6.2.2.1.1, the left Kan lift

$$\mathrm{Lift}_R\colon \mathbf{Rel}(X,B) \to \mathbf{Rel}(X,A)$$

along a relation of the form $R = \mathrm{Gr}(f)$ exists. Is there a explicit description of it, similarly to the explicit description of right Kan lifts given in Proposition 6.2.4.1.1?
This question also appears as [MO 461592].

00NV 6.2.3 Right Kan Extensions in **Rel**

Let $R\colon A \nrightarrow B$ be a relation.

00NW **Proposition 6.2.3.1.1.** The right Kan extension

$$\mathrm{Ran}_R\colon \mathrm{Rel}(A,X) \to \mathrm{Rel}(B,X)$$

along R in **Rel** exists and is given by

$$\mathrm{Ran}_R(S) \stackrel{\mathrm{def}}{=} \int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^{-2}, S_a^{-1})$$

for each $S \in \mathrm{Rel}(A,X)$, so that the following conditions are equivalent:

1. We have $b \sim_{\mathrm{Ran}_R(S)} x$.

2. For each $a \in A$, if $a \sim_R b$, then $a \sim_S x$.

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(A,X)}(S\diamond R, T) &\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}((S\diamond R)^x_a, T^x_a)\\
&\cong \int_{a\in A}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{b\in B} S^x_b \times R^b_a\right), T^x_a\right)\\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^x_b \times R^b_a, T^x_a)\\
&\cong \int_{a\in A}\int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^x_b, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^x_a))\\
&\cong \int_{b\in B}\int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(S^x_b, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^x_a))\\
&\cong \int_{b\in B}\int_{x\in X} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^x_b, \int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R^b_a, T^x_a)\right)\\
&\cong \mathrm{Hom}_{\mathbf{Rel}(B,X)}\left(S, \int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^{-2}, T_a^{-1})\right)
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(B,X)$ and each $T \in \mathbf{Rel}(A,X)$, showing that

$$\int_{a\in A}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^{-2}, T_a^{-1})$$

is right adjoint to the precomposition functor $-\diamond R$, being thus the right Kan extension along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Item 1 of Proposition 5.1.1.1.5.

2. Definition 6.3.12.1.1.

3. **??** of **??**.

4. Proposition 1.2.2.1.5.

5. ?? of ??.

6. ?? of ??.

7. Item 1 of Proposition 5.1.1.1.5.

This finishes the proof. □

00NX 6.2.4 Right Kan Lifts in Rel

Let $R\colon A \nrightarrow B$ be a relation.

00NY **Proposition 6.2.4.1.1.** The right Kan lift

$$\mathrm{Rift}_R\colon \mathrm{Rel}(X,B) \to \mathrm{Rel}(X,A)$$

along R in **Rel** exists and is given by

$$\mathrm{Rift}_R(S) \stackrel{\mathrm{def}}{=} \int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-_1}, S^b_{-_2}\right)$$

for each $S \in \mathrm{Rel}(X,B)$, so that the following conditions are equivalent:

1. We have $x \sim_{\mathrm{Rift}_R(S)} a$.

2. For each $b \in B$, if $a \sim_R b$, then $x \sim_S b$.

Proof. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Rel}(X,B)}(R \diamond S, T) &\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left((R\diamond S)^b_x, T^b_x\right)\\
&\cong \int_{x\in X}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(\left(\int^{a\in A} R^b_a \times S^a_x\right), T^b_x\right)\\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a \times S^a_x, T^b_x\right)\\
&\cong \int_{x\in X}\int_{b\in B}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^a_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^b_x\right)\right)\\
&\cong \int_{x\in X}\int_{a\in A}\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^a_x, \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^b_x\right)\right)\\
&\cong \int_{x\in X}\int_{a\in A} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(S^a_x, \int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, T^b_x\right)\right)\\
&\cong \mathrm{Hom}_{\mathbf{Rel}(X,A)}\left(S, \int_{b\in B}\mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-_1}, T^b_{-_2}\right)\right)
\end{aligned}$$

naturally in each $S \in \mathbf{Rel}(X,A)$ and each $T \in \mathbf{Rel}(X,B)$, showing that

$$\int_{b\in B} \mathbf{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-_1}, S^b_{-_2}\right)$$

is right adjoint to the postcomposition functor $R \diamond -$, being thus the right Kan lift along R. Here we have used the following results, respectively (i.e. for each $\cong$ sign):

1. Item 1 of Proposition 5.1.1.1.5.
2. Definition 6.3.12.1.1.
3. **??** of **??**.
4. Proposition 1.2.2.1.5.
5. **??** of **??**.
6. **??** of **??**.
7. Item 1 of Proposition 5.1.1.1.5.

This finishes the proof. $\square$

00NZ 6.3 More Constructions With Relations

00P0 6.3.1 The Graph of a Function

Let $f\colon A \to B$ be a function.

00P1 **Definition 6.3.1.1.1.** The **graph of** f is the relation $\mathrm{Gr}(f)\colon A \nrightarrow B$ defined as follows:[1]

- Viewing relations from A to B as subsets of $A \times B$, we define
$$\mathrm{Gr}(f) \overset{\text{def}}{=} \{(a, f(a)) \in A \times B \mid a \in A\}.$$
- Viewing relations from A to B as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define
$$[\mathrm{Gr}(f)](a,b) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } b = f(a), \\ \mathsf{false} & \text{otherwise} \end{cases}$$
for each $(a,b) \in A \times B$.
- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define
$$[\mathrm{Gr}(f)](a) \overset{\text{def}}{=} \{f(a)\}$$
for each $a \in A$, i.e. we define $\mathrm{Gr}(f)$ as the composition
$$A \xrightarrow{f} B \xrightarrow{\chi_B} \mathcal{P}(B).$$

[1] *Further Notation:* We write $\mathrm{Gr}(A)$ for $\mathrm{Gr}(\mathrm{id}_A)$, and call it the **graph** of A.

00P2 **Proposition 6.3.1.1.2.** Let $f\colon A \to B$ be a function.

00P3 1. *Functoriality.* The assignment $A \mapsto \mathrm{Gr}(A)$ defines a functor

$$\mathrm{Gr}\colon \mathsf{Sets} \to \mathrm{Rel}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\mathrm{Gr}(A) \stackrel{\text{def}}{=} A.$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$\mathrm{Gr}_{A,B}\colon \mathsf{Sets}(A, B) \to \underbrace{\mathrm{Rel}(\mathrm{Gr}(A), \mathrm{Gr}(B))}_{\stackrel{\text{def}}{=}\mathrm{Rel}(A,B)}$$

of Gr at (A, B) is defined by

$$\mathrm{Gr}_{A,B}(f) \stackrel{\text{def}}{=} \mathrm{Gr}(f),$$

where $\mathrm{Gr}(f)$ is the graph of f as in Definition 6.3.1.1.1.

In particular:

- *Preservation of Identities.* We have

$$\mathrm{Gr}(\mathrm{id}_A) = \chi_A$$

for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition.* We have

$$\mathrm{Gr}(g \circ f) = \mathrm{Gr}(g) \diamond \mathrm{Gr}(f)$$

for each pair of functions $f\colon A \to B$ and $g\colon B \to C$.

00P4 2. *Adjointness Inside* **Rel**. We have an adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon \quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

in **Rel**, where f^{-1} is the inverse of f of Definition 6.3.2.1.1.

00P5 3. *Adjointness.* We have an adjunction

$$(\mathrm{Gr} \dashv \mathcal{P}_*)\colon \quad \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left, "\mathrm{Gr}"] \arrow[r, phantom, "\perp"] & \mathrm{Rel}, \arrow[l, bend left, "\mathcal{P}_*"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Rel}(\mathrm{Gr}(A), B) \cong \mathsf{Sets}(A, \mathcal{P}(B))$$

natural in $A \in \mathrm{Obj}(\mathsf{Sets})$ and $B \in \mathrm{Obj}(\mathrm{Rel})$.

00P6 4. *Interaction With Inverses.* We have

$$\mathrm{Gr}(f)^\dagger = f^{-1},$$
$$\left(f^{-1}\right)^\dagger = \mathrm{Gr}(f).$$

00P7 5. *Cocontinuity.* The functor $\mathrm{Gr}\colon \mathsf{Sets} \to \mathrm{Rel}$ of Item 1 preserves colimits.

00P8 6. *Characterisations.* Let $R\colon A \nrightarrow B$ be a relation. The following conditions are equivalent:

00P9 (a) There exists a function $f\colon A \to B$ such that $R = \mathrm{Gr}(f)$.

00PA (b) The relation R is total and functional.

(c) The weak and strong inverse images of R agree, i.e. we have
00PB $R^{-1} = R_{-1}$.

00PC (d) The relation R has a right adjoint $R^\dagger$ in Rel.

Proof. Item 1, Functoriality: Clear.
Item 2, Adjointness Inside **Rel**: We need to check that there are inclusions

$$\chi_A \subset f^{-1} \diamond \mathrm{Gr}(f),$$
$$\mathrm{Gr}(f) \diamond f^{-1} \subset \chi_B.$$

These correspond respectively to the following conditions:

1. For each $a \in A$, there exists some $b \in B$ such that $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$.

2. For each $a, b \in A$, if $a \sim_{\mathrm{Gr}(f)} b$ and $b \sim_{f^{-1}} a$, then $a = b$.

In other words, the first condition states that the image of any $a \in A$ by f is nonempty, whereas the second condition states that f is not multivalued. As f is a function, both of these statements are true, and we are done.

Item 3, Adjointness: The stated bijection follows from Remark 5.1.1.1.4, with naturality being clear.
Item 4, Interaction With Inverses: Clear.
Item 5, Cocontinuity: Omitted.
Item 6, Characterisations: We claim that Items 6a to 6d are indeed equivalent:

- *Item 6a* $\iff$ *Item 6b*. This is shown in the proof of **??** of **??**.
- *Item 6b* $\implies$ *Item 6c*. If R is total and functional, then, for each $a \in A$, the set $R(a)$ is a singleton, implying that

  $$R^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\},$$
  $$R_{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

  are equal for all $V \in \mathcal{P}(B)$, as the conditions $R(a) \cap V \neq \emptyset$ and $R(a) \subset V$ are equivalent when $R(a)$ is a singleton.
- *Item 6c* $\implies$ *Item 6b*. We claim that R is indeed total and functional:
  - *Totality.* If we had $R(a) = \emptyset$ for some $a \in A$, then we would have $a \in R_{-1}(\emptyset)$, so that $R_{-1}(\emptyset) \neq \emptyset$. But since $R^{-1}(\emptyset) = \emptyset$, this would imply $R_{-1}(\emptyset) \neq R^{-1}(\emptyset)$, a contradiction. Thus $R(a) \neq \emptyset$ for all $a \in A$ and R is total.
  - *Functionality.* If $R^{-1} = R_{-1}$, then we have

    $$\begin{aligned}\{a\} &= R^{-1}(\{b\})\\ &= R_{-1}(\{b\})\end{aligned}$$

    for each $b \in R(a)$ and each $a \in A$, and thus $R(a) \subset \{b\}$. But since R is total, we must have $R(a) = \{b\}$, and thus we see that R is functional.
- *Item 6a* $\iff$ *Item 6d*. This follows from Proposition 5.3.3.1.1.

This finishes the proof. □

00PD 6.3.2 The Inverse of a Function

Let $f \colon A \to B$ be a function.

00PE **Definition 6.3.2.1.1.** The **inverse of** f is the relation $f^{-1} \colon B \nrightarrow A$ defined as follows:

- Viewing relations from B to A as subsets of $B \times A$, we define

$$f^{-1} \stackrel{\text{def}}{=} \left\{ \left(b, f^{-1}(b)\right) \in B \times A \;\middle|\; a \in A \right\},$$

where

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

for each $b \in B$.

- Viewing relations from B to A as functions $B \times A \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$f^{-1}(b, a) \stackrel{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if there exists } a \in A \text{ with } f(a) = b, \\ \mathsf{false} & \text{otherwise} \end{cases}$$

for each $(b, a) \in B \times A$.

- Viewing relations from B to A as functions $B \to \mathcal{P}(A)$, we define

$$f^{-1}(b) \stackrel{\text{def}}{=} \{a \in A \mid f(a) = b\}$$

for each $b \in B$.

00PF **Proposition 6.3.2.1.2.** Let $f \colon A \to B$ be a function.

00PG 1. *Functoriality.* The assignment $A \mapsto A$, $f \mapsto f^{-1}$ defines a functor

$$(-)^{-1} \colon \mathsf{Sets} \to \mathrm{Rel}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$\left[(-)^{-1}\right](A) \stackrel{\text{def}}{=} A.$$

- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathsf{Sets})$, the action on Hom-sets

$$(-)^{-1}_{A,B} \colon \mathsf{Sets}(A, B) \to \mathrm{Rel}(A, B)$$

of $(-)^{-1}$ at (A, B) is defined by

$$(-)^{-1}_{A,B}(f) \stackrel{\text{def}}{=} \left[(-)^{-1}\right](f),$$

where f^{-1} is the inverse of f as in Definition 6.3.2.1.1.

In particular:

- *Preservation of Identities.* We have

$$\mathrm{id}_A^{-1} = \chi_A$$

for each $A \in \mathrm{Obj}(\mathsf{Sets})$.

- *Preservation of Composition.* We have

$$(g \circ f)^{-1} = g^{-1} \diamond f^{-1}$$

for pair of functions $f\colon A \to B$ and $g\colon B \to C$.

00PH 2. *Adjointness Inside* **Rel**. We have an adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon\quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

in **Rel**.

00PJ 3. *Interaction With Inverses of Relations.* We have

$$\left(f^{-1}\right)^{\dagger} = \mathrm{Gr}(f),$$
$$\mathrm{Gr}(f)^{\dagger} = f^{-1}.$$

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Adjointness Inside* **Rel**: This is proved in Item 2 of Proposition 6.3.1.1.2.
Item 3, *Interaction With Inverses of Relations*: Clear. □

00PK 6.3.3 Representable Relations

Let A and B be sets.

00PL **Definition 6.3.3.1.1.** Let $f\colon A \to B$ and $g\colon B \to A$ be functions.[2]

1. The **representable relation associated to** f is the relation

[2]More generally, given functions

$$f\colon A \to C,$$
$$g\colon B \to D$$

and a relation $B \nrightarrow D$, we may consider the composite relation

$$A \times B \xrightarrow{f \times g} C \times D \xrightarrow{R} \{\mathsf{true}, \mathsf{false}\},$$

for which we have $a \sim_{R \circ (f \times g)} b$ iff $f(a) \sim_R g(b)$.

$\chi_f \colon A \nrightarrow B$ defined as the composition

$$A \times B \xrightarrow{f \times \mathrm{id}_B} B \times B \xrightarrow{\chi_B} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $a \sim_{\chi_f} b$ iff $f(a) = b$.

2. The **corepresentable relation associated to** g is the relation $\chi^g \colon B \nrightarrow A$ defined as the composition

$$B \times A \xrightarrow{g \times \mathrm{id}_A} A \times A \xrightarrow{\chi_A} \{\mathsf{true}, \mathsf{false}\},$$

i.e. given by declaring $b \sim_{\chi^g} a$ iff $g(b) = a$.

00PM 6.3.4 The Domain and Range of a Relation

Let A and B be sets.

00PN **Definition 6.3.4.1.1.** Let $R \subset A \times B$ be a relation.[3,4]

1. The **domain of** R is the subset $\mathrm{dom}(R)$ of A defined by

$$\mathrm{dom}(R) \overset{\mathrm{def}}{=} \left\{ a \in A \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \\ \text{such that } a \sim_R b \end{array} \right\}.$$

2. The **range of** R is the subset $\mathrm{range}(R)$ of B defined by

$$\mathrm{range}(R) \overset{\mathrm{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists some } a \in A \\ \text{such that } a \sim_R b \end{array} \right\}.$$

[3]Following **??**, we may compute the (characteristic functions associated to the) domain and range of a relation using the following colimit formulas:

$$\begin{aligned} \chi_{\mathrm{dom}(R)}(a) &\cong \operatorname*{colim}_{b \in B}\left(R^b_a\right) \qquad (a \in A) \\ &\cong \bigvee_{b \in B} R^b_a, \\ \chi_{\mathrm{range}(R)}(b) &\cong \operatorname*{colim}_{a \in A}\left(R^b_a\right) \qquad (b \in B) \\ &\cong \bigvee_{a \in A} R^b_a, \end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Definition 1.2.2.1.3.

[4]Viewing R as a function $R \colon A \to \mathcal{P}(B)$, we have

$$\begin{aligned} \mathrm{dom}(R) &\cong \operatorname*{colim}_{y \in Y}(R(y)) \\ &\cong \bigcup_{y \in Y} R(y), \\ \mathrm{range}(R) &\cong \operatorname*{colim}_{x \in X}(R(x)) \\ &\cong \bigcup_{x \in X} R(x), \end{aligned}$$

00PP 6.3.5 Binary Unions of Relations

Let A and B be sets and let R and S be relations from A to B.

00PQ **Definition 6.3.5.1.1.** The **union of** R **and** S[5] is the relation $R \cup S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[6]
$$R \cup S \overset{\text{def}}{=} \{(a, b) \in B \times A \mid \text{we have } a \sim_R b \text{ or } a \sim_S b\}.$$
- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define
$$[R \cup S](a) \overset{\text{def}}{=} R(a) \cup S(a)$$
for each $a \in A$.

00PR **Proposition 6.3.5.1.2.** Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

00PS 1. *Interaction With Inverses.* We have
$$(R \cup S)^\dagger = R^\dagger \cup S^\dagger.$$

00PT 2. *Interaction With Composition.* We have
$$(S_1 \diamond R_1) \cup (S_2 \diamond R_2) \overset{\text{poss.}}{\neq} (S_1 \cup S_2) \diamond (R_1 \cup R_2).$$

Proof. *Item 1, Interaction With Inverses*: Clear.
Item 2, Interaction With Composition: Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cup (S_2 \diamond R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
   or
      i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
3. The condition for $(S_1 \cup S_2) \diamond (R_1 \cup R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ or $a \sim_{R_2} b$;
   and
      i. $b \sim_{S_1} c$ or $b \sim_{S_2} c$.

These two conditions may fail to agree (counterexample omitted), and thus the two resulting relations on $A \times C$ may differ. □

[5] *Further Terminology:* Also called the **binary union of** R **and** S, for emphasis.
[6] This is the same as the union of R and S as subsets of $A \times B$.

00PU 6.3.6 Unions of Families of Relations

Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

00PV **Definition 6.3.6.1.1.** The **union of the family** $\{R_i\}_{i\in I}$ is the relation $\bigcup_{i\in I} R_i$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[7]

$$\bigcup_{i\in I} R_i \overset{\text{def}}{=} \left\{ (a,b) \in (A \times B)^{\times I} \;\middle|\; \begin{array}{l} \text{there exists some } i \in I \\ \text{such that } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcup_{i\in I} R_i\right](a) \overset{\text{def}}{=} \bigcup_{i\in I} R_i(a)$$

  for each $a \in A$.

00PW **Proposition 6.3.6.1.2.** Let A and B be sets and let $\{R_i\}_{i\in I}$ be a family of relations from A to B.

00PX 1. *Interaction With Inverses.* We have

$$\left(\bigcup_{i\in I} R_i\right)^{\dagger} = \bigcup_{i\in I} R_i^{\dagger}.$$

Proof. *Item 1*, *Interaction With Inverses*: Clear. □

00PY 6.3.7 Binary Intersections of Relations

Let A and B be sets and let R and S be relations from A to B.

00PZ **Definition 6.3.7.1.1.** The **intersection of R and S**[8] is the relation $R \cap S$ from A to B defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[9]

$$R \cap S \overset{\text{def}}{=} \{(a,b) \in B \times A \mid \text{we have } a \sim_R b \text{ and } a \sim_S b\}.$$

[7] This is the same as the union of $\{R_i\}_{i\in I}$ as a collection of subsets of $A \times B$.

[8] *Further Terminology:* Also called the **binary intersection of R and S**, for emphasis.

[9] This is the same as the intersection of R and S as subsets of $A \times B$.

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$[R \cap S](a) \stackrel{\text{def}}{=} R(a) \cap S(a)$$

for each $a \in A$.

00Q0 **Proposition 6.3.7.1.2.** Let R, S, R_1, and R_2 be relations from A to B, and let S_1 and S_2 be relations from B to C.

00Q1 1. *Interaction With Inverses.* We have

$$(R \cap S)^\dagger = R^\dagger \cap S^\dagger.$$

00Q2 2. *Interaction With Composition.* We have

$$(S_1 \diamond R_1) \cap (S_2 \diamond R_2) = (S_1 \cap S_2) \diamond (R_1 \cap R_2).$$

Proof. *Item 1, Interaction With Inverses*: Clear.
Item 2, Interaction With Composition: Unwinding the definitions, we see that:

1. The condition for $(S_1 \diamond R_1) \cap (S_2 \diamond R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ and $b \sim_{S_1} c$;

      and

      i. $a \sim_{R_2} b$ and $b \sim_{S_2} c$;
3. The condition for $(S_1 \cap S_2) \diamond (R_1 \cap R_2)$ is:
   (a) There exists some $b \in B$ such that:
      i. $a \sim_{R_1} b$ and $a \sim_{R_2} b$;

      and

      i. $b \sim_{S_1} c$ and $b \sim_{S_2} c$.

These two conditions agree, and thus so do the two resulting relations on $A \times C$. □

00Q3 6.3.8 Intersections of Families of Relations

Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

Definition 6.3.8.1.1. The **intersection of the family** $\{R_i\}_{i \in I}$ is the relation $\bigcup_{i \in I} R_i$ defined as follows:

- Viewing relations from A to B as subsets of $A \times B$, we define[10]

$$\bigcup_{i \in I} R_i \stackrel{\text{def}}{=} \left\{ (a,b) \in (A \times B)^{\times I} \;\middle|\; \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a \sim_{R_i} b \end{array} \right\}.$$

- Viewing relations from A to B as functions $A \to \mathcal{P}(B)$, we define

$$\left[\bigcap_{i \in I} R_i\right](a) \stackrel{\text{def}}{=} \bigcap_{i \in I} R_i(a)$$

  for each $a \in A$.

00Q5 **Proposition 6.3.8.1.2.** Let A and B be sets and let $\{R_i\}_{i \in I}$ be a family of relations from A to B.

00Q6 1. *Interaction With Inverses.* We have

$$\left(\bigcap_{i \in I} R_i\right)^{\dagger} = \bigcap_{i \in I} R_i^{\dagger}.$$

Proof. *Item 1, Interaction With Inverses*: Clear. □

00Q7 6.3.9 Binary Products of Relations

Let A, B, X, and Y be sets, let $R\colon A \nrightarrow B$ be a relation from A to B, and let $S\colon X \nrightarrow Y$ be a relation from X to Y.

00Q8 **Definition 6.3.9.1.1.** The **product of R and S**[11] is the relation $R \times S$ from $A \times X$ to $B \times Y$ defined as follows:

- Viewing relations from $A \times X$ to $B \times Y$ as subsets of $(A \times X) \times (B \times Y)$, we define $R \times S$ as the Cartesian product of R and S as subsets of $A \times X$ and $B \times Y$.[12]

- Viewing relations from $A \times X$ to $B \times Y$ as functions $A \times X \to \mathcal{P}(B \times Y)$, we define $R \times S$ as the composition

$$A \times X \xrightarrow{R \times S} \mathcal{P}(B) \times \mathcal{P}(Y) \overset{\mathcal{P}^{\otimes}_{B,Y}}{\hookrightarrow} \mathcal{P}(B \times Y)$$

  in Sets, i.e. by

$$[R \times S](a,x) \stackrel{\text{def}}{=} R(a) \times S(x)$$

  for each $(a,x) \in A \times X$.

[10] This is the same as the intersection of $\{R_i\}_{i \in I}$ as a collection of subsets of $A \times B$.
[11] *Further Terminology:* Also called the **binary product of R and S**, for emphasis.
[12] That is, $R \times S$ is the relation given by declaring $(a,x) \sim_{R \times S} (b,y)$ iff $a \sim_R b$ and

00Q9 **Proposition 6.3.9.1.2.** Let A, B, X, and Y be sets.

00QA 1. *Interaction With Inverses.* Let

$$\begin{aligned} R &\colon A \nrightarrow A, \\ S &\colon X \nrightarrow X \end{aligned}$$

We have

$$(R \times S)^{\dagger} = R^{\dagger} \times S^{\dagger}.$$

00QB 2. *Interaction With Composition.* Let

$$\begin{aligned} R_1 &\colon A \nrightarrow B, \\ S_1 &\colon B \nrightarrow C, \\ R_2 &\colon X \nrightarrow Y, \\ S_2 &\colon Y \nrightarrow Z \end{aligned}$$

be relations. We have

$$(S_1 \diamond R_1) \times (S_2 \diamond R_2) = (S_1 \times S_2) \diamond (R_1 \times R_2).$$

Proof. *Item 1, Interaction With Inverses*: Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(R \times S)^{\dagger}} (b, y)$ iff:
   - We have $(b, y) \sim_{R \times S} (a, x)$, i.e. iff:
     - We have $b \sim_R a$;
     - We have $y \sim_S x$;
2. We have $(a, x) \sim_{R^{\dagger} \times S^{\dagger}} (b, y)$ iff:
   - We have $a \sim_{R^{\dagger}} b$ and $x \sim_{S^{\dagger}} y$, i.e. iff:
     - We have $b \sim_R a$;
     - We have $y \sim_S x$.

These two conditions agree, and thus the two resulting relations on $A \times X$ are equal.
Item 2, Interaction With Composition: Unwinding the definitions, we see that:

1. We have $(a, x) \sim_{(S_1 \diamond R_1) \times (S_2 \diamond R_2)} (c, z)$ iff:
   (a) We have $a \sim_{S_1 \diamond R_1} c$ and $x \sim_{S_2 \diamond R_2} z$, i.e. iff:

i. There exists some $b \in B$ such that $a \sim_{R_1} b$ and $b \sim_{S_1} c$;
ii. There exists some $y \in Y$ such that $x \sim_{R_2} y$ and $y \sim_{S_2} z$;

2. We have $(a, x) \sim_{(S_1 \times S_2) \diamond (R_1 \times R_2)} (c, z)$ iff:
   (a) There exists some $(b, y) \in B \times Y$ such that $(a, x) \sim_{R_1 \times R_2} (b, y)$ and $(b, y) \sim_{S_1 \times S_2} (c, z)$, i.e. such that:
      i. We have $a \sim_{R_1} b$ and $x \sim_{R_2} y$;
      ii. We have $b \sim_{S_1} c$ and $y \sim_{S_2} z$.

These two conditions agree, and thus the two resulting relations from $A \times X$ to $C \times Z$ are equal. □

00QC 6.3.10 Products of Families of Relations

Let $\{A_i\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be families of sets, and let $\{R_i : A_i \nrightarrow B_i\}_{i \in I}$ be a family of relations.

00QD **Definition 6.3.10.1.1.** The **product of the family** $\{R_i\}_{i \in I}$ is the relation $\prod_{i \in I} R_i$ from $\prod_{i \in I} A_i$ to $\prod_{i \in I} B_i$ defined as follows:

- Viewing relations as subsets, we define $\prod_{i \in I} R_i$ as its product as a family of sets, i.e. we have

$$\prod_{i \in I} R_i \overset{\text{def}}{=} \left\{ (a_i, b_i)_{i \in I} \in \prod_{i \in I} (A_i \times B_i) \;\middle|\; \begin{array}{l} \text{for each } i \in I, \\ \text{we have } a_i \sim_{R_i} b_i \end{array} \right\}.$$

- Viewing relations as functions to powersets, we define

$$\left[\prod_{i \in I} R_i \right] ((a_i)_{i \in I}) \overset{\text{def}}{=} \prod_{i \in I} R_i(a_i)$$

  for each $(a_i)_{i \in I} \in \prod_{i \in I} R_i$.

00QE 6.3.11 The Inverse of a Relation

Let A, B, and C be sets and let $R \subset A \times B$ be a relation.

00QF **Definition 6.3.11.1.1.** The **inverse of** R[13] is the relation $R^\dagger$ defined as follows:

- Viewing relations as subsets, we define

$$R^\dagger \overset{\text{def}}{=} \{(b, a) \in B \times A \mid \text{we have } b \sim_R a\}.$$

$x \sim_S y$.

[13] *Further Terminology:* Also called the **opposite of** R, the **transpose of** R, or

- Viewing relations as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\left[R^{\dagger}\right]^{a}_{b} \stackrel{\text{def}}{=} R^{b}_{a}$$

for each $(b, a) \in B \times A$.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$\begin{aligned}\left[R^{\dagger}\right](b) &\stackrel{\text{def}}{=} R^{\dagger}(\{b\})\\ &\stackrel{\text{def}}{=} \{a \in A \mid b \in R(a)\}\end{aligned}$$

for each $b \in B$, where $R^{\dagger}(\{b\})$ is the fibre of R over $\{b\}$.

00QG **Example 6.3.11.1.2.** Here are some examples of inverses of relations.

00QH 1. *Less Than Equal Signs.* We have $(\leq)^{\dagger} = \geq$.

00QJ 2. *Greater Than Equal Signs.* Dually to Item 1, we have $(\geq)^{\dagger} = \leq$.

00QK 3. *Functions.* Let $f \colon A \to B$ be a function. We have

$$\begin{aligned}\mathrm{Gr}(f)^{\dagger} &= f^{-1},\\ \left(f^{-1}\right)^{\dagger} &= \mathrm{Gr}(f).\end{aligned}$$

00QL **Proposition 6.3.11.1.3.** Let $R \colon A \nrightarrow B$ and $S \colon B \nrightarrow C$ be relations.

00QM 1. *Functoriality.* The assignment $R \mapsto R^{\dagger}$ defines a functor (i.e. morphism of posets)

$$(-)^{\dagger} \colon \mathbf{Rel}(A, B) \to \mathbf{Rel}(B, A).$$

In particular, given relations $R, S \colon A \rightrightarrows B$, we have:

($\star$) If $R \subset S$, then $R^{\dagger} \subset S^{\dagger}$.

00QN 2. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\mathrm{dom}\left(R^{\dagger}\right) &= \mathrm{range}(R),\\ \mathrm{range}\left(R^{\dagger}\right) &= \mathrm{dom}(R).\end{aligned}$$

00QP 3. *Interaction With Composition I.* We have

$$(S \diamond R)^{\dagger} = R^{\dagger} \diamond S^{\dagger}.$$

00QQ 4. *Interaction With Composition II.* We have

$$\chi_B \subset R \diamond R^\dagger,$$
$$\chi_A \subset R^\dagger \diamond R.$$

00QR 5. *Invertibility.* We have

$$\left(R^\dagger\right)^\dagger = R.$$

00QS 6. *Identity.* We have

$$\chi_A^\dagger = \chi_A.$$

Proof. *Item 1*, *Functoriality*: Clear.
Item 2, *Interaction With Ranges and Domains*: Clear.
Item 3, *Interaction With Composition I*: Clear.
Item 4, *Interaction With Composition II*: Clear.
Item 5, *Invertibility*: Clear.
Item 6, *Identity*: Clear. □

00QT 6.3.12 Composition of Relations

Let A, B, and C be sets and let $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$ be relations.

00QU **Definition 6.3.12.1.1.** The **composition of** R **and** S is the relation $S \diamond R$ defined as follows:

- Viewing relations from A to C as subsets of $A \times C$, we define

$$S \diamond R \overset{\text{def}}{=} \left\{ (a, c) \in A \times C \;\middle|\; \begin{array}{l} \text{there exists some } b \in B \text{ such} \\ \text{that } a \sim_R b \text{ and } b \sim_S c \end{array} \right\}.$$

- Viewing relations as functions $A \times B \to \{\mathsf{true}, \mathsf{false}\}$, we define

$$\begin{aligned} (S \diamond R)^{-1}_{-2} &\overset{\text{def}}{=} \int^{b \in B} S^{-1}_b \times R^b_{-2} \\ &= \bigvee_{b \in B} S^{-1}_b \times R^b_{-2}, \end{aligned}$$

where the join $\bigvee$ is taken in the poset $(\{\mathsf{true}, \mathsf{false}\}, \preceq)$ of Definition 1.2.2.1.3.

the **converse of** R.

- Viewing relations as functions $A \to \mathcal{P}(B)$, we define

$$S \diamond R \overset{\text{def}}{=} \mathrm{Lan}_{\chi_B}(S) \circ R, \qquad \begin{tikzcd} & B \arrow[r, "S"] \arrow[d, "\chi_B"'] \arrow[dr, phantom, "\Swarrow"] & \mathcal{P}(C), \\ A \arrow[r, "R"'] & \mathcal{P}(B) \arrow[ur, bend right, "\mathrm{Lan}_{\chi_B}(S)"'] & \end{tikzcd}$$

  where $\mathrm{Lan}_{\chi_B}(S)$ is computed by the formula

$$\begin{aligned}[\mathrm{Lan}_{\chi_B}(S)](V) &\cong \int^{y \in B} \chi_{\mathcal{P}(B)}(\chi_y, V) \odot S_y \\ &\cong \int^{y \in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y \in B} \chi_V(y) \odot S_y \\ &\cong \bigcup_{y \in V} S_y\end{aligned}$$

  for each $V \in \mathcal{P}(B)$. In other words, $S \diamond R$ is defined by[14]

$$\begin{aligned}[S \diamond R](a) &\overset{\text{def}}{=} S(R(a)) \\ &\overset{\text{def}}{=} \bigcup_{x \in R(a)} S(x).\end{aligned}$$

  for each $a \in A$.

00QV **Example 6.3.12.1.2.** Here are some examples of composition of relations.

1. *Composing Less/Greater Than Equal With Greater/Less Than Equal Signs.* We have

$$\begin{aligned}\leq \diamond \geq &= \sim_{\text{triv}}, \\ \geq \diamond \leq &= \sim_{\text{triv}}.\end{aligned}$$

2. *Composing Less/Greater Than Equal Signs With Less/Greater Than Equal Signs.* We have

$$\begin{aligned}\leq \diamond \leq &= \leq, \\ \geq \diamond \geq &= \geq.\end{aligned}$$

[14] That is: the relation R may send $a \in A$ to a number of elements $\{b_i\}_{i \in I}$ in B, and then the relation S may send the image of each of the b_i's to a number of elements $\{S(b_i)\}_{i \in I} = \left\{\{c_{j_i}\}_{j_i \in J_i}\right\}_{i \in I}$ in C.

00QW **Proposition 6.3.12.1.3.** Let $R\colon A \nrightarrow B$, $S\colon B \nrightarrow C$, and $T\colon C \nrightarrow D$ be relations.

00QX 1. *Interaction With Ranges and Domains.* We have

$$\begin{aligned}\mathrm{dom}(S \diamond R) &\subset \mathrm{dom}(R),\\ \mathrm{range}(S \diamond R) &\subset \mathrm{range}(S).\end{aligned}$$

00QY 2. *Associativity.* We have

$$(T \diamond S) \diamond R = T \diamond (S \diamond R).$$

00QZ 3. *Unitality.* We have

$$\begin{aligned}\chi_B \diamond R &= R,\\ R \diamond \chi_A &= R.\end{aligned}$$

00R0 4. *Interaction With Inverses.* We have

$$(S \diamond R)^\dagger = R^\dagger \diamond S^\dagger.$$

00R1 5. *Interaction With Composition.* We have

$$\begin{aligned}\chi_B &\subset R \diamond R^\dagger,\\ \chi_A &\subset R^\dagger \diamond R.\end{aligned}$$

Proof. *Item 1*, *Interaction With Ranges and Domains*: Clear.
Item 2, *Associativity*: Indeed, we have

$$\begin{aligned}(T \diamond S) \diamond R &\stackrel{\text{def}}{=} \left(\int^{c\in C} T^{-1}_{c} \times S^{c}_{-_2}\right) \diamond R\\ &\stackrel{\text{def}}{=} \int^{b\in B}\left(\int^{c\in C} T^{-1}_{c} \times S^{c}_{b}\right) \diamond R^{b}_{-_2}\\ &= \int^{b\in B}\int^{c\in C} (T^{-1}_{c} \times S^{c}_{b}) \diamond R^{b}_{-_2}\\ &= \int^{c\in C}\int^{b\in B} (T^{-1}_{c} \times S^{c}_{b}) \diamond R^{b}_{-_2}\\ &= \int^{c\in C}\int^{b\in B} T^{-1}_{c} \times \left(S^{c}_{b} \diamond R^{b}_{-_2}\right)\\ &= \int^{c\in C} T^{-1}_{c} \times \left(\int^{b\in B} S^{c}_{b} \diamond R^{b}_{-_2}\right)\\ &\stackrel{\text{def}}{=} \int^{c\in C} T^{-1}_{c} \times (S \diamond R)^{c}_{-_2}\\ &\stackrel{\text{def}}{=} T \diamond (S \diamond R).\end{aligned}$$

In the language of relations, given $a \in A$ and $d \in D$, the stated equality witnesses the equivalence of the following two statements:

1. We have $a \sim_{(T \diamond S) \diamond R} d$, i.e. there exists some $b \in B$ such that:
   (a) We have $a \sim_R b$;
   (b) We have $b \sim_{T \diamond S} d$, i.e. there exists some $c \in C$ such that:
      i. We have $b \sim_S c$;
      ii. We have $c \sim_T d$;
2. We have $a \sim_{T \diamond (S \diamond R)} d$, i.e. there exists some $c \in C$ such that:
   (a) We have $a \sim_{S \diamond R} c$, i.e. there exists some $b \in B$ such that:
      i. We have $a \sim_R b$;
      ii. We have $b \sim_S c$;
   (b) We have $c \sim_T d$;

both of which are equivalent to the statement

- There exist $b \in B$ and $c \in C$ such that $a \sim_R b \sim_S c \sim_T d$.

Item 3, *Unitality*: Indeed, we have

$$\begin{aligned} \chi_B \diamond R &\stackrel{\text{def}}{=} \int^{x \in B} (\chi_B)^{-1}_{x} \times R^{x}_{-2} \\ &= \bigvee_{x \in B} (\chi_B)^{-1}_{x} \times R^{x}_{-2} \\ &= \bigvee_{\substack{x \in B \\ x = -1}} R^{x}_{-2} \\ &= R^{-1}_{-2}, \end{aligned}$$

and

$$\begin{aligned} R \diamond \chi_A &\stackrel{\text{def}}{=} \int^{x \in A} R^{-1}_{x} \times (\chi_A)^{x}_{-2} \\ &= \bigvee_{x \in B} R^{-1}_{x} \times (\chi_A)^{x}_{-2} \\ &= \bigvee_{\substack{x \in B \\ x = -2}} R^{-1}_{x} \\ &= R^{-1}_{-2}. \end{aligned}$$

In the language of relations, given $a \in A$ and $b \in B$:

- The equality
$$\chi_B \diamond R = R$$
witnesses the equivalence of the following two statements:
  1. We have $a \sim_b B$.
  2. There exists some $b' \in B$ such that:
     (a) We have $a \sim_R b'$
     (b) We have $b' \sim_{\chi_B} b$, i.e. $b' = b$.

- The equality
$$R \diamond \chi_A = R$$
witnesses the equivalence of the following two statements:
  1. There exists some $a' \in A$ such that:
     (a) We have $a \sim_{\chi_B} a'$, i.e. $a = a'$.
     (b) We have $a' \sim_R b$
  2. We have $a \sim_b B$.

Item 4, Interaction With Inverses: Clear.
Item 5, Interaction With Composition: Clear. □

00R2 6.3.13 The Collage of a Relation

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

00R3 **Definition 6.3.13.1.1.** The **collage of** R[15] is the poset $\mathbf{Coll}(R) \overset{\text{def}}{=} \left(\mathrm{Coll}(R), \preceq_{\mathbf{Coll}(R)}\right)$ consisting of:

- *The Underlying Set.* The set $\mathrm{Coll}(R)$ defined by
$$\mathrm{Coll}(R) \overset{\text{def}}{=} A \coprod B.$$

- *The Partial Order.* The partial order
$$\preceq_{\mathbf{Coll}(R)}\colon \mathrm{Coll}(R) \times \mathrm{Coll}(R) \to \{\mathsf{true}, \mathsf{false}\}$$
on $\mathrm{Coll}(R)$ defined by
$$\preceq(a,b) \overset{\text{def}}{=} \begin{cases} \mathsf{true} & \text{if } a = b \text{ or } a \sim_R b, \\ \mathsf{false} & \text{otherwise.} \end{cases}$$

00R4 **Proposition 6.3.13.1.2.** Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation from A to B.

[15] *Further Terminology:* Also called the **cograph of** R.

00R5 1. *Functoriality I.* The assignment $R \mapsto \mathbf{Coll}(R)$ defines a functor[16]

$$\mathbf{Coll}\colon \mathbf{Rel}(A, B) \to \mathsf{Pos}_{/\Delta^1}(A, B),$$

where

- *Action on Objects.* For each $R \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, we have

  $$[\mathbf{Coll}](R) \stackrel{\text{def}}{=} (\mathbf{Coll}(R), \phi_R)$$

  for each $R \in \mathbf{Rel}(A, B)$, where

  – The poset $\mathbf{Coll}(R)$ is the collage of R of Definition 6.3.13.1.1.

  – The morphism $\phi_R \colon \mathbf{Coll}(R) \to \Delta^1$ is given by

  $$\phi_R(x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \in B \end{cases}$$

  for each $x \in \mathbf{Coll}(R)$.

- *Action on Morphisms.* For each $R, S \in \mathrm{Obj}(\mathbf{Rel}(A, B))$, the action on Hom-sets

  $$\mathbf{Coll}_{R,S}\colon \mathrm{Hom}_{\mathbf{Rel}(A,B)}(R, S) \to \mathsf{Pos}(\mathbf{Coll}(R), \mathbf{Coll}(S))$$

  of $\mathbf{Coll}$ at (R, S) is given by sending an inclusion

  $$\iota\colon R \subset S$$

[16] Here $\mathsf{Pos}_{/\Delta^1}(A, B)$ is the category defined as the pullback

$$\mathsf{Pos}_{/\Delta^1}(A, B) \stackrel{\text{def}}{=} \mathsf{pt} \underset{[A], \mathsf{Pos}, \mathrm{fib}_0}{\times} \mathsf{Pos}_{/\Delta^1} \underset{\mathrm{fib}_1, \mathsf{Pos}, [B]}{\times} \mathsf{pt},$$

as in the diagram

$$\begin{tikzcd}
 & & & \mathsf{Pos}_{/\Delta^1}(A, B) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & & \\
 & & \mathsf{Pos}_{/\Delta^1} \underset{\mathsf{Pos}}{\times} \mathsf{pt} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \mathsf{pt} \underset{\mathsf{Pos}}{\times} \mathsf{Pos}_{/\Delta^1} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & \mathsf{pt} \arrow[dr, "{[A]}"'] & & \mathsf{Pos}_{/\Delta^1} \arrow[dl, "\mathrm{fib}_{[0]}"] \arrow[dr, "\mathrm{fib}_{[1]}"'] & & \mathsf{pt}. \arrow[dl, "{[B]}"] & \\
 & & \mathsf{Pos} & & \mathsf{Pos} & &
\end{tikzcd}$$

Explicitly, an object of $\mathsf{Pos}_{/\Delta^1}(A, B)$ is a pair (X, ϕ_X) consisting of

- A poset X;
- A morphism $\phi_X : X \to \Delta^1$;

to the morphism

$$\mathbf{Coll}(\iota)\colon \mathbf{Coll}(R) \to \mathbf{Coll}(S)$$

of posets over Δ^1 defined by

$$[\mathbf{Coll}(\iota)](x) \stackrel{\text{def}}{=} x$$

for each $x \in \mathbf{Coll}(R)$.[17]

00R6 2. *Equivalence.* The functor of Item 1 is an equivalence of categories.

Proof. Item 1, Functoriality: Clear.
Item 2, Equivalence: Omitted. □

00R7 6.4 Functoriality of Powersets

00R8 6.4.1 Direct Images

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

00R9 **Definition 6.4.1.1.1.** The **direct image function associated to** R is the function

$$R_*\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[18,19]

$$\begin{aligned} R_*(U) &\stackrel{\text{def}}{=} R(U) \\ &\stackrel{\text{def}}{=} \bigcup_{a\in U} R(a) \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{there exists some } a \in U \\ \text{such that } b \in R(a) \end{array} \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

00RA **Remark 6.4.1.1.2.** Identifying subsets of A with relations from pt to A via Item 3 of Proposition 2.4.3.1.6, we see that the direct image function associated to R is equivalently the function

$$R_*\colon \underbrace{\mathcal{P}(A)}_{\cong\mathrm{Rel}(\mathrm{pt},A)} \to \underbrace{\mathcal{P}(B)}_{\cong\mathrm{Rel}(\mathrm{pt},B)}$$

---

such that $\phi_X^{-1}(0) = A$ and $\phi_X^{-1}(0) = B$, with morphisms between such objects being morphisms of posets over Δ^1.

[17]Note that this is indeed a morphism of posets: if $x \preceq_{\mathbf{Coll}(R)} y$, then $x = y$ or $x \sim_R y$, so we have either $x = y$ or $x \sim_S y$ (as $R \subset S$), and thus $x \preceq_{\mathbf{Coll}(S)} y$.

[18]*Further Terminology:* The set $R(U)$ is called the **direct image of** U **by** R.

[19]We also have

$$R_*(U) = B \setminus R_!(A \setminus U);$$

defined by

$$R_*(U) \stackrel{\text{def}}{=} R \diamond U$$

for each $U \in \mathcal{P}(A)$, where $R \diamond U$ is the composition

$$\mathrm{pt} \overset{U}{\nrightarrow} A \overset{R}{\nrightarrow} B.$$

00RB **Proposition 6.4.1.1.3.** Let $R\colon A \nrightarrow B$ be a relation.

00RC 1. *Functoriality.* The assignment $U \mapsto R_*(U)$ defines a functor

$$R_*\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_*](U) \stackrel{\text{def}}{=} R_*(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $R_*(U) \subset R_*(V)$.

00RD 2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1})\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R_*(U) \subset V$.
  - We have $U \subset R_{-1}(V)$.

00RE 3. *Preservation of Colimits.* We have an equality of sets

$$R_*\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} R_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$R_*(U) \cup R_*(V) = R_*(U \cup V),$$
$$R_*(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

00RF 4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R_*\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} R_*(U_i),$$

natural in $\{U_i\}_{i\in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$R_*(U \cap V) \subset R_*(U) \cap R_*(V),$$
$$R_*(A) \subset B,$$

natural in $U, V \in \mathcal{P}(A)$.

00RG 5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$R_{*|U,V}^{\otimes} : R_*(U) \cup R_*(V) \xrightarrow{=} R_*(U \cup V),$$
$$R_{*|\mathbb{1}}^{\otimes} : \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(A)$.

00RH 6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R_*, R_*^{\otimes}, R_{*|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$R_{*|U,V}^{\otimes} : R_*(U \cap V) \subset R_*(U) \cap R_*(V),$$
$$R_{*|\mathbb{1}}^{\otimes} : R_*(A) \subset B,$$

natural in $U, V \in \mathcal{P}(A)$.

00RJ 7. *Relation to Direct Images With Compact Support.* We have

$$R_*(U) = B \setminus R_!(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. Item 1, Functoriality: Clear.

Item 2, Adjointness: This follows from **??** of **??**.
Item 3, Preservation of Colimits: This follows from Item 2 and **??** of **??**.
Item 4, Oplax Preservation of Limits: Omitted.
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Relation to Direct Images With Compact Support: The proof proceeds in the same way as in the case of functions (**??** of Proposition 2.4.4.1.4): applying Item 7 of Proposition 6.4.4.1.3 to $A \setminus U$, we have

$$\begin{aligned} R_!(A \setminus U) &= B \setminus R_*(A \setminus (A \setminus U)) \\ &= B \setminus R_*(U). \end{aligned}$$

Taking complements, we then obtain

$$\begin{aligned} R_*(U) &= B \setminus (B \setminus R_*(U)), \\ &= B \setminus R_!(A \setminus U), \end{aligned}$$

which finishes the proof. □

00RK **Proposition 6.4.1.1.4.** Let $R\colon A \nrightarrow B$ be a relation.

00RL 1. *Functionality I.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*\colon \mathrm{Rel}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

00RM 2. *Functionality II.* The assignment $R \mapsto R_*$ defines a function

$$(-)_*\colon \mathrm{Rel}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

00RN 3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[20]

$$(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)}.$$

00RP 4. *Interaction With Composition.* For each pair of composable

see Item 7 of Proposition 6.4.1.1.3.

[20]That is, the postcomposition function

$$(\chi_A)_*\colon \mathrm{Rel}(\mathrm{pt}, A) \to \mathrm{Rel}(\mathrm{pt}, A)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},A)}$.

relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have[21]

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathcal{P}(B) \arrow[d, "S_*"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned} (\chi_A)_*(U) &\stackrel{\text{def}}{=} \bigcup_{a\in U} \chi_A(a) \\ &\stackrel{\text{def}}{=} \bigcup_{a\in U} \{a\} \\ &= U \\ &\stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{P}(A)}(U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_* = \mathrm{id}_{\mathcal{P}(A)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned} (S \diamond R)_*(U) &\stackrel{\text{def}}{=} \bigcup_{a\in U} [S \diamond R](a) \\ &\stackrel{\text{def}}{=} \bigcup_{a\in U} S(R(a)) \\ &\stackrel{\text{def}}{=} \bigcup_{a\in U} S_*(R(a)) \\ &= S_*\left(\bigcup_{a\in U} R(a)\right) \\ &\stackrel{\text{def}}{=} S_*(R_*(U)) \\ &\stackrel{\text{def}}{=} [S_* \circ R_*](U) \end{aligned}$$

for each $U \in \mathcal{P}(A)$, where we used Item 3 of Proposition 6.4.1.1.3. Thus $(S \diamond R)_* = S_* \circ R_*$. □

[21] That is, we have

$$(S \diamond R)_* = S_* \circ R_*, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt}, A) \arrow[r, "R_*"] \arrow[dr, "(S\diamond R)_*"'] & \mathrm{Rel}(\mathrm{pt}, B) \arrow[d, "S_*"] \\ & \mathrm{Rel}(\mathrm{pt}, C). \end{tikzcd}$$

00RQ 6.4.2 Strong Inverse Images

Let A and B be sets and let $R\colon A \mathrel{\rightarrow\!\!\!\!\!\!\mapsfromchar} B$ be a relation.

00RR **Definition 6.4.2.1.1.** The **strong inverse image function associated to** R is the function

$$R_{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[22]

$$R_{-1}(V) \overset{\text{def}}{=} \{a \in A \mid R(a) \subset V\}$$

for each $V \in \mathcal{P}(B)$.

00RS **Remark 6.4.2.1.2.** Identifying subsets of B with relations from pt to B via Item 3 of Proposition 2.4.3.1.6, we see that the inverse image function associated to R is equivalently the function

$$R_{-1}\colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(\mathrm{pt},B)} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(\mathrm{pt},A)}$$

defined by

$$R_{-1}(V) \overset{\text{def}}{=} \mathrm{Rift}_R(V), \qquad \begin{tikzcd} & A \arrow[d, "R", "\mid"'] \arrow[dl, phantom, "\Swarrow"] \\ \mathrm{pt} \arrow[r, "V"', "\mid"] \arrow[ur, dashed, "\mathrm{Rift}_R(V)"] & B, \end{tikzcd}$$

and being explicitly computed by

$$\begin{aligned} R_{-1}(V) &\overset{\text{def}}{=} \mathrm{Rift}_R(V) \\ &\cong \int_{b \in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-1}, V^b_{-2}\right), \end{aligned}$$

where we have used Proposition 6.2.4.1.1.

[22] *Further Terminology:* The set $R_{-1}(V)$ is called the **strong inverse image of** V **by** R.

Proof. We have

$$
\begin{aligned}
\mathrm{Rift}_R(V) &\cong \int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_{-1}, V^b_{-2}\right)\\
&= \left\{ a\in A \;\middle|\; \int_{b\in B} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}\left(R^b_a, V^b_\star\right) = \mathsf{true} \right\}\\
&= \left\{ a\in A \;\middle|\; \begin{array}{l} \text{for each } b\in B\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } R^b_a = \mathsf{false}\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad \text{(a) We have } R^b_a = \mathsf{true}\\ \qquad \text{(b) We have } V^b_\star = \mathsf{true} \end{array} \right\}\\
&= \left\{ a\in A \;\middle|\; \begin{array}{l} \text{for each } b\in B\text{, at least one of the}\\ \text{following conditions hold:}\\ \quad 1.\ \text{We have } b\notin R(a)\\ \quad 2.\ \text{The following conditions hold:}\\ \qquad \text{(a) We have } b\in R(a)\\ \qquad \text{(b) We have } b\in V \end{array} \right\}\\
&= \{a\in A \mid \text{for each } b\in R(a)\text{, we have } b\in V\}\\
&= \{a\in A \mid R(a)\subset V\}\\
&\overset{\text{def}}{=} R_{-1}(V).
\end{aligned}
$$

This finishes the proof. □

00RT **Proposition 6.4.2.1.3.** Let $R\colon A \nrightarrow B$ be a relation.

00RU 1. *Functoriality.* The assignment $V \mapsto R_{-1}(V)$ defines a functor

$$R_{-1}\colon (\mathcal{P}(B),\subset) \to (\mathcal{P}(A),\subset)$$

where

- *Action on Objects.* For each $V\in\mathcal{P}(B)$, we have

$$[R_{-1}](V) \overset{\text{def}}{=} R_{-1}(V).$$

- *Action on Morphisms.* For each $U,V\in\mathcal{P}(B)$:
  - If $U\subset V$, then $R_{-1}(U)\subset R_{-1}(V)$.

00RV 2. *Adjointness.* We have an adjunction

$$(R_* \dashv R_{-1})\colon \quad \begin{tikzcd} \mathcal{P}(A) \arrow[r, bend left, "R_*"] \arrow[r, phantom, "\perp"] & \mathcal{P}(B), \arrow[l, bend left, "R_{-1}"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}(R_*(U), V) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_{-1}(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R_*(U) \subset V$.
  - We have $U \subset R_{-1}(V)$.

00RW 3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_{-1}(U_i) \subset R_{-1}\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V),$$
$$\emptyset \subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

00RX 4. *Preservation of Limits.* We have an equality of sets

$$R_{-1}\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$R_{-1}(U \cap V) = R_{-1}(U) \cap R_{-1}(V),$$
$$R_{-1}(B) = B,$$

natural in $U, V \in \mathcal{P}(B)$.

00RY 5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(R_{-1}, R_{-1}^{\otimes}, R_{-1|\mathbb{1}}^{\otimes}\right)\colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R^{\otimes}_{-1|U,V} \colon R_{-1}(U) \cup R_{-1}(V) \subset R_{-1}(U \cup V),$$
$$R^{\otimes}_{-1|\mathbb{1}} \colon \emptyset \subset R_{-1}(\emptyset),$$

natural in $U, V \in \mathcal{P}(B)$.

00RZ 6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_{-1}, R^{\otimes}_{-1}, R^{\otimes}_{-1|\mathbb{1}}\right) \colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R^{\otimes}_{-1|U,V} \colon R_{-1}(U \cap V) \xrightarrow{=} R_{-1}(U) \cap R_{-1}(V),$$
$$R^{\otimes}_{-1|\mathbb{1}} \colon R_{-1}(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(B)$.

00S0 7. *Interaction With Weak Inverse Images I.* We have

$$R_{-1}(V) = A \setminus R^{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

00S1 8. *Interaction With Weak Inverse Images II.* Let $R \colon A \nrightarrow B$ be a relation from A to B.

00S2 (a) If R is a total relation, then we have an inclusion of sets

$$R_{-1}(V) \subset R^{-1}(V)$$

natural in $V \in \mathcal{P}(B)$.

00S3 (b) If R is total and functional, then the above inclusion is in fact an equality.

00S4 (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This follows from **??** of **??**.
Item 3, Lax Preservation of Colimits: Omitted.
Item 4, Preservation of Limits: This follows from Item 2 and **??** of **??**.
Item 5, Symmetric Lax Monoidality With Respect to Unions: This follows from Item 3.

Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.

Item 7, Interaction With Weak Inverse Images I: We claim we have an equality

$$R_{-1}(B \setminus V) = A \setminus R^{-1}(V).$$

Indeed, we have

$$\begin{aligned} R_{-1}(B \setminus V) &= \{a \in A \mid R(a) \subset B \setminus V\}, \\ A \setminus R^{-1}(V) &= \{a \in A \mid R(a) \cap V = \emptyset\}. \end{aligned}$$

Taking $V = B \setminus V$ then implies the original statement.

Item 8, Interaction With Weak Inverse Images II: Item 8a is clear, while Items 8b and 8c follow from Item 6 of Proposition 6.3.1.1.2. □

00S5 **Proposition 6.4.2.1.4.** Let $R \colon A \nrightarrow B$ be a relation.

00S6 1. *Functionality I.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1} \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

00S7 2. *Functionality II.* The assignment $R \mapsto R_{-1}$ defines a function

$$(-)_{-1} \colon \mathsf{Sets}(A, B) \to \mathsf{Pos}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

00S8 3. *Interaction With Identities.* For each $A \in \text{Obj}(\mathsf{Sets})$, we have

$$(\text{id}_A)_{-1} = \text{id}_{\mathcal{P}(A)}.$$

00S9 4. *Interaction With Composition.* For each pair of composable relations $R \colon A \nrightarrow B$ and $S \colon B \nrightarrow C$, we have

$$(S \diamond R)_{-1} = R_{-1} \circ S_{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S_{-1}"] \arrow[dr, "(S\diamond R)_{-1}"'] & \mathcal{P}(B) \arrow[d, "R_{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

Proof. Item 1, Functionality I: Clear.

Item 2, Functionality II: Clear.

Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned} (\chi_A)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid \chi_A(a) \subset U\} \\ &\stackrel{\text{def}}{=} \{a \in A \mid \{a\} \subset U\} \\ &= U \end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_{-1} = \mathrm{id}_{\mathcal{P}(A)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}(S \diamond R)_{-1}(U) &\stackrel{\text{def}}{=} \{a \in A \mid [S \diamond R](a) \subset U\}\\ &\stackrel{\text{def}}{=} \{a \in A \mid S(R(a)) \subset U\}\\ &\stackrel{\text{def}}{=} \{a \in A \mid S_*(R(a)) \subset U\}\\ &= \{a \in A \mid R(a) \subset S_{-1}(U)\}\\ &\stackrel{\text{def}}{=} R_{-1}(S_{-1}(U))\\ &\stackrel{\text{def}}{=} [R_{-1} \circ S_{-1}](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 6.4.2.1.3, which implies that the conditions

- We have $S_*(R(a)) \subset U$.
- We have $R(a) \subset S_{-1}(U)$.

are equivalent. Thus $(S \diamond R)_{-1} = R_{-1} \circ S_{-1}$. □

00SA 6.4.3 Weak Inverse Images

Let A and B be sets and let $R \colon A \nrightarrow B$ be a relation.

00SB **Definition 6.4.3.1.1.** The **weak inverse image function associated to** R[23] is the function

$$R^{-1} \colon \mathcal{P}(B) \to \mathcal{P}(A)$$

defined by[24]

$$R^{-1}(V) \stackrel{\text{def}}{=} \{a \in A \mid R(a) \cap V \neq \emptyset\}$$

for each $V \in \mathcal{P}(B)$.

00SC **Remark 6.4.3.1.2.** Identifying subsets of B with relations from B to pt via Item 3 of Proposition 2.4.3.1.6, we see that the weak inverse image function associated to R is equivalently the function

$$R^{-1} \colon \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})} \to \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})}$$

defined by

$$R^{-1}(V) \stackrel{\text{def}}{=} V \diamond R$$

[23] *Further Terminology:* Also called simply the **inverse image function associated to** R.

[24] *Further Terminology:* The set $R^{-1}(V)$ is called the **weak inverse image of** V **by** R or simply the **inverse image of** V **by** R.

for each $V \in \mathcal{P}(A)$, where $R \diamond V$ is the composition

$$A \overset{R}{\nrightarrow} B \overset{V}{\nrightarrow} \mathrm{pt}.$$

Explicitly, we have

$$\begin{aligned} R^{-1}(V) &\overset{\text{def}}{=} V \diamond R \\ &\overset{\text{def}}{=} \int^{b \in B} V_b^{-1} \times R^b_{-_2}. \end{aligned}$$

Proof. We have

$$\begin{aligned} V \diamond R &\overset{\text{def}}{=} \int^{b \in B} V_b^{-1} \times R^b_{-_2} \\ &= \left\{ a \in A \;\middle|\; \int^{b \in B} V_b^{\star} \times R^b_a = \mathsf{true} \right\} \\ &= \left\{ a \in A \;\middle|\; \begin{array}{l} \text{there exists } b \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1. \text{ We have } V_b^{\star} = \mathsf{true} \\ \quad 2. \text{ We have } R^b_a = \mathsf{true} \end{array} \right\} \\ &= \left\{ a \in A \;\middle|\; \begin{array}{l} \text{there exists } b \in B \text{ such that the} \\ \text{following conditions hold:} \\ \quad 1. \text{ We have } b \in V \\ \quad 2. \text{ We have } b \in R(a) \end{array} \right\} \\ &= \{a \in A \mid \text{there exists } b \in V \text{ such that } b \in R(a)\} \\ &= \{a \in A \mid R(a) \cap V \neq \emptyset\} \\ &\overset{\text{def}}{=} R^{-1}(V) \end{aligned}$$

This finishes the proof. □

00SD **Proposition 6.4.3.1.3.** Let $R \colon A \nrightarrow B$ be a relation.

00SE 1. *Functoriality.* The assignment $V \mapsto R^{-1}(V)$ defines a functor

$$R^{-1} \colon (\mathcal{P}(B), \subset) \to (\mathcal{P}(A), \subset)$$

where

- *Action on Objects.* For each $V \in \mathcal{P}(B)$, we have

$$\left[R^{-1}\right](V) \overset{\text{def}}{=} R^{-1}(V).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(B)$:
  - If $U \subset V$, then $R^{-1}(U) \subset R^{-1}(V)$.

00SF 2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_!\right) \colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A) \arrow[l, bend left, "R_!"] \end{tikzcd},$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R^{-1}(U) \subset V$.
  - We have $U \subset R_!(V)$.

00SG 3. *Preservation of Colimits.* We have an equality of sets

$$R^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have equalities

$$R^{-1}(U) \cup R^{-1}(V) = R^{-1}(U \cup V),$$
$$R^{-1}(\emptyset) = \emptyset,$$

natural in $U, V \in \mathcal{P}(B)$.

00SH 4. *Oplax Preservation of Limits.* We have an inclusion of sets

$$R^{-1}\left(\bigcap_{i \in I} U_i\right) \subset \bigcap_{i \in I} R^{-1}(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(B)^{\times I}$. In particular, we have inclusions

$$R^{-1}(U \cap V) \subset R^{-1}(U) \cap R^{-1}(V),$$
$$R^{-1}(A) \subset B,$$

natural in $U, V \in \mathcal{P}(B)$.

00SJ 5. *Symmetric Strict Monoidality With Respect to Unions.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right)\colon (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with equalities

$$R^{-1,\otimes}_{U,V}\colon R^{-1}(U) \cup R^{-1}(V) \xrightarrow{=} R^{-1}(U \cup V),$$
$$R^{-1,\otimes}_{\mathbb{1}}\colon \emptyset \xrightarrow{=} \emptyset,$$

natural in $U, V \in \mathcal{P}(B)$.

00SK 6. *Symmetric Oplax Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric oplax monoidal structure

$$\left(R^{-1}, R^{-1,\otimes}, R^{-1,\otimes}_{\mathbb{1}}\right)\colon (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with inclusions

$$R^{-1,\otimes}_{U,V}\colon R^{-1}(U \cap V) \subset R^{-1}(U) \cap R^{-1}(V),$$
$$R^{-1,\otimes}_{\mathbb{1}}\colon R^{-1}(A) \subset B,$$

natural in $U, V \in \mathcal{P}(B)$.

00SL 7. *Interaction With Strong Inverse Images I.* We have

$$R^{-1}(V) = A \setminus R_{-1}(B \setminus V)$$

for each $V \in \mathcal{P}(B)$.

00SM 8. *Interaction With Strong Inverse Images II.* Let $R\colon A \nrightarrow B$ be a relation from A to B.

00SN (a) If R is a total relation, then we have an inclusion of sets

$$R_{-1}(V) \subset R^{-1}(V)$$

natural in $V \in \mathcal{P}(B)$.

00SP (b) If R is total and functional, then the above inclusion is in fact an equality.

00SQ (c) Conversely, if we have $R_{-1} = R^{-1}$, then R is total and functional.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This follows from **??** of **??**.

Item 3, Preservation of Colimits: This follows from Item 2 and **??** of **??**.
Item 4, Oplax Preservation of Limits: Omitted.
Item 5, Symmetric Strict Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Oplax Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Interaction With Strong Inverse Images I: This follows from Item 7 of Proposition 6.4.2.1.3.
Item 8, Interaction With Strong Inverse Images II: This was proved in Item 8 of Proposition 6.4.2.1.3. □

00SR **Proposition 6.4.3.1.4.** Let $R\colon A \nrightarrow B$ be a relation.

00SS 1. *Functionality I.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(A,B) \to \mathsf{Sets}(\mathcal{P}(A),\mathcal{P}(B)).$$

00ST 2. *Functionality II.* The assignment $R \mapsto R^{-1}$ defines a function

$$(-)^{-1}\colon \mathrm{Rel}(A,B) \to \mathsf{Pos}((\mathcal{P}(A),\subset),(\mathcal{P}(B),\subset)).$$

00SU 3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have[25]

$$(\chi_A)^{-1} = \mathrm{id}_{\mathcal{P}(A)}.$$

00SV 4. *Interaction With Composition.* For each pair of composable relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, we have[26]

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathcal{P}(C) \arrow[r, "S^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathcal{P}(B) \arrow[d, "R^{-1}"] \\ & \mathcal{P}(A). \end{tikzcd}$$

[25] That is, the postcomposition

$$(\chi_A)^{-1}\colon \mathrm{Rel}(\mathrm{pt},A) \to \mathrm{Rel}(\mathrm{pt},A)$$

is equal to $\mathrm{id}_{\mathrm{Rel}(\mathrm{pt},A)}$.

[26] That is, we have

$$(S \diamond R)^{-1} = R^{-1} \circ S^{-1}, \qquad \begin{tikzcd} \mathrm{Rel}(\mathrm{pt},C) \arrow[r, "R^{-1}"] \arrow[dr, "(S\diamond R)^{-1}"'] & \mathrm{Rel}(\mathrm{pt},B) \arrow[d, "S^{-1}"] \\ & \mathrm{Rel}(\mathrm{pt},A). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: This follows from Item 5 of Proposition 8.1.6.1.2.
Item 4, Interaction With Composition: This follows from Item 2 of Proposition 8.1.6.1.2. □

00SW 6.4.4 Direct Images With Compact Support

Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

00SX **Definition 6.4.4.1.1.** The **direct image with compact support function associated to** R is the function

$$R_!\colon \mathcal{P}(A) \to \mathcal{P}(B)$$

defined by[27,28]

$$\begin{aligned} R_!(U) &\overset{\text{def}}{=} \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A, \text{ if we have} \\ b \in R(a), \text{ then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; R^{-1}(b) \subset U \right\} \end{aligned}$$

for each $U \in \mathcal{P}(A)$.

00SY **Remark 6.4.4.1.2.** Identifying subsets of B with relations from pt to B via Item 3 of Proposition 2.4.3.1.6, we see that the direct image with compact support function associated to R is equivalently the function

$$R_!\colon \underbrace{\mathcal{P}(A)}_{\cong \mathrm{Rel}(A,\mathrm{pt})} \to \underbrace{\mathcal{P}(B)}_{\cong \mathrm{Rel}(B,\mathrm{pt})}$$

defined by

$$R_!(U) \overset{\text{def}}{=} \mathrm{Ran}_R(U), \qquad \begin{tikzcd} & B \arrow[d, dashed, "\mathrm{Ran}_R(U)"] \\ A \arrow[ur, "R"] \arrow[r, "U"'] \arrow[ur, phantom, shift right=3, "\Leftarrow"] & \mathrm{pt}, \end{tikzcd}$$

[27] *Further Terminology:* The set $R_!(U)$ is called the **direct image with compact support of U by R.**

[28] We also have

$$R_!(U) = B \setminus R_*(A \setminus U);$$

see Item 7 of Proposition 6.4.4.1.3.

being explicitly computed by

$$\begin{aligned} R^*(U) &\overset{\text{def}}{=} \mathrm{Ran}_R(U) \\ &\cong \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^{-2}, U_a^{-1}), \end{aligned}$$

where we have used Proposition 6.2.3.1.1.

Proof. We have

$$\begin{aligned} \mathrm{Ran}_R(V) &\cong \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^{-2}, U_a^{-1}) \\ &= \left\{ b \in B \;\middle|\; \int_{a\in A} \mathrm{Hom}_{\{\mathsf{t},\mathsf{f}\}}(R_a^{b}, U_a^{\star}) = \mathsf{true} \right\} \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } R_a^b = \mathsf{false} \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } R_a^b = \mathsf{true} \\ \qquad \text{(b) We have } U_a^\star = \mathsf{true} \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A\text{, at least one of the} \\ \text{following conditions hold:} \\ \quad 1.\ \text{We have } b \notin R(A) \\ \quad 2.\ \text{The following conditions hold:} \\ \qquad \text{(a) We have } b \in R(a) \\ \qquad \text{(b) We have } a \in U \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; \begin{array}{l} \text{for each } a \in A\text{, if we have} \\ b \in R(a)\text{, then } a \in U \end{array} \right\} \\ &= \left\{ b \in B \;\middle|\; R^{-1}(b) \subset U \right\} \\ &\overset{\text{def}}{=} R^{-1}(U). \end{aligned}$$

This finishes the proof. □

00SZ **Proposition 6.4.4.1.3.** Let $R\colon A \nrightarrow B$ be a relation.

00T0 1. *Functoriality.* The assignment $U \mapsto R_!(U)$ defines a functor

$$R_!\colon (\mathcal{P}(A), \subset) \to (\mathcal{P}(B), \subset)$$

where

- *Action on Objects.* For each $U \in \mathcal{P}(A)$, we have

$$[R_!](U) \overset{\text{def}}{=} R_!(U).$$

- *Action on Morphisms.* For each $U, V \in \mathcal{P}(A)$:
  - If $U \subset V$, then $R_!(U) \subset R_!(V)$.

00T1 2. *Adjointness.* We have an adjunction

$$\left(R^{-1} \dashv R_!\right)\colon \quad \begin{tikzcd} \mathcal{P}(B) \arrow[r, bend left, "R^{-1}"] \arrow[r, phantom, "\perp"] & \mathcal{P}(A), \arrow[l, bend left, "R_!"] \end{tikzcd}$$

witnessed by a bijections of sets

$$\mathrm{Hom}_{\mathcal{P}(A)}\left(R^{-1}(U), V\right) \cong \mathrm{Hom}_{\mathcal{P}(A)}(U, R_!(V)),$$

natural in $U \in \mathcal{P}(A)$ and $V \in \mathcal{P}(B)$, i.e. such that:

($\star$) The following conditions are equivalent:
  - We have $R^{-1}(U) \subset V$.
  - We have $U \subset R_!(V)$.

00T2 3. *Lax Preservation of Colimits.* We have an inclusion of sets

$$\bigcup_{i \in I} R_!(U_i) \subset R_!\left(\bigcup_{i \in I} U_i\right),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have inclusions

$$R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$\emptyset \subset R_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

00T3 4. *Preservation of Limits.* We have an equality of sets

$$R_!\left(\bigcap_{i \in I} U_i\right) = \bigcap_{i \in I} R_!(U_i),$$

natural in $\{U_i\}_{i \in I} \in \mathcal{P}(A)^{\times I}$. In particular, we have equalities

$$R_!(U \cap V) = R_!(U) \cap R_!(V),$$
$$R_!(A) = B,$$

natural in $U, V \in \mathcal{P}(A)$.

00T4 5. *Symmetric Lax Monoidality With Respect to Unions.* The direct image with compact support function of Item 1 has a symmetric lax monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cup, \emptyset) \to (\mathcal{P}(B), \cup, \emptyset),$$

being equipped with inclusions

$$R_{!|U,V}^{\otimes} : R_!(U) \cup R_!(V) \subset R_!(U \cup V),$$
$$R_{!|\mathbb{1}}^{\otimes} : \emptyset \subset R_!(\emptyset),$$

natural in $U, V \in \mathcal{P}(A)$.

00T5 6. *Symmetric Strict Monoidality With Respect to Intersections.* The direct image function of Item 1 has a symmetric strict monoidal structure

$$\left(R_!, R_!^{\otimes}, R_{!|\mathbb{1}}^{\otimes}\right) : (\mathcal{P}(A), \cap, A) \to (\mathcal{P}(B), \cap, B),$$

being equipped with equalities

$$R_{!|U,V}^{\otimes} : R_!(U \cap V) \xrightarrow{=} R_!(U) \cap R_!(V),$$
$$R_{!|\mathbb{1}}^{\otimes} : R_!(A) \xrightarrow{=} B,$$

natural in $U, V \in \mathcal{P}(A)$.

00T6 7. *Relation to Direct Images.* We have

$$R_!(U) = B \setminus R_*(A \setminus U)$$

for each $U \in \mathcal{P}(A)$.

Proof. Item 1, Functoriality: Clear.
Item 2, Adjointness: This follows from **??** of **??**.
Item 3, Lax Preservation of Colimits: Omitted.
Item 4, Preservation of Limits: This follows from Item 2 and **??** of **??**.
Item 5, Symmetric Lax Monoidality With Respect to Unions: This follows from Item 3.
Item 6, Symmetric Strict Monoidality With Respect to Intersections: This follows from Item 4.
Item 7, Relation to Direct Images: This follows from Item 7 of Proposition 6.4.1.1.3. Alternatively, we may prove it directly as follows, with the proof proceeding in the same way as in the case of functions (Item 9 of Proposition 2.4.6.1.6).
We claim that $R_!(U) = B \setminus R_*(A \setminus U)$:

- *The First Implication.* We claim that

$$R_!(U) \subset B \setminus R_*(A \setminus U).$$

Let $b \in R_!(U)$. We need to show that $b \notin R_*(A \setminus U)$, i.e. that there is no $a \in A \setminus U$ such that $b \in R(a)$.

This is indeed the case, as otherwise we would have $a \in R^{-1}(b)$ and $a \notin U$, contradicting $R^{-1}(b) \subset U$ (which holds since $b \in R_!(U)$).

Thus $b \in B \setminus R_*(A \setminus U)$.

- *The Second Implication.* We claim that

$$B \setminus R_*(A \setminus U) \subset R_!(U).$$

Let $b \in B \setminus R_*(A \setminus U)$. We need to show that $b \in R_!(U)$, i.e. that $R^{-1}(b) \subset U$.

Since $b \notin R_*(A \setminus U)$, there exists no $a \in A \setminus U$ such that $b \in R(a)$, and hence $R^{-1}(b) \subset U$.

Thus $b \in R_!(U)$.

This finishes the proof. □

00T7 **Proposition 6.4.4.1.4.** Let $R \colon A \mathrel{\rightarrow\mkern-11mu\mapstochar\mkern5mu} B$ be a relation.

00T8 1. *Functionality I.* The assignment $R \mapsto R_!$ defines a function

$$(-)_! \colon \mathsf{Sets}(A, B) \to \mathsf{Sets}(\mathcal{P}(A), \mathcal{P}(B)).$$

00T9 2. *Functionality II.* The assignment $R \mapsto R_!$ defines a function

$$(-)_! \colon \mathsf{Sets}(A, B) \to \mathrm{Hom}_{\mathsf{Pos}}((\mathcal{P}(A), \subset), (\mathcal{P}(B), \subset)).$$

00TA 3. *Interaction With Identities.* For each $A \in \mathrm{Obj}(\mathsf{Sets})$, we have

$$(\mathrm{id}_A)_! = \mathrm{id}_{\mathcal{P}(A)}.$$

00TB 4. *Interaction With Composition.* For each pair of composable relations $R \colon A \mathrel{\rightarrow\mkern-11mu\mapstochar\mkern5mu} B$ and $S \colon B \mathrel{\rightarrow\mkern-11mu\mapstochar\mkern5mu} C$, we have

$$(S \diamond R)_! = S_! \circ R_!, \qquad \begin{tikzcd} \mathcal{P}(A) \arrow[r, "R_!"] \arrow[dr, "(S \diamond R)_!"'] & \mathcal{P}(B) \arrow[d, "S_!"] \\ & \mathcal{P}(C). \end{tikzcd}$$

Proof. *Item 1, Functionality I*: Clear.
Item 2, Functionality II: Clear.
Item 3, Interaction With Identities: Indeed, we have

$$\begin{aligned}(\chi_A)_!(U) &\overset{\text{def}}{=} \left\{a \in A \;\middle|\; \chi_A^{-1}(a) \subset U\right\}\\ &\overset{\text{def}}{=} \{a \in A \mid \{a\} \subset U\}\\ &= U\end{aligned}$$

for each $U \in \mathcal{P}(A)$. Thus $(\chi_A)_! = \mathrm{id}_{\mathcal{P}(A)}$.
Item 4, Interaction With Composition: Indeed, we have

$$\begin{aligned}(S \diamond R)_!(U) &\overset{\text{def}}{=} \left\{c \in C \;\middle|\; [S \diamond R]^{-1}(c) \subset U\right\}\\ &\overset{\text{def}}{=} \left\{c \in C \;\middle|\; S^{-1}\Big(R^{-1}(c)\Big) \subset U\right\}\\ &= \left\{c \in C \;\middle|\; R^{-1}(c) \subset S_!(U)\right\}\\ &\overset{\text{def}}{=} R_!(S_!(U))\\ &\overset{\text{def}}{=} [R_! \circ S_!](U)\end{aligned}$$

for each $U \in \mathcal{P}(C)$, where we used Item 2 of Proposition 6.4.4.1.3, which implies that the conditions

- We have $S^{-1}(R^{-1}(c)) \subset U$.
- We have $R^{-1}(c) \subset S_!(U)$.

are equivalent. Thus $(S \diamond R)_! = S_! \circ R_!$. □

00TC 6.4.5 Functoriality of Powersets

00TD **Proposition 6.4.5.1.1.** The assignment $X \mapsto \mathcal{P}(X)$ defines functors[29]

$$\begin{aligned}\mathcal{P}_* &\colon \mathrm{Rel} \to \mathsf{Sets},\\ \mathcal{P}_{-1} &\colon \mathrm{Rel}^{\mathsf{op}} \to \mathsf{Sets},\\ \mathcal{P}^{-1} &\colon \mathrm{Rel}^{\mathsf{op}} \to \mathsf{Sets},\\ \mathcal{P}_! &\colon \mathrm{Rel} \to \mathsf{Sets}\end{aligned}$$

where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathrm{Rel})$, we have

$$\begin{aligned}\mathcal{P}_*(A) &\overset{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_{-1}(A) &\overset{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}^{-1}(A) &\overset{\text{def}}{=} \mathcal{P}(A),\\ \mathcal{P}_!(A) &\overset{\text{def}}{=} \mathcal{P}(A).\end{aligned}$$

[29]The functor $\mathcal{P}_* \colon \mathrm{Rel} \to \mathsf{Sets}$ admits a left adjoint; see Item 3 of

- *Action on Morphisms.* For each morphism $R\colon A \nrightarrow B$ of Rel, the images

$$\begin{aligned}\mathcal{P}_*(R)&\colon \mathcal{P}(A)\to\mathcal{P}(B),\\ \mathcal{P}_{-1}(R)&\colon \mathcal{P}(B)\to\mathcal{P}(A),\\ \mathcal{P}^{-1}(R)&\colon \mathcal{P}(B)\to\mathcal{P}(A),\\ \mathcal{P}_!(R)&\colon \mathcal{P}(A)\to\mathcal{P}(B)\end{aligned}$$

of R by $\mathcal{P}_*$, $\mathcal{P}_{-1}$, $\mathcal{P}^{-1}$, and $\mathcal{P}_!$ are defined by

$$\begin{aligned}\mathcal{P}_*(R)&\overset{\text{def}}{=} R_*,\\ \mathcal{P}_{-1}(R)&\overset{\text{def}}{=} R_{-1},\\ \mathcal{P}^{-1}(R)&\overset{\text{def}}{=} R^{-1},\\ \mathcal{P}_!(R)&\overset{\text{def}}{=} R_!,\end{aligned}$$

as in Definitions 6.4.1.1.1, 6.4.2.1.1, 6.4.3.1.1 and 6.4.4.1.1.

Proof. This follows from Items 3 and 4 of Proposition 6.4.1.1.4, Items 3 and 4 of Proposition 6.4.2.1.4, Items 3 and 4 of Proposition 6.4.3.1.4, and Items 3 and 4 of Proposition 6.4.4.1.4. □

6.4.6 Functoriality of Powersets: Relations on Powersets

00TE Let A and B be sets and let $R\colon A \nrightarrow B$ be a relation.

00TF **Definition 6.4.6.1.1.** The **relation on powersets associated to** R is the relation

$$\mathcal{P}(R)\colon \mathcal{P}(A) \nrightarrow \mathcal{P}(B)$$

defined by[30]

$$\mathcal{P}(R)^V_U \overset{\text{def}}{=} \mathbf{Rel}(\chi_{\mathrm{pt}}, V\diamond R\diamond U)$$

for each $U\in\mathcal{P}(A)$ and each $V\in\mathcal{P}(B)$.

00TG **Remark 6.4.6.1.2.** In detail, we have $U \sim_{\mathcal{P}(R)} V$ iff the following equivalent conditions hold:

- We have $\chi_{\mathrm{pt}} \subset V\diamond R\diamond U$.
- We have $(V\diamond R\diamond U)^\star_\star = \mathsf{true}$, i.e. we have

$$\int^{a\in A}\int^{b\in B} V^\star_b \times R^b_a \times U^a_\star = \mathsf{true}.$$

Proposition 6.3.1.1.2.

[30]Illustration:

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "U"', "\shortmid" marking] \arrow[rrr, bend left, "\chi_{\mathrm{pt}}", "\shortmid" marking] & A \arrow[r, "R"', "\shortmid" marking] & B \arrow[r, "V"', "\shortmid" marking] & \mathrm{pt}. \end{tikzcd}$$

- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $U^a_\star = \mathsf{true}$.
  - We have $R^b_a = \mathsf{true}$.
  - We have $V^\star_b = \mathsf{true}$.
- There exists some $a \in A$ and some $b \in B$ such that:
  - We have $a \in U$.
  - We have $a \sim_R b$.
  - We have $b \in V$.

00TH **Proposition 6.4.6.1.3.** The assignment $R \mapsto \mathcal{P}(R)$ defines a functor

$$\mathcal{P} \colon \mathrm{Rel} \to \mathrm{Rel}.$$

Proof. Omitted. □

Appendices

6.A Other Chapters

Chapter 7

Equivalence Relations and Apartness Relations

00TJ This chapter contains some material about reflexive, symmetric, transitive, equivalence, and apartness relations.

Contents

00TK 7.1 Reflexive Relations

00TL 7.1.1 Foundations

Let A be a set.

00TM **Definition 7.1.1.1.1.** A **reflexive relation** is equivalently:[1]

- An $\mathbb{E}_0$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A,A)),\chi_A)$.
- A pointed object in $(\mathbf{Rel}(A,A),\chi_A)$.

00TN **Remark 7.1.1.1.2.** In detail, a relation R on A is **reflexive** if we have an inclusion

$$\eta_R\colon \chi_A \subset R$$

of relations in $\mathbf{Rel}(A,A)$, i.e. if, for each $a\in A$, we have $a\sim_R a$.

00TP **Definition 7.1.1.1.3.** Let A be a set.

1. The **set of reflexive relations on** A is the subset $\mathrm{Rel}^{\mathrm{refl}}(A,A)$
00TQ of $\mathrm{Rel}(A,A)$ spanned by the reflexive relations.

2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathsf{refl}}(A,A)$ of
00TR $\mathbf{Rel}(A,A)$ spanned by the reflexive relations.

00TS **Proposition 7.1.1.1.4.** Let R and S be relations on A.

00TT 1. *Interaction With Inverses.* If R is reflexive, then so is $R^\dagger$.

00TU 2. *Interaction With Composition.* If R and S are reflexive, then so is $S\diamond R$.

Proof. *Item 1*, *Interaction With Inverses*: Clear.
Item 2, *Interaction With Composition*: Clear. □

00TV 7.1.2 The Reflexive Closure of a Relation

Let R be a relation on A.

00TW **Definition 7.1.2.1.1.** The **reflexive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{refl}}$[2] satisfying the following universal property:[3]

(⋆) Given another reflexive relation $\sim_S$ on A such that $R\subset S$, there exists an inclusion $\sim_R^{\mathrm{refl}}\subset\sim_S$.

00TX **Construction 7.1.2.1.2.** Concretely, $\sim_R^{\mathrm{refl}}$ is the free pointed object on R in $(\mathbf{Rel}(A,A),\chi_A)$[4], being given by

$$\begin{aligned} R^{\mathrm{refl}} &\overset{\mathrm{def}}{=} R \textstyle\coprod^{\mathbf{Rel}(A,A)} \Delta_A\\ &= R\cup\Delta_A\\ &= \{(a,b)\in A\times A \mid \text{we have } a\sim_R b \text{ or } a=b\}. \end{aligned}$$

[1] Note that since $\mathbf{Rel}(A,A)$ is posetal, reflexivity is a property of a relation, rather than extra structure.
[2] *Further Notation:* Also written R^{refl}.
[3] *Slogan:* The reflexive closure of R is the smallest reflexive relation containing R.
[4] Or, equivalently, the free $\mathbb{E}_0$-monoid on R in $(\mathrm{N}_\bullet(\mathbf{Rel}(A,A)),\chi_A)$.

Proof. Clear. □

00TY **Proposition 7.1.2.1.3.** Let R be a relation on A.

00TZ 1. *Adjointness.* We have an adjunction

$$\Big((-)^{\mathrm{refl}} \dashv 忘\Big)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{refl}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathsf{refl}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathsf{refl}}\Big(R^{\mathrm{refl}}, S\Big) \cong \mathbf{Rel}(R,S),$$

natural in $R \in \mathrm{Obj}\Big(\mathbf{Rel}^{\mathsf{refl}}(A,A)\Big)$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

00U0 2. *The Reflexive Closure of a Reflexive Relation.* If R is reflexive, then $R^{\mathrm{refl}} = R$.

00U1 3. *Idempotency.* We have

$$\Big(R^{\mathrm{refl}}\Big)^{\mathrm{refl}} = R^{\mathrm{refl}}.$$

00U2 4. *Interaction With Inverses.* We have

$$\Big(R^{\dagger}\Big)^{\mathrm{refl}} = \Big(R^{\mathrm{refl}}\Big)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

00U3 5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{refl}} = S^{\mathrm{refl}} \diamond R^{\mathrm{refl}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{refl}} \times (-)^{\mathrm{refl}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{refl}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the reflexive closure of a relation, stated in Definition 7.1.2.1.1.
Item 2, The Reflexive Closure of a Reflexive Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: Clear.
Item 5, Interaction With Composition: This follows from Item 2 of Proposition 7.1.1.1.4. □

00U4 7.2 Symmetric Relations

00U5 7.2.1 Foundations

Let A be a set.

00U6 **Definition 7.2.1.1.1.** A relation R on A is **symmetric** if we have $R^{\dagger} = R$.

00U7 **Remark 7.2.1.1.2.** In detail, a relation R is symmetric if it satisfies the following condition:

($\star$) For each $a, b \in A$, if $a \sim_R b$, then $b \sim_R a$.

00U8 **Definition 7.2.1.1.3.** Let A be a set.

1. The **set of symmetric relations on** A is the subset $\mathrm{Rel}^{\mathrm{symm}}(A, A)$
00U9 of $\mathrm{Rel}(A, A)$ spanned by the symmetric relations.

2. The **poset of relations on** A is is the subposet $\mathbf{Rel}^{\mathsf{symm}}(A, A)$ of
00UA $\mathbf{Rel}(A, A)$ spanned by the symmetric relations.

00UB **Proposition 7.2.1.1.4.** Let R and S be relations on A.

00UC 1. *Interaction With Inverses.* If R is symmetric, then so is $R^{\dagger}$.

00UD 2. *Interaction With Composition.* If R and S are symmetric, then so is $S \diamond R$.

Proof. *Item 1, Interaction With Inverses*: Clear.
Item 2, Interaction With Composition: Clear. □

00UE 7.2.2 The Symmetric Closure of a Relation

Let R be a relation on A.

00UF **Definition 7.2.2.1.1.** The **symmetric closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{symm}}$[5] satisfying the following universal property:[6]

($\star$) Given another symmetric relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{symm}} \subset \sim_S$.

00UG **Construction 7.2.2.1.2.** Concretely, $\sim_R^{\mathrm{symm}}$ is the symmetric relation on A defined by

$$\begin{aligned} R^{\mathrm{symm}} &\overset{\mathrm{def}}{=} R \cup R^{\dagger} \\ &= \{(a, b) \in A \times A \mid \text{we have } a \sim_R b \text{ or } b \sim_R a\}. \end{aligned}$$

[5] *Further Notation:* Also written R^{symm}.
[6] *Slogan:* The symmetric closure of R is the smallest symmetric relation containing

Proof. Clear. □

00UH **Proposition 7.2.2.1.3.** Let R be a relation on A.

00UJ 1. *Adjointness.* We have an adjunction

$$\left((-)^{\mathrm{symm}} \dashv 忘\right)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{symm}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathsf{symm}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathsf{symm}}(R^{\mathrm{symm}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathsf{symm}}(A,A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,A))$.

00UK 2. *The Symmetric Closure of a Symmetric Relation.* If R is symmetric, then $R^{\mathrm{symm}} = R$.

00UL 3. *Idempotency.* We have

$$(R^{\mathrm{symm}})^{\mathrm{symm}} = R^{\mathrm{symm}}.$$

00UM 4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{symm}} = \left(R^{\mathrm{symm}}\right)^{\dagger}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{symm}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{symm}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

00UN 5. *Interaction With Composition.* We have

$$(S \diamond R)^{\mathrm{symm}} = S^{\mathrm{symm}} \diamond R^{\mathrm{symm}}, \qquad \begin{tikzcd} \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{symm}} \times (-)^{\mathrm{symm}}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{symm}}"] \\ \mathrm{Rel}(A,A) \times \mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the symmetric closure of a relation, stated in Definition 7.2.2.1.1.
Item 2, The Symmetric Closure of a Symmetric Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: Clear.
Item 5, Interaction With Composition: This follows from Item 2 of Proposition 7.2.1.1.4. □

00UP 7.3 Transitive Relations

00UQ 7.3.1 Foundations

Let A be a set.

00UR **Definition 7.3.1.1.1.** A **transitive relation** is equivalently:[7]

- A non-unital $\mathbb{E}_1$-monoid in $(\mathrm{N}_\bullet(\mathbf{Rel}(A, A)), \diamond)$.
- A non-unital monoid in $(\mathbf{Rel}(A, A), \diamond)$.

00US **Remark 7.3.1.1.2.** In detail, a relation R on A is **transitive** if we have an inclusion

$$\mu_R \colon R \diamond R \subset R$$

of relations in $\mathbf{Rel}(A, A)$, i.e. if, for each $a, c \in A$, the following condition is satisfied:

($\star$) If there exists some $b \in A$ such that $a \sim_R b$ and $b \sim_R c$, then $a \sim_R c$.

00UT **Definition 7.3.1.1.3.** Let A be a set.

1. The **set of transitive relations from A to B** is the subset
00UU $\mathrm{Rel}^{\mathrm{trans}}(A)$ of $\mathrm{Rel}(A, A)$ spanned by the transitive relations.
2. The **poset of relations from A to B** is is the subposet $\mathbf{Rel}^{\mathsf{trans}}(A)$
00UV of $\mathbf{Rel}(A, A)$ spanned by the transitive relations.

00UW **Proposition 7.3.1.1.4.** Let R and S be relations on A.

00UX 1. *Interaction With Inverses.* If R is transitive, then so is $R^\dagger$.

00UY 2. *Interaction With Composition.* If R and S are transitive, then $S \diamond R$ ***may fail to be transitive***.

Proof. *Item 1, Interaction With Inverses*: Clear.
Item 2, Interaction With Composition: See [MSE 2096272].[8] □

R.

[7] Note that since $\mathbf{Rel}(A, A)$ is posetal, transitivity is a property of a relation, rather than extra structure.

[8] *Intuition:* Transitivity for R and S fails to imply that of $S \diamond R$ because the composition operation for relations intertwines R and S in an incompatible way:

1. If $a \sim_{S \diamond R} c$ and $c \sim_{S \diamond r} e$, then:
   (a) There is some $b \in A$ such that:
      i. $a \sim_R b$;
      ii. $b \sim_S c$;

00UZ 7.3.2 The Transitive Closure of a Relation

Let R be a relation on A.

00V0 **Definition 7.3.2.1.1.** The **transitive closure** of $\sim_R$ is the relation $\sim_R^{\mathrm{trans}}$[9] satisfying the following universal property:[10]

($\star$) Given another transitive relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim_R^{\mathrm{trans}} \subset \sim_S$.

00V1 **Construction 7.3.2.1.2.** Concretely, $\sim_R^{\mathrm{trans}}$ is the free non-unital monoid on R in $(\mathbf{Rel}(A,A),\diamond)$[11], being given by

$$\begin{aligned} R^{\mathrm{trans}} &\overset{\mathrm{def}}{=} \coprod_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \bigcup_{n=1}^{\infty} R^{\diamond n} \\ &\overset{\mathrm{def}}{=} \left\{ (a,b) \in A \times B \ \middle|\ \begin{gathered} \text{there exists some } (x_1,\ldots,x_n) \in R^{\times n} \\ \text{such that } a \sim_R x_1 \sim_R \cdots \sim_R x_n \sim_R b \end{gathered} \right\}. \end{aligned}$$

Proof. Clear. □

00V2 **Proposition 7.3.2.1.3.** Let R be a relation on A.

00V3 1. *Adjointness.* We have an adjunction

$$\Big((-)^{\mathrm{trans}} \dashv 忘\Big)\colon\quad \begin{tikzcd} \mathbf{Rel}(A,A) \arrow[r, bend left, "(-)^{\mathrm{trans}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathsf{trans}}(A,A), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathsf{trans}}\Big(R^{\mathrm{trans}}, S\Big) \cong \mathbf{Rel}(R,S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathsf{trans}}(A,A))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

00V4 2. *The Transitive Closure of a Transitive Relation.* If R is transitive, then $R^{\mathrm{trans}} = R$.

(b) There is some $d \in A$ such that:

i. $c \sim_R d$;

ii. $d \sim_S e$.

[9] *Further Notation:* Also written R^{trans}.

[10] *Slogan:* The transitive closure of R is the smallest transitive relation containing R.

[11] Or, equivalently, the free non-unital $\mathbb{E}_1$-monoid on R in $(\mathrm{N}_\bullet(\mathbf{Rel}(A,A)),\diamond)$.

00V5 3. *Idempotency.* We have

$$\left(R^{\mathrm{trans}}\right)^{\mathrm{trans}}=R^{\mathrm{trans}}.$$

00V6 4. *Interaction With Inverses.* We have

$$\left(R^{\dagger}\right)^{\mathrm{trans}}=\left(R^{\mathrm{trans}}\right)^{\dagger},\qquad \begin{tikzcd} \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"] \arrow[d, "(-)^{\dagger}"'] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\dagger}"] \\ \mathrm{Rel}(A,A) \arrow[r, "(-)^{\mathrm{trans}}"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

00V7 5. *Interaction With Composition.* We have

$$(S\diamond R)^{\mathrm{trans}}\overset{\text{poss.}}{\neq}S^{\mathrm{trans}}\diamond R^{\mathrm{trans}},\qquad \begin{tikzcd} \mathrm{Rel}(A,A)\times\mathrm{Rel}(A,A) \arrow[r, "\diamond"] \arrow[d, "(-)^{\mathrm{trans}}\times(-)^{\mathrm{trans}}"'] \arrow[dr, phantom, "\times"] & \mathrm{Rel}(A,A) \arrow[d, "(-)^{\mathrm{trans}}"] \\ \mathrm{Rel}(A,A)\times\mathrm{Rel}(A,A) \arrow[r, "\diamond"'] & \mathrm{Rel}(A,A). \end{tikzcd}$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the transitive closure of a relation, stated in Definition 7.3.2.1.1.
Item 2, The Transitive Closure of a Transitive Relation: Clear.
Item 3, Idempotency: This follows from Item 2.
Item 4, Interaction With Inverses: We have

$$\begin{aligned}\left(R^{\dagger}\right)^{\mathrm{trans}}&=\bigcup_{n=1}^{\infty}\left(R^{\dagger}\right)^{\diamond n}\\&=\bigcup_{n=1}^{\infty}\left(R^{\diamond n}\right)^{\dagger}\\&=\left(\bigcup_{n=1}^{\infty}R^{\diamond n}\right)^{\dagger}\\&=\left(R^{\mathrm{trans}}\right)^{\dagger},\end{aligned}$$

where we have used, respectively:

1. Construction 7.3.2.1.2.
2. Item 4 of Proposition 6.3.12.1.3.
3. Item 1 of Proposition 6.3.6.1.2.
4. Construction 7.3.2.1.2.

Item 5, Interaction With Composition: This follows from Item 2 of Proposition 7.3.1.1.4. □

00V8 7.4 Equivalence Relations

00V9 7.4.1 Foundations

Let A be a set.

00VA **Definition 7.4.1.1.1.** A relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.[12]

00VB **Example 7.4.1.1.2.** The **kernel of a function** $f\colon A \to B$ is the equivalence relation $\sim_{\mathrm{Ker}(f)}$ on A obtained by declaring $a \sim_{\mathrm{Ker}(f)} b$ iff $f(a) = f(b)$.[13]

00VC **Definition 7.4.1.1.3.** Let A and B be sets.

1. The **set of equivalence relations from** A **to** B is the subset
00VD $\mathrm{Rel}^{\mathrm{eq}}(A, B)$ of $\mathrm{Rel}(A, B)$ spanned by the equivalence relations.

2. The **poset of relations from** A **to** B is is the subposet $\mathbf{Rel}^{\mathrm{eq}}(A, B)$
00VE of $\mathbf{Rel}(A, B)$ spanned by the equivalence relations.

00VF 7.4.2 The Equivalence Closure of a Relation

Let R be a relation on A.

00VG **Definition 7.4.2.1.1.** The **equivalence closure**[14] of $\sim_R$ is the relation $\sim^{\mathrm{eq}}_R$[15] satisfying the following universal property:[16]

($\star$) Given another equivalence relation $\sim_S$ on A such that $R \subset S$, there exists an inclusion $\sim^{\mathrm{eq}}_R \subset \sim_S$.

[12] *Further Terminology:* If instead R is just symmetric and transitive, then it is called a **partial equivalence relation**.

[13] The kernel $\mathrm{Ker}(f)\colon A \nrightarrow A$ of f is the underlying functor of the monad induced by the adjunction $\mathrm{Gr}(f) \dashv f^{-1}\colon A \rightleftarrows B$ in **Rel** of Item 2 of Proposition 6.3.1.1.2.

[14] *Further Terminology:* Also called the **equivalence relation associated to** $\sim_R$.

[15] *Further Notation:* Also written R^{eq}.

[16] *Slogan:* The equivalence closure of R is the smallest equivalence relation containing R.

00VH **Construction 7.4.2.1.2.** Concretely, $\sim_R^{\mathrm{eq}}$ is the equivalence relation on A defined by

$$\begin{aligned} R^{\mathrm{eq}} &\overset{\mathrm{def}}{=} \left(\left(R^{\mathrm{refl}}\right)^{\mathrm{symm}}\right)^{\mathrm{trans}} \\ &= \left((R^{\mathrm{symm}})^{\mathrm{trans}}\right)^{\mathrm{refl}} \\ &= \left\{ (a,b) \in A \times B \;\middle|\; \begin{array}{l} \text{there exists } (x_1, \ldots, x_n) \in R^{\times n} \text{ satisfying at} \\ \text{least one of the following conditions:} \\ \quad 1. \text{ The following conditions are satisfied:} \\ \qquad \text{(a) We have } a \sim_R x_1 \text{ or } x_1 \sim_R a; \\ \qquad \text{(b) We have } x_i \sim_R x_{i+1} \text{ or } x_{i+1} \sim_R x_i \\ \qquad\quad \text{for each } 1 \le i \le n-1; \\ \qquad \text{(c) We have } b \sim_R x_n \text{ or } x_n \sim_R b; \\ \quad 2. \text{ We have } a = b. \end{array} \right\}. \end{aligned}$$

Proof. From the universal properties of the reflexive, symmetric, and transitive closures of a relation (Definitions 7.1.2.1.1, 7.2.2.1.1 and 7.3.2.1.1), we see that it suffices to prove that:

00VJ 1. The symmetric closure of a reflexive relation is still reflexive.

00VK 2. The transitive closure of a symmetric relation is still symmetric.

which are both clear. □

00VL **Proposition 7.4.2.1.3.** Let R be a relation on A.

00VM 1. *Adjointness.* We have an adjunction

$$((-)^{\mathrm{eq}} \dashv 忘)\colon \quad \begin{tikzcd} \mathbf{Rel}(A,B) \arrow[r, bend left, "(-)^{\mathrm{eq}}"] \arrow[r, phantom, "\perp"] & \mathbf{Rel}^{\mathrm{eq}}(A,B), \arrow[l, bend left, "忘"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathbf{Rel}^{\mathrm{eq}}(R^{\mathrm{eq}}, S) \cong \mathbf{Rel}(R, S),$$

natural in $R \in \mathrm{Obj}(\mathbf{Rel}^{\mathrm{eq}}(A,B))$ and $S \in \mathrm{Obj}(\mathbf{Rel}(A,B))$.

00VN 2. *The Equivalence Closure of an Equivalence Relation.* If R is an equivalence relation, then $R^{\mathrm{eq}} = R$.

00VP 3. *Idempotency.* We have

$$(R^{\mathrm{eq}})^{\mathrm{eq}} = R^{\mathrm{eq}}.$$

Proof. *Item 1, Adjointness*: This is a rephrasing of the universal property of the equivalence closure of a relation, stated in Definition 7.4.2.1.1.
Item 2, The Equivalence Closure of an Equivalence Relation: Clear.
Item 3, Idempotency: This follows from Item 2. □

00VQ 7.5 Quotients by Equivalence Relations

00VR 7.5.1 Equivalence Classes

Let A be a set, let R be a relation on A, and let $a \in A$.

00VS **Definition 7.5.1.1.1.** The **equivalence class associated to** a is the set $[a]$ defined by

$$\begin{aligned}[a] &\overset{\text{def}}{=} \{x \in X \mid x \sim_R a\} \\ &= \{x \in X \mid a \sim_R x\}. &&\text{(since } R \text{ is symmetric)}\end{aligned}$$

00VT 7.5.2 Quotients of Sets by Equivalence Relations

Let A be a set and let R be a relation on A.

00VU **Definition 7.5.2.1.1.** The **quotient of** X **by** R is the set $X/{\sim_R}$ defined by

$$X/{\sim_R} \overset{\text{def}}{=} \{[a] \in \mathcal{P}(X) \mid a \in X\}.$$

00VV **Remark 7.5.2.1.2.** The reason we define quotient sets for equivalence relations only is that each of the properties of being an equivalence relation—reflexivity, symmetry, and transitivity—ensures that the equivalences classes $[a]$ of X under R are well-behaved:

- *Reflexivity.* If R is reflexive, then, for each $a \in X$, we have $a \in [a]$.
- *Symmetry.* The equivalence class $[a]$ of an element a of X is defined by

  $$[a] \overset{\text{def}}{=} \{x \in X \mid x \sim_R a\},$$

  but we could equally well define

  $$[a]' \overset{\text{def}}{=} \{x \in X \mid a \sim_R x\}$$

  instead. This is not a problem when R is symmetric, as we then have $[a] = [a]'$.[17]
- *Transitivity.* If R is transitive, then $[a]$ and $[b]$ are disjoint iff $a \nsim_R b$, and equal otherwise.

[17]When categorifying equivalence relations, one finds that $[a]$ and $[a]'$ correspond to

00VW **Proposition 7.5.2.1.3.** Let $f\colon X \to Y$ be a function and let R be a relation on X.

00VX 1. *As a Coequaliser.* We have an isomorphism of sets

$$X/\sim_R^{\mathrm{eq}} \cong \mathrm{CoEq}\left(R \hookrightarrow X \times X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X\right),$$

where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

00VY 2. *As a Pushout.* We have an isomorphism of sets[18]

$$X/\sim_R^{\mathrm{eq}} \cong X \textstyle\coprod_{\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2)} X, \qquad \begin{tikzcd} X/\sim_R^{\mathrm{eq}} & X \arrow[l] \\ X \arrow[u] & \mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2). \arrow[l] \arrow[u] \arrow[ul, phantom, "\ulcorner", very near end] \end{tikzcd}$$

where $\sim_R^{\mathrm{eq}}$ is the equivalence relation generated by $\sim_R$.

00VZ 3. *The First Isomorphism Theorem for Sets.* We have an isomorphism of sets[19,20]

$$X/\sim_{\mathrm{Ker}(f)} \cong \mathrm{Im}(f).$$

presheaves and copresheaves; see **??**.

[18]Dually, we also have an isomorphism of sets

$$\mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2) \cong X \times_{X/\sim_R^{\mathrm{eq}}} X, \qquad \begin{tikzcd} \mathrm{Eq}(\mathrm{pr}_1,\mathrm{pr}_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ X \arrow[r] & X/\sim_R^{\mathrm{eq}}. \end{tikzcd}$$

[19]*Further Terminology:* The set $X/\sim_{\mathrm{Ker}(f)}$ is often called the **coimage of** f, and denoted by $\mathrm{Coim}(f)$.

[20]In a sense this is a result relating the monad in **Rel** induced by f with the comonad in **Rel** induced by f, as the kernel and image

$$\begin{aligned} \mathrm{Ker}(f)&\colon X \nrightarrow X,\\ \mathrm{Im}(f)&\subset Y \end{aligned}$$

of f are the underlying functors of (respectively) the induced monad and comonad of the adjunction

$$\left(\mathrm{Gr}(f) \dashv f^{-1}\right)\colon\quad \begin{tikzcd} A \arrow[r, bend left, "\mathrm{Gr}(f)"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

of Item 2 of Proposition 6.3.1.1.2.

00W0 4. *Descending Functions to Quotient Sets, I.* Let R be an equivalence relation on X. The following conditions are equivalent:

(a) There exists a map

$$\overline{f}\colon X/{\sim_R} \to Y$$

making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R} \arrow[ur, dotted, "\overline{f}"', "\exists"] & \end{tikzcd}$$

commute.

(b) We have $R \subset \mathrm{Ker}(f)$.

(c) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

00W1 5. *Descending Functions to Quotient Sets, II.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then $\overline{f}$ is the *unique* map making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ X/{\sim_R} \arrow[ur, dashed, "\overline{f}"', "\exists!"] & \end{tikzcd}$$

commute.

00W2 6. *Descending Functions to Quotient Sets, III.* Let R be an equivalence relation on X. We have a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(X/{\sim_R}, Y) \cong \mathrm{Hom}^R_{\mathsf{Sets}}(X, Y),$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$, given by the assignment $f \mapsto \overline{f}$ of Items 4 and 5, where $\mathrm{Hom}^R_{\mathsf{Sets}}(X, Y)$ is the set defined by

$$\mathrm{Hom}^R_{\mathsf{Sets}}(X, Y) \overset{\text{def}}{=} \left\{ f \in \mathrm{Hom}_{\mathsf{Sets}}(X, Y) \;\middle|\; \begin{array}{l} \text{for each } x, y \in X, \\ \text{if } x \sim_R y, \text{ then} \\ f(x) = f(y) \end{array} \right\}.$$

00W3 7. *Descending Functions to Quotient Sets, IV.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

(a) The map $\overline{f}$ is an injection.

(b) We have $R = \mathrm{Ker}(f)$.

(c) For each $x, y \in X$, we have $x \sim_R y$ iff $f(x) = f(y)$.

00W4 8. *Descending Functions to Quotient Sets, V.* Let R be an equivalence relation on X. If the conditions of Item 4 hold, then the following conditions are equivalent:

(a) The map $f\colon X \to Y$ is surjective.

(b) The map $\overline{f}\colon X/{\sim_R} \to Y$ is surjective.

00W5 9. *Descending Functions to Quotient Sets, VI.* Let R be a relation on X and let $\sim_R^{\mathrm{eq}}$ be the equivalence relation associated to R. The following conditions are equivalent:

00W6 (a) The map f satisfies the equivalent conditions of Item 4:

- There exists a map

$$\overline{f}\colon X/{\sim_R^{\mathrm{eq}}} \to Y$$

making the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "q"'] & Y \\
X/{\sim_R^{\mathrm{eq}}} \arrow[ur, dotted, "\overline{f}"'] \arrow[ur, phantom, "\exists", near end] &
\end{tikzcd}$$

commute.

- For each $x, y \in X$, if $x \sim_R^{\mathrm{eq}} y$, then $f(x) = f(y)$.

00W7 (b) For each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$.

Proof. Item 1, As a Coequaliser: Omitted.
Item 2, As a Pushout: Omitted.
Item 3, The First Isomorphism Theorem for Sets: Clear.
Item 4, Descending Functions to Quotient Sets, I: See [Pro24o].
Item 5, Descending Functions to Quotient Sets, II: See [Pro24aa].
Item 6, Descending Functions to Quotient Sets, III: This follows from Items 5 and 6.
Item 7, Descending Functions to Quotient Sets, IV: See [Pro24n].
Item 8, Descending Functions to Quotient Sets, V: See [Pro24m].
Item 9, Descending Functions to Quotient Sets, VI: The implication Item 9a $\Longrightarrow$ Item 9b is clear.
Conversely, suppose that, for each $x, y \in X$, if $x \sim_R y$, then $f(x) = f(y)$. Spelling out the definition of the equivalence closure of R, we see that the condition $x \sim_R^{\mathrm{eq}} y$ unwinds to the following:

(⋆) There exist $(x_1,\ldots,x_n)\in R^{\times n}$ satisfying at least one of the following conditions:

1. The following conditions are satisfied:
   (a) We have $x\sim_R x_1$ or $x_1\sim_R x$;
   (b) We have $x_i\sim_R x_{i+1}$ or $x_{i+1}\sim_R x_i$ for each $1\leq i\leq n-1$;
   (c) We have $y\sim_R x_n$ or $x_n\sim_R y$;
2. We have $x=y$.

Now, if $x=y$, then $f(x)=f(y)$ trivially; otherwise, we have

$$\begin{aligned} f(x) &= f(x_1), \\ f(x_1) &= f(x_2), \\ &\vdots \\ f(x_{n-1}) &= f(x_n), \\ f(x_n) &= f(y), \end{aligned}$$

and $f(x)=f(y)$, as we wanted to show. □

Appendices

7.A Other Chapters

Part III

Category Theory

Chapter 8

Categories

00W8 This chapter contains some elementary material about categories, functors, and natural transformations. Notably, we discuss and explore:

1. Categories (Section 8.1).
2. The quadruple adjunction $\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}}$ between the category of categories and the category of sets (Section 8.2).
3. Groupoids, categories in which all morphisms admit inverses (Section 8.3).
4. Functors (Section 8.4).
5. The conditions one may impose on functors in decreasing order of importance:
   (a) Section 8.5 introduces the foundationally important conditions one may impose on functors, such as faithfulness, conservativity, essential surjectivity, etc.
   (b) Section 8.6 introduces more conditions one may impose on functors that are still important but less omni-present than those of Section 8.5, such as being dominant, being a monomorphism, being pseudomonic, etc.
   (c) Section 8.7 introduces some rather rare or uncommon conditions one may impose on functors that are nevertheless still useful to explicit record in this chapter.
6. Natural transformations (Section 8.8).
7. The various categorical and 2-categorical structures formed by categories, functors, and natural transformations (Section 8.9).

Contents

00W9 8.1 Categories

00WA 8.1.1 Foundations

00WB **Definition 8.1.1.1.1.** A **category** $(\mathcal{C}, \circ^{\mathcal{C}}, \mathbb{1}^{\mathcal{C}})$ consists of:

- *Objects.* A class $\mathrm{Obj}(\mathcal{C})$ of **objects**.
- *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a class $\mathrm{Hom}_{\mathcal{C}}(A, B)$, called the **class of morphisms of $\mathcal{C}$ from A to B**.

- *Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, a map of sets
$$\mathbb{1}^{\mathcal{C}}_A \colon \mathrm{pt} \to \mathrm{Hom}_{\mathcal{C}}(A, A),$$
called the **unit map of $\mathcal{C}$ at** A, determining a morphism
$$\mathrm{id}_A \colon A \to A$$
of $\mathcal{C}$, called the **identity morphism of** A.
- *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, a map of sets
$$\circ^{\mathcal{C}}_{A,B,C} \colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C),$$
called the **composition map of $\mathcal{C}$ at** (A, B, C).

such that the following conditions are satisfied:

1. *Associativity.* The diagram
$$\begin{tikzcd}
& \mathrm{Hom}_{\mathcal{C}}(C,D) \times (\mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B)) \arrow[dr, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(C,D)} \times \circ^{\mathcal{C}}_{A,B,C}"] & \\
(\mathrm{Hom}_{\mathcal{C}}(C,D) \times \mathrm{Hom}_{\mathcal{C}}(B,C)) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(C,D),\mathrm{Hom}_{\mathcal{C}}(B,C),\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] \arrow[d, "\circ^{\mathcal{C}}_{B,C,D} \times \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] & & \mathrm{Hom}_{\mathcal{C}}(C,D) \times \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "\circ^{\mathcal{C}}_{A,C,D}"] \\
\mathrm{Hom}_{\mathcal{C}}(B,D) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[rr, "\circ^{\mathcal{C}}_{A,B,D}"'] & & \mathrm{Hom}_{\mathcal{C}}(A,D)
\end{tikzcd}$$
commutes, i.e. for each composable triple (f, g, h) of morphisms of $\mathcal{C}$, we have
$$(f \circ g) \circ h = f \circ (g \circ h).$$
2. *Left Unitality.* The diagram
$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "\mathbb{1}^{\mathcal{C}}_B \times \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(B,B) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B)
\end{tikzcd}$$
commutes, i.e. for each morphism $f \colon A \to B$ of $\mathcal{C}$, we have
$$\mathrm{id}_B \circ f = f.$$

3. *Right Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A,B)\times\mathrm{pt} \arrow[d, "\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}\times\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(A,B)\times\mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "\circ^{\mathcal{C}}_{A,A,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B)
\end{tikzcd}$$

commutes, i.e. for each morphism $f\colon A\to B$ of $\mathcal{C}$, we have

$$f\circ\mathrm{id}_A = f.$$

00WC **Notation 8.1.1.1.2.** Let $\mathcal{C}$ be a category.

00WD 1. We also write $\mathcal{C}(A,B)$ for $\mathrm{Hom}_{\mathcal{C}}(A,B)$.

00WE 2. We write $\mathrm{Mor}(\mathcal{C})$ for the class of all morphisms of $\mathcal{C}$.

00WF **Definition 8.1.1.1.3.** Let κ be a regular cardinal. A category $\mathcal{C}$ is

00WG 1. **Locally small** if, for each $A,B\in\mathrm{Obj}(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is a set.

00WH 2. **Locally essentially small** if, for each $A,B\in\mathrm{Obj}(\mathcal{C})$, the class

$$\mathrm{Hom}_{\mathcal{C}}(A,B)/\{\text{isomorphisms}\}$$

is a set.

00WJ 3. **Small** if $\mathcal{C}$ is locally small and $\mathrm{Obj}(\mathcal{C})$ is a set.

00WK 4. **κ-Small** if $\mathcal{C}$ is locally small, $\mathrm{Obj}(\mathcal{C})$ is a set, and we have $\#\mathrm{Obj}(\mathcal{C})<\kappa$.

00WL 8.1.2 Examples of Categories

00WM **Example 8.1.2.1.1.** The **punctual category**[1] is the category pt where

- *Objects.* We have
$$\mathrm{Obj}(\mathsf{pt})\overset{\mathrm{def}}{=}\{\star\}.$$

- *Morphisms.* The unique Hom-set of pt is defined by

$$\mathrm{Hom}_{\mathsf{pt}}(\star,\star)\overset{\mathrm{def}}{=}\{\mathrm{id}_\star\}.$$

[1] *Further Terminology:* Also called the **singleton category**.

- *Identities.* The unit map

$$\mathbb{1}^{\mathsf{pt}}_{\star} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{pt}}(\star, \star)$$

of pt at $\star$ is defined by

$$\mathrm{id}^{\mathsf{pt}}_{\star} \overset{\mathrm{def}}{=} \mathrm{id}_{\star}.$$

- *Composition.* The composition map

$$\circ^{\mathsf{pt}}_{\star,\star,\star} \colon \mathrm{Hom}_{\mathsf{pt}}(\star, \star) \times \mathrm{Hom}_{\mathsf{pt}}(\star, \star) \to \mathrm{Hom}_{\mathsf{pt}}(\star, \star)$$

of pt at $(\star, \star, \star)$ is given by the bijection $\mathrm{pt} \times \mathrm{pt} \cong \mathrm{pt}$.

00WN **Example 8.1.2.1.2.** We have an isomorphism of categories[2]

$$\mathsf{Mon} \cong \mathsf{pt} \underset{\mathsf{Sets}}{\times} \mathsf{Cats}, \qquad \begin{tikzcd} \mathsf{Mon} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{Cats} \arrow[d, "\mathrm{Obj}"] \\ \mathsf{pt} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets} \end{tikzcd}$$

via the delooping functor $\mathsf{B} \colon \mathsf{Mon} \to \mathsf{Cats}$ of ?? of ??, exhibiting monoids as exactly those categories having a single object.

Proof. Omitted. □

00WP **Example 8.1.2.1.3.** The **empty category** is the category $\emptyset_{\mathsf{cat}}$ where

- *Objects.* We have

$$\mathrm{Obj}(\emptyset_{\mathsf{cat}}) \overset{\mathrm{def}}{=} \emptyset.$$

- *Morphisms.* We have

$$\mathrm{Mor}(\emptyset_{\mathsf{cat}}) \overset{\mathrm{def}}{=} \emptyset.$$

- *Identities and Composition.* Having no objects, $\emptyset_{\mathsf{cat}}$ has no unit nor composition maps.

[2] This can be enhanced to an isomorphism of 2-categories

$$\mathsf{Mon}_{\mathsf{2disc}} \cong \mathsf{pt}_{\mathsf{bi}} \underset{\mathsf{Sets}_{\mathsf{2disc}}}{\times} \mathsf{Cats}_{2,*}, \qquad \begin{tikzcd} \mathsf{Mon}_{\mathsf{2disc}} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathsf{Cats}_{2,*} \arrow[d, "\mathrm{Obj}"] \\ \mathsf{pt}_{\mathsf{bi}} \arrow[r, "{[\mathrm{pt}]}"'] & \mathsf{Sets}_{\mathsf{2disc}} \end{tikzcd}$$

00WQ **Example 8.1.2.1.4.** The **nth ordinal category** is the category $\mathbb{n}$ where[3]

- *Objects.* We have
$$\mathrm{Obj}(\mathbb{n}) \stackrel{\mathrm{def}}{=} \{[0], \ldots, [n]\}.$$
- *Morphisms.* For each $[i], [j] \in \mathrm{Obj}(\mathbb{n})$, we have
$$\mathrm{Hom}_{\mathbb{n}}([i],[j]) \stackrel{\mathrm{def}}{=} \begin{cases} \left\{\mathrm{id}_{[i]}\right\} & \text{if } [i] = [j], \\ \{[i] \to [j]\} & \text{if } [j] < [i], \\ \emptyset & \text{if } [j] > [i]. \end{cases}$$
- *Identities.* For each $[i] \in \mathrm{Obj}(\mathbb{n})$, the unit map
$$\mathbb{1}^{\mathbb{n}}_{[i]} \colon \mathrm{pt} \to \mathrm{Hom}_{\mathbb{n}}([i],[i])$$
of $\mathbb{n}$ at $[i]$ is defined by
$$\mathrm{id}^{\mathbb{n}}_{[i]} \stackrel{\mathrm{def}}{=} \mathrm{id}_{[i]}.$$
- *Composition.* For each $[i], [j], [k] \in \mathrm{Obj}(\mathbb{n})$, the composition map
$$\circ^{\mathbb{n}}_{[i],[j],[k]} \colon \mathrm{Hom}_{\mathbb{n}}([j],[k]) \times \mathrm{Hom}_{\mathbb{n}}([i],[j]) \to \mathrm{Hom}_{\mathbb{n}}([i],[k])$$
of $\mathbb{n}$ at $([i],[j],[k])$ is defined by
$$\begin{aligned} \mathrm{id}_{[i]} \circ \mathrm{id}_{[i]} &= \mathrm{id}_{[i]}, \\ ([j] \to [k]) \circ ([i] \to [j]) &= ([i] \to [k]). \end{aligned}$$

between the discrete 2-category $\mathsf{Mon}_{\mathsf{2disc}}$ on Mon and the 2-category of pointed categories with one object.

[3]In other words, $\mathbb{n}$ is the category associated to the poset
$$[0] \to [1] \to \cdots \to [n-1] \to [n].$$
The category $\mathbb{n}$ for $n \geq 2$ may also be defined in terms of $\mathbb{0}$ and joins (**??**): we have

00WR **Example 8.1.2.1.5.** Here we list some of the other categories appearing throughout this work.

00WS 1. The category Sets_* of pointed sets of Definition 3.1.3.1.1.

00WT 2. The category Rel of sets and relations of Definition 5.2.1.1.1.

00WU 3. The category $\mathsf{Span}(A, B)$ of spans from a set A to a set B of **??**.

00WV 4. The category $\mathsf{ISets}(K)$ of K-indexed sets of **??**.

00WW 5. The category ISets of indexed sets of **??**.

00WX 6. The category $\mathsf{FibSets}(K)$ of K-fibred sets of **??**.

00WY 7. The category $\mathsf{FibSets}$ of fibred sets of **??**.

00WZ 8. Categories of functors $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ as in Definition 8.9.1.1.1.

00X0 9. The category of categories Cats of Definition 8.9.2.1.1.

00X1 10. The category of groupoids Grpd of Definition 8.9.4.1.1.

00X2 8.1.3 Posetal Categories

00X3 **Definition 8.1.3.1.1.** Let $(X, \preceq_X)$ be a poset.

1. The **posetal category associated to** $(X, \preceq_X)$ is the category
00X4 X_{pos} where

   - *Objects.* We have

$$\mathrm{Obj}(X_{\mathsf{pos}}) \overset{\text{def}}{=} X.$$

isomorphisms of categories

$$\begin{aligned}
\mathbb{1} &\cong \mathbb{0} \star \mathbb{0},\\
\mathbb{2} &\cong \mathbb{1} \star \mathbb{0}\\
&\cong (\mathbb{0} \star \mathbb{0}) \star \mathbb{0},\\
\mathbb{3} &\cong \mathbb{2} \star \mathbb{0}\\
&\cong (\mathbb{1} \star \mathbb{0}) \star \mathbb{0}\\
&\cong ((\mathbb{0} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0},\\
\mathbb{4} &\cong \mathbb{3} \star \mathbb{0}\\
&\cong (\mathbb{2} \star \mathbb{0}) \star \mathbb{0}\\
&\cong ((\mathbb{1} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0}\\
&\cong (((\mathbb{0} \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0}) \star \mathbb{0},
\end{aligned}$$

and so on.

- *Morphisms.* For each $a, b \in \mathrm{Obj}(X_{\mathsf{pos}})$, we have

$$\mathrm{Hom}_{X_{\mathsf{pos}}}(a, b) \stackrel{\text{def}}{=} \begin{cases} \mathrm{pt} & \text{if } a \preceq_X b, \\ \emptyset & \text{otherwise.} \end{cases}$$

- *Identities.* For each $a \in \mathrm{Obj}(X_{\mathsf{pos}})$, the unit map

$$\mathbb{1}_a^{X_{\mathsf{pos}}} \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{pos}}}(a, a)$$

  of X_{pos} at a is given by the identity map.

- *Composition.* For each $a, b, c \in \mathrm{Obj}(X_{\mathsf{pos}})$, the composition map

$$\circ_{a,b,c}^{X_{\mathsf{pos}}} \colon \mathrm{Hom}_{X_{\mathsf{pos}}}(b, c) \times \mathrm{Hom}_{X_{\mathsf{pos}}}(a, b) \to \mathrm{Hom}_{X_{\mathsf{pos}}}(a, c)$$

  of X_{pos} at (a, b, c) is defined as either the inclusion $\emptyset \hookrightarrow \mathrm{pt}$ or the identity map of pt, depending on whether we have $a \preceq_X b$, $b \preceq_X c$, and $a \preceq_X c$.

00X5 2. A category $\mathcal{C}$ is **posetal** [4] if $\mathcal{C}$ is equivalent to X_{pos} for some poset $(X, \preceq_X)$.

00X6 **Proposition 8.1.3.1.2.** Let $(X, \preceq_X)$ be a poset and let $\mathcal{C}$ be a category.

00X7 1. *Functoriality.* The assignment $(X, \preceq_X) \mapsto X_{\mathsf{pos}}$ defines a functor

$$(-)_{\mathsf{pos}} \colon \mathsf{Pos} \to \mathsf{Cats}.$$

00X8 2. *Fully Faithfulness.* The functor $(-)_{\mathsf{pos}}$ of Item 1 is fully faithful.

00X9 3. *Characterisations.* The following conditions are equivalent:

00XA (a) The category $\mathcal{C}$ is posetal.

(b) For each $A, B \in \mathrm{Obj}(\mathcal{C})$ and each $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, we have
00XB $f = g$.

Proof. Item 1, Functoriality: Omitted.
Item 2, Fully Faithfulness: Omitted.
Item 3, Characterisations: Clear. □

00XC 8.1.4 Subcategories

Let $\mathcal{C}$ be a category.

[4] *Further Terminology:* Also called a **thin** category or a $(0, 1)$**-category**.

00XD **Definition 8.1.4.1.1.** A **subcategory** of $\mathcal{C}$ is a category $\mathcal{A}$ satisfying the following conditions:

1. *Objects.* We have $\mathrm{Obj}(\mathcal{A}) \subset \mathrm{Obj}(\mathcal{C})$.
2. *Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{A})$, we have
$$\mathrm{Hom}_{\mathcal{A}}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B).$$
3. *Identities.* For each $A \in \mathrm{Obj}(\mathcal{A})$, we have
$$\mathbb{1}^{\mathcal{A}}_{A} = \mathbb{1}^{\mathcal{C}}_{A}.$$
4. *Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{A})$, we have
$$\circ^{\mathcal{A}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,C}.$$

00XE **Definition 8.1.4.1.2.** A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **full** if the canonical inclusion functor $\mathcal{A} \to \mathcal{C}$ is full, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{A})$, the inclusion
$$\iota_{A,B} \colon \mathrm{Hom}_{\mathcal{A}}(A, B) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(A, B)$$
is surjective (and thus bijective).

00XF **Definition 8.1.4.1.3.** A subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is **strictly full** if it satisfies the following conditions:

1. *Fullness.* The subcategory $\mathcal{A}$ is full.
2. *Closedness Under Isomorphisms.* The class $\mathrm{Obj}(\mathcal{A})$ is closed under isomorphisms.[5]

00XG **Definition 8.1.4.1.4.** A subcategory $\mathcal{A}$ of $\mathcal{C}$ is **wide**[6] if $\mathrm{Obj}(\mathcal{A}) = \mathrm{Obj}(\mathcal{C})$.

00XH 8.1.5 Skeletons of Categories

00XJ **Definition 8.1.5.1.1.** A[7] **skeleton** of a category $\mathcal{C}$ is a full subcategory $\mathsf{Sk}(\mathcal{C})$ with one object from each isomorphism class of objects of $\mathcal{C}$.

00XK **Definition 8.1.5.1.2.** A category $\mathcal{C}$ is **skeletal** if $\mathcal{C} \cong \mathsf{Sk}(\mathcal{C})$.[8]

00XL **Proposition 8.1.5.1.3.** Let $\mathcal{C}$ be a category.

[5]That is, given $A \in \mathrm{Obj}(\mathcal{A})$ and $C \in \mathrm{Obj}(\mathcal{C})$, if $C \cong A$, then $C \in \mathrm{Obj}(\mathcal{A})$.

[6]*Further Terminology:* Also called **lluf**.

[7]Due to Item 3 of Proposition 8.1.5.1.3, we often refer to any such full subcategory $\mathsf{Sk}(\mathcal{C})$ of $\mathcal{C}$ as *the* skeleton of $\mathcal{C}$.

[8]That is, $\mathcal{C}$ is **skeletal** if isomorphic objects of $\mathcal{C}$ are equal.

00XM 1. *Existence.* Assuming the axiom of choice, $\mathsf{Sk}(\mathcal{C})$ always exists.

00XN 2. *Pseudofunctoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Sk}(\mathcal{C})$ defines a pseudofunctor

$$\mathsf{Sk}\colon \mathsf{Cats}_2 \to \mathsf{Cats}_2.$$

00XP 3. *Uniqueness Up to Equivalence.* Any two skeletons of $\mathcal{C}$ are equivalent.

00XQ 4. *Inclusions of Skeletons Are Equivalences.* The inclusion

$$\iota_{\mathcal{C}}\colon \mathsf{Sk}(\mathcal{C}) \hookrightarrow \mathcal{C}$$

of a skeleton of $\mathcal{C}$ into $\mathcal{C}$ is an equivalence of categories.

Proof. *Item 1, Existence*: See [nLab23, Section "Existence of Skeletons of Categories"].
Item 2, Pseudofunctoriality: See [nLab23, Section "Skeletons as an Endo-Pseudofunctor on $\mathfrak{Cat}$"].
Item 3, Uniqueness Up to Equivalence: Clear.
Item 4, Inclusions of Skeletons Are Equivalences: Clear. □

00XR 8.1.6 Precomposition and Postcomposition

Let $\mathcal{C}$ be a category and let $A, B, C \in \mathrm{Obj}(\mathcal{C})$.

00XS **Definition 8.1.6.1.1.** Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$.

00XT 1. The **precomposition function associated to** f is the function

$$f^*\colon \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

defined by

$$f^*(\phi) \stackrel{\text{def}}{=} \phi \circ f$$

for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(B, C)$.

00XU 2. The **postcomposition function associated to** g is the function

$$g_*\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

defined by

$$g_*(\phi) \stackrel{\text{def}}{=} g \circ \phi$$

for each $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

00XV **Proposition 8.1.6.1.2.** Let $A, B, C, D \in \mathrm{Obj}(\mathcal{C})$ and let $f\colon A \to B$ and $g\colon B \to C$ be morphisms of $\mathcal{C}$.

00XW 1. *Interaction Between Precomposition and Postcomposition.* We have

$$g_* \circ f^* = f^* \circ g_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[r, "g_*"] \arrow[d, "f^*"'] & \mathrm{Hom}_{\mathcal{C}}(B,D) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[r, "g_*"'] & \mathrm{Hom}_{\mathcal{C}}(A,D). \end{tikzcd}$$

00XX 2. *Interaction With Composition I.* We have

$$(g \circ f)^* = f^* \circ g^*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,A) \arrow[r, "f_*"] \arrow[dr, "(g\circ f)_*"'] & \mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(X,C), \end{tikzcd}$$

$$(g \circ f)_* = g_* \circ f_*, \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C,X) \arrow[r, "g^*"] \arrow[dr, "(g\circ f)^*"'] & \mathrm{Hom}_{\mathcal{C}}(B,X) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,X). \end{tikzcd}$$

00XY 3. *Interaction With Composition II.* We have

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "{[f]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "g_*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C) \end{tikzcd} \qquad \begin{gathered} [g \circ f] = g_* \circ [f], \\ [g \circ f] = f^* \circ [g], \end{gathered} \qquad \begin{tikzcd} \mathrm{pt} \arrow[r, "{[g]}"] \arrow[dr, "{[g\circ f]}"'] & \mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[d, "f^*"] \\ & \mathrm{Hom}_{\mathcal{C}}(A,C). \end{tikzcd}$$

00XZ 4. *Interaction With Composition III.* We have

$$f^* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{X,B,C} \circ (f^* \times \mathsf{id}), \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "\mathsf{id}\times f^*"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "f^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(X,B) \arrow[r, "\circ^{\mathcal{C}}_{X,B,C}"'] & \mathrm{Hom}_{\mathcal{C}}(X,C), \end{tikzcd}$$

$$g_* \circ \circ^{\mathcal{C}}_{A,B,C} = \circ^{\mathcal{C}}_{A,B,D} \circ (\mathsf{id} \times g_*), \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "g_*\times \mathsf{id}"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "g^*"] \\ \mathrm{Hom}_{\mathcal{C}}(B,D) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,D}"'] & \mathrm{Hom}_{\mathcal{C}}(A,D). \end{tikzcd}$$

00Y0 5. *Interaction With Identities.* We have

$$\begin{aligned}(\mathrm{id}_A)^* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)},\\ (\mathrm{id}_B)_* &= \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}.\end{aligned}$$

Proof. *Item 1, Interaction Between Precomposition and Postcomposition*: Clear.
Item 2, Interaction With Composition I: Clear.
Item 3, Interaction With Composition II: Clear.
Item 4, Interaction With Composition III: Clear.
Item 5, Interaction With Identities: Clear. □

00Y1 8.2 The Quadruple Adjunction With Sets

00Y2 8.2.1 Statement

Let $\mathcal{C}$ be a category.

00Y3 **Proposition 8.2.1.1.1.** We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})\colon \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=15, "(-)_{\mathsf{disc}}"] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}, \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=15, "\mathrm{Obj}"] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Sets}}(\pi_0(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{disc}}),\\ \mathrm{Hom}_{\mathsf{Cats}}(X_{\mathsf{disc}}, \mathcal{C}) &\cong \mathrm{Hom}_{\mathsf{Sets}}(X, \mathrm{Obj}(\mathcal{C})),\\ \mathrm{Hom}_{\mathsf{Sets}}(\mathrm{Obj}(\mathcal{C}), X) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, X_{\mathsf{indisc}}),\end{aligned}$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $X \in \mathrm{Obj}(\mathsf{Sets})$, where

- The functor
$$\pi_0\colon \mathsf{Cats} \to \mathsf{Sets},$$
the **connected components functor**, is the functor sending a category to its set of connected components of Definition 8.2.2.2.1.

- The functor
$$(-)_{\mathsf{disc}}\colon \mathsf{Sets} \to \mathsf{Cats},$$
the **discrete category functor**, is the functor sending a set to its associated discrete category of Item 1.

- The functor

$$\mathrm{Obj}\colon \mathsf{Cats} \to \mathsf{Sets},$$

the **object functor**, is the functor sending a category to its set of objects.

- The functor

$$(-)_{\mathsf{indisc}}\colon \mathsf{Sets} \to \mathsf{Cats},$$

the **indiscrete category functor**, is the functor sending a set to its associated indiscrete category of Item 1.

Proof. Omitted. □

00Y4
8.2.2 Connected Components and Connected Categories

00Y5
8.2.2.1 Connected Components of Categories

Let $\mathcal{C}$ be a category.

00Y6 **Definition 8.2.2.1.1.** A **connected component** of $\mathcal{C}$ is a full subcategory $\mathcal{I}$ of $\mathcal{C}$ satisfying the following conditions:[9]

1. *Non-Emptiness.* We have $\mathrm{Obj}(\mathcal{I}) \neq \emptyset$.

2. *Connectedness.* There exists a zigzag of arrows between any two objects of $\mathcal{I}$.

00Y7
8.2.2.2 Sets of Connected Components of Categories

Let $\mathcal{C}$ be a category.

00Y8 **Definition 8.2.2.2.1.** The **set of connected components of** $\mathcal{C}$ is the set $\pi_0(\mathcal{C})$ whose elements are the connected components of $\mathcal{C}$.

00Y9 **Proposition 8.2.2.2.2.** Let $\mathcal{C}$ be a category.

00YA 1. *Functoriality.* The assignment $\mathcal{C} \mapsto \pi_0(\mathcal{C})$ defines a functor

$$\pi_0\colon \mathsf{Cats} \to \mathsf{Sets}.$$

00YB 2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})\colon \begin{tikzcd} \mathsf{Sets} \arrow[r, shift left, "(-)_{\mathsf{disc}}"{description}] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, shift left, "\mathrm{Obj}"{description}] \end{tikzcd}$$

[9]In other words, a **connected component** of $\mathcal{C}$ is an element of the set $\mathrm{Obj}(\mathcal{C})/\!\sim$

00YC 3. *Interaction With Groupoids.* If $\mathcal{C}$ is a groupoid, then we have an isomorphism of categories

$$\pi_0(\mathcal{C}) \cong \mathrm{K}(\mathcal{C}),$$

where $\mathrm{K}(\mathcal{C})$ is the set of isomorphism classes of $\mathcal{C}$ of **??**.

00YD 4. *Preservation of Colimits.* The functor π_0 of Item 1 preserves colimits. In particular, we have bijections of sets

$$\pi_0(\mathcal{C} \textstyle\coprod \mathcal{D}) \cong \pi_0(\mathcal{C}) \textstyle\coprod \pi_0(\mathcal{D}),$$
$$\pi_0(\mathcal{C} \textstyle\coprod_{\mathcal{E}} \mathcal{D}) \cong \pi_0(\mathcal{C}) \textstyle\coprod_{\pi_0(\mathcal{E})} \pi_0(\mathcal{D}),$$
$$\pi_0\left(\mathrm{CoEq}\left(\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{D}\right)\right) \cong \mathrm{CoEq}\left(\pi_0(\mathcal{C}) \underset{\pi_0(G)}{\overset{\pi_0(F)}{\rightrightarrows}} \pi_0(\mathcal{D})\right),$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

00YE 5. *Symmetric Strong Monoidality With Respect to Coproducts.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\coprod}, \pi_{0|\mathbb{1}}^{\coprod}\right) \colon (\mathsf{Cats}, \textstyle\coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Sets}, \textstyle\coprod, \emptyset),$$

being equipped with isomorphisms

$$\pi_{0|\mathcal{C},\mathcal{D}}^{\coprod} \colon \pi_0(\mathcal{C}) \textstyle\coprod \pi_0(\mathcal{D}) \xrightarrow{\cong} \pi_0(\mathcal{C} \textstyle\coprod \mathcal{D}),$$
$$\pi_{0|\mathbb{1}}^{\coprod} \colon \emptyset \xrightarrow{\cong} \pi_0(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

00YF 6. *Symmetric Strong Monoidality With Respect to Products.* The connected components functor of Item 1 has a symmetric strong monoidal structure

$$\left(\pi_0, \pi_0^{\times}, \pi_{0|\mathbb{1}}^{\times}\right) \colon (\mathsf{Cats}, \times, \mathsf{pt}) \to (\mathsf{Sets}, \times, \mathrm{pt}),$$

being equipped with isomorphisms

$$\pi_{0|\mathcal{C},\mathcal{D}}^{\times} \colon \pi_0(\mathcal{C}) \times \pi_0(\mathcal{D}) \xrightarrow{\cong} \pi_0(\mathcal{C} \times \mathcal{D}),$$
$$\pi_{0|\mathbb{1}}^{\times} \colon \mathrm{pt} \xrightarrow{\cong} \pi_0(\mathsf{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This is proved in Proposition 8.2.1.1.1.
Item 3, Interaction With Groupoids: Clear.
Item 4, Preservation of Colimits: This follows from Item 2 and **??** of **??**.
Item 5, Symmetric Strong Monoidality With Respect to Coproducts: Clear.
Item 6, Symmetric Strong Monoidality With Respect to Products: Clear. □

00YG 8.2.2.3 Connected Categories

00YH **Definition 8.2.2.3.1.** A category $\mathcal{C}$ is **connected** if $\pi_0(\mathcal{C}) \cong \mathrm{pt}$.[10,11]

00YJ 8.2.3 Discrete Categories

00YK **Definition 8.2.3.1.1.** Let X be a set.

00YL 1. The **discrete category on** X is the category X_{disc} where

- *Objects.* We have
$$\mathrm{Obj}(X_{\mathsf{disc}}) \overset{\text{def}}{=} X.$$
- *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{disc}})$, we have
$$\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \overset{\text{def}}{=} \begin{cases} \mathrm{id}_A & \text{if } A = B, \\ \emptyset & \text{if } A \neq B. \end{cases}$$
- *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{disc}})$, the unit map
$$\mathbb{1}_A^{X_{\mathsf{disc}}} \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{disc}}}(A, A)$$
of X_{disc} at A is defined by
$$\mathrm{id}_A^{X_{\mathsf{disc}}} \overset{\text{def}}{=} \mathrm{id}_A.$$
- *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{disc}})$, the composition map
$$\circ_{A,B,C}^{X_{\mathsf{disc}}} \colon \mathrm{Hom}_{X_{\mathsf{disc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) \to \mathrm{Hom}_{X_{\mathsf{disc}}}(A, C)$$
of X_{disc} at (A, B, C) is defined by
$$\mathrm{id}_A \circ \mathrm{id}_A \overset{\text{def}}{=} \mathrm{id}_A.$$

with $\sim$ the equivalence relation generated by the relation $\sim'$ obtained by declaring $A \sim' B$ iff there exists a morphism of $\mathcal{C}$ from A to B.

[10] *Further Terminology:* A category is **disconnected** if it is not connected.
[11] *Example:* A groupoid is connected iff any two of its objects are isomorphic.

2. A category $\mathcal{C}$ is **discrete** if it is equivalent to X_{disc} for some set X.

00YM

00YN **Proposition 8.2.3.1.2.** Let X be a set.

00YP 1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{disc}}$ defines a functor

$$(-)_{\mathsf{disc}} \colon \mathsf{Sets} \to \mathsf{Cats}.$$

00YQ 2. *Adjointness.* We have a quadruple adjunction

$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}})\colon \begin{tikzcd} \mathsf{Sets} \arrow[r, bend left=15, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=15, "\mathrm{Obj}" description] \end{tikzcd}$$

00YR 3. *Symmetric Strong Monoidality With Respect to Coproducts.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}}, (-)^{\coprod}_{\mathsf{disc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \coprod, \emptyset) \to (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}),$$

being equipped with isomorphisms

$$(-)^{\coprod}_{\mathsf{disc}|X,Y} \colon X_{\mathsf{disc}} \coprod Y_{\mathsf{disc}} \xrightarrow{\cong} (X \coprod Y)_{\mathsf{disc}},$$
$$(-)^{\coprod}_{\mathsf{disc}|\mathbb{1}} \colon \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \emptyset_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

00YS 4. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{disc}}, (-)^{\times}_{\mathsf{disc}}, (-)^{\times}_{\mathsf{disc}|\mathbb{1}}\right) \colon (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathsf{pt}),$$

being equipped with isomorphisms

$$(-)^{\times}_{\mathsf{disc}|X,Y} \colon X_{\mathsf{disc}} \times Y_{\mathsf{disc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{disc}},$$
$$(-)^{\times}_{\mathsf{disc}|\mathbb{1}} \colon \mathsf{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathsf{disc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. *Item 1, Functoriality*: Clear.
Item 2, Adjointness: This is proved in Proposition 8.2.1.1.1.
Item 3, Symmetric Strong Monoidality With Respect to Coproducts: Clear.
Item 4, Symmetric Strong Monoidality With Respect to Products: Clear.
□

00YT 8.2.4 Indiscrete Categories

00YU **Definition 8.2.4.1.1.** Let X be a set.

00YV 1. The **indiscrete category on** X [12] is the category X_{indisc} where

   - *Objects.* We have
   $$\mathrm{Obj}(X_{\mathsf{indisc}}) \stackrel{\text{def}}{=} X.$$
   - *Morphisms.* For each $A, B \in \mathrm{Obj}(X_{\mathsf{indisc}})$, we have
   $$\begin{aligned}\mathrm{Hom}_{X_{\mathsf{disc}}}(A, B) &\stackrel{\text{def}}{=} \{[A] \to [B]\} \\ &\cong \mathrm{pt}.\end{aligned}$$
   - *Identities.* For each $A \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the unit map
   $$\mathbb{1}^{X_{\mathsf{indisc}}}_A \colon \mathrm{pt} \to \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, A)$$
   of X_{indisc} at A is defined by
   $$\mathrm{id}^{X_{\mathsf{indisc}}}_A \stackrel{\text{def}}{=} \{[A] \to [A]\}.$$
   - *Composition.* For each $A, B, C \in \mathrm{Obj}(X_{\mathsf{indisc}})$, the composition map
   $$\circ^{X_{\mathsf{indisc}}}_{A,B,C} \colon \mathrm{Hom}_{X_{\mathsf{indisc}}}(B, C) \times \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, B) \to \mathrm{Hom}_{X_{\mathsf{indisc}}}(A, C)$$
   of X_{disc} at (A, B, C) is defined by
   $$([B] \to [C]) \circ ([A] \to [B]) \stackrel{\text{def}}{=} ([A] \to [C]).$$

2. A category $\mathcal{C}$ is **indiscrete** if it is equivalent to X_{indisc} for some
00YW set X.

00YX **Proposition 8.2.4.1.2.** Let X be a set.

00YY 1. *Functoriality.* The assignment $X \mapsto X_{\mathsf{indisc}}$ defines a functor
$$(-)_{\mathsf{indisc}} \colon \mathsf{Sets} \to \mathsf{Cats}.$$

00YZ 2. *Adjointness.* We have a quadruple adjunction
$$(\pi_0 \dashv (-)_{\mathsf{disc}} \dashv \mathrm{Obj} \dashv (-)_{\mathsf{indisc}}) \colon \begin{tikzcd} \mathsf{Sets} \arrow[r, "(-)_{\mathsf{disc}}" description] \arrow[r, bend right=60, "(-)_{\mathsf{indisc}}"'] & \mathsf{Cats}. \arrow[l, bend right=60, "\pi_0"'] \arrow[l, bend left=20, "\mathrm{Obj}" description] \end{tikzcd}$$

[12] *Further Terminology:* Sometimes called the **chaotic category on** X.

00Z0 3. *Symmetric Strong Monoidality With Respect to Products.* The functor of Item 1 has a symmetric strong monoidal structure

$$\left((-)_{\mathsf{indisc}}, (-)^{\times}_{\mathsf{indisc}}, (-)^{\times}_{\mathsf{indisc}|\mathbb{1}}\right) : (\mathsf{Sets}, \times, \mathrm{pt}) \to (\mathsf{Cats}, \times, \mathsf{pt}),$$

being equipped with isomorphisms

$$(-)^{\times}_{\mathsf{indisc}|X,Y} : X_{\mathsf{indisc}} \times Y_{\mathsf{indisc}} \xrightarrow{\cong} (X \times Y)_{\mathsf{indisc}},$$
$$(-)^{\times}_{\mathsf{indisc}|\mathbb{1}} : \mathsf{pt} \xrightarrow{\cong} \mathrm{pt}_{\mathsf{indisc}},$$

natural in $X, Y \in \mathrm{Obj}(\mathsf{Sets})$.

Proof. Item 1, Functoriality: Clear.
Item 2, Adjointness: This is proved in Proposition 8.2.1.1.1.
Item 3, Symmetric Strong Monoidality With Respect to Products: Clear. □

00Z1 8.3 Groupoids

00Z2 8.3.1 Foundations

Let $\mathcal{C}$ be a category.

00Z3 **Definition 8.3.1.1.1.** A morphism $f : A \to B$ of $\mathcal{C}$ is an **isomorphism** if there exists a morphism $f^{-1} : B \to A$ of $\mathcal{C}$ such that

$$f \circ f^{-1} = \mathrm{id}_B,$$
$$f^{-1} \circ f = \mathrm{id}_A.$$

00Z4 **Notation 8.3.1.1.2.** We write $\mathrm{Iso}_{\mathcal{C}}(A, B)$ for the set of all isomorphisms in $\mathcal{C}$ from A to B.

00Z5 **Definition 8.3.1.1.3.** A **groupoid** is a category in which every morphism is an isomorphism.

00Z6 8.3.2 The Groupoid Completion of a Category

Let $\mathcal{C}$ be a category.

00Z7 **Definition 8.3.2.1.1.** The **groupoid completion of $\mathcal{C}$**[13] is the pair $(\mathrm{K}_0(\mathcal{C}), \iota_{\mathcal{C}})$ consisting of

- A groupoid $\mathrm{K}_0(\mathcal{C})$;

[13] *Further Terminology:* Also called the **Grothendieck groupoid of $\mathcal{C}$** or the **Grothendieck groupoid completion of $\mathcal{C}$**.

- A functor $\iota_{\mathcal{C}} : \mathcal{C} \to \mathrm{K}_0(\mathcal{C})$;

satisfying the following universal property:[14]

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathrm{K}_0(\mathcal{C}) \xrightarrow{\exists!} \mathcal{G}$ making the diagram

$$\begin{tikzcd} & \mathrm{K}_0(\mathcal{C}) \arrow[d, dashed, "\exists!"] \\ \mathcal{C} \arrow[ur, "\iota_{\mathcal{C}}"] \arrow[r, "i"'] & \mathcal{G} \end{tikzcd}$$

commute.

00Z8 **Construction 8.3.2.1.2.** Concretely, the groupoid completion of $\mathcal{C}$ is the Gabriel–Zisman localisation $\mathrm{Mor}(\mathcal{C})^{-1}\mathcal{C}$ of $\mathcal{C}$ at the set $\mathrm{Mor}(\mathcal{C})$ of all morphisms of $\mathcal{C}$; see **??**.
(To be expanded upon later on.)

Proof. Omitted. □

00Z9 **Proposition 8.3.2.1.3.** Let $\mathcal{C}$ be a category.

00ZA 1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a functor

$$\mathrm{K}_0 : \mathsf{Cats} \to \mathsf{Grpd}.$$

00ZB 2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathrm{K}_0(\mathcal{C})$ defines a 2-functor

$$\mathrm{K}_0 : \mathsf{Cats}_2 \to \mathsf{Grpd}_2.$$

00ZC 3. *Adjointness.* We have an adjunction

$$(\mathrm{K}_0 \dashv \iota) : \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, shift left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp"] & \mathsf{Grpd}, \arrow[l, shift left, hook, "\iota"] \end{tikzcd}$$

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}),$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$, forming, together with the functor Core of Item 1 of Proposition 8.3.3.1.4, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core}) : \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathrm{K}_0"] \arrow[r, bend right, "\mathsf{Core}"'] & \mathsf{Grpd}, \arrow[l, hook, "\iota"'] \end{tikzcd}$$

[14] See Item 5 of Proposition 8.3.2.1.3 for an explicit construction.

witnessed by bijections of sets

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}),\mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C},\mathcal{G}),\\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G},\mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G},\mathsf{Core}(\mathcal{D})),\end{aligned}$$

natural in $\mathcal{C},\mathcal{D}\in\mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G}\in\mathrm{Obj}(\mathsf{Grpd})$.

00ZD 4. *2-Adjointness.* We have a 2-adjunction

$$(\mathrm{K}_0 \dashv \iota)\colon \quad \begin{tikzcd}\mathsf{Cats} \arrow[r, shift left, "\mathrm{K}_0"] \arrow[r, phantom, "\perp_2"] & \mathsf{Grpd} \arrow[l, shift left, "\iota"]\end{tikzcd},$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathrm{K}_0(\mathcal{C}),\mathcal{G}) \cong \mathsf{Fun}(\mathcal{C},\mathcal{G}),$$

natural in $\mathcal{C}\in\mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G}\in\mathrm{Obj}(\mathsf{Grpd})$, forming, together with the 2-functor Core of Item 2 of Proposition 8.3.3.1.4, a triple 2-adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core})\colon \quad \begin{tikzcd}\mathsf{Cats} \arrow[r, bend left, "\mathrm{K}_0"] \arrow[r, bend right, "\mathsf{Core}"'] & \mathsf{Grpd} \arrow[l, "\iota" description]\end{tikzcd},$$

witnessed by isomorphisms of categories

$$\begin{aligned}\mathsf{Fun}(\mathrm{K}_0(\mathcal{C}),\mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C},\mathcal{G}),\\ \mathsf{Fun}(\mathcal{G},\mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G},\mathsf{Core}(\mathcal{D})),\end{aligned}$$

natural in $\mathcal{C},\mathcal{D}\in\mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G}\in\mathrm{Obj}(\mathsf{Grpd})$.

00ZE 5. *Interaction With Classifying Spaces.* We have an isomorphism of groupoids

$$\mathrm{K}_0(\mathcal{C}) \cong \Pi_{\leq 1}(|\mathrm{N}_\bullet(\mathcal{C})|),$$

natural in $\mathcal{C}\in\mathrm{Obj}(\mathsf{Cats})$; i.e. the diagram

$$\begin{tikzcd}\mathsf{Cats} \arrow[r, "\mathrm{K}_0"] \arrow[d, "\mathrm{N}_\bullet"'] \arrow[dr, phantom, "\wr"] & \mathsf{Grp} \\ \mathsf{sSets} \arrow[r, "|-|"'] & \mathsf{Top} \arrow[u, "\Pi_{\leq 1}"']\end{tikzcd}$$

commutes up to natural isomorphism.

00ZF 6. *Symmetric Strong Monoidality With Respect to Coproducts.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathrm{K}_0, \mathrm{K}_0^{\coprod}, \mathrm{K}_{0|\mathbb{1}}^{\coprod}\right) : (\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}) \to (\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}})$$

being equipped with isomorphisms

$$\mathrm{K}_{0|\mathcal{C},\mathcal{D}}^{\coprod} : \mathrm{K}_0(\mathcal{C}) \coprod \mathrm{K}_0(\mathcal{D}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{C} \coprod \mathcal{D}),$$
$$\mathrm{K}_{0|\mathbb{1}}^{\coprod} : \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \mathrm{K}_0(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

00ZG 7. *Symmetric Strong Monoidality With Respect to Products.* The groupoid completion functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathrm{K}_0, \mathrm{K}_0^{\times}, \mathrm{K}_{0|\mathbb{1}}^{\times}\right) : (\mathsf{Cats}, \times, \mathsf{pt}) \to (\mathsf{Grpd}, \times, \mathsf{pt})$$

being equipped with isomorphisms

$$\mathrm{K}_{0|\mathcal{C},\mathcal{D}}^{\times} : \mathrm{K}_0(\mathcal{C}) \times \mathrm{K}_0(\mathcal{D}) \xrightarrow{\cong} \mathrm{K}_0(\mathcal{C} \times \mathcal{D}),$$
$$\mathrm{K}_{0|\mathbb{1}}^{\times} : \mathsf{pt} \xrightarrow{\cong} \mathrm{K}_0(\mathsf{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, 2-Functoriality: Omitted.
Item 3, Adjointness: Omitted.
Item 4, 2-Adjointness: Omitted.
Item 5, Interaction With Classifying Spaces: See Corollary 18.33 of `https://web.ma.utexas.edu/users/dafr/M392C-2012/Notes/lecture18.pdf`.
Item 6, Symmetric Strong Monoidality With Respect to Coproducts: Omitted.
Item 7, Symmetric Strong Monoidality With Respect to Products: Omitted. □

00ZH 8.3.3 The Core of a Category

Let $\mathcal{C}$ be a category.

00ZJ **Definition 8.3.3.1.1.** The **core** of $\mathcal{C}$ is the pair $(\mathsf{Core}(\mathcal{C}), \iota_{\mathcal{C}})$ consisting of

- A groupoid $\mathsf{Core}(\mathcal{C})$;

- A functor $\iota_{\mathcal{C}} \colon \mathsf{Core}(\mathcal{C}) \hookrightarrow \mathcal{C}$;

satisfying the following universal property:

(UP) Given another such pair $(\mathcal{G}, i)$, there exists a unique functor $\mathcal{G} \xrightarrow{\exists!} \mathsf{Core}(\mathcal{C})$ making the diagram

$$\begin{tikzcd} & \mathsf{Core}(\mathcal{C}) \arrow[d, hook, "\iota_{\mathcal{C}}"] \\ \mathcal{G} \arrow[ur, dashed, "\exists!"] \arrow[r, "i"'] & \mathcal{C} \end{tikzcd}$$

commute.

00ZK **Notation 8.3.3.1.2.** We also write $\mathcal{C}^{\simeq}$ for $\mathsf{Core}(\mathcal{C})$.

00ZL **Construction 8.3.3.1.3.** The core of $\mathcal{C}$ is the wide subcategory of $\mathcal{C}$ spanned by the isomorphisms of $\mathcal{C}$, i.e. the category $\mathsf{Core}(\mathcal{C})$ where[15]

1. *Objects.* We have
$$\mathrm{Obj}(\mathsf{Core}(\mathcal{C})) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathcal{C}).$$
2. *Morphisms.* The morphisms of $\mathsf{Core}(\mathcal{C})$ are the isomorphisms of $\mathcal{C}$.

Proof. This follows from the fact that functors preserve isomorphisms (Item 1 of Proposition 8.4.1.1.6). □

00ZM **Proposition 8.3.3.1.4.** Let $\mathcal{C}$ be a category.

00ZN 1. *Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a functor
$$\mathsf{Core} \colon \mathsf{Cats} \to \mathsf{Grpd}.$$

00ZP 2. *2-Functoriality.* The assignment $\mathcal{C} \mapsto \mathsf{Core}(\mathcal{C})$ defines a 2-functor
$$\mathsf{Core} \colon \mathsf{Cats}_2 \to \mathsf{Grpd}_2.$$

00ZQ 3. *Adjointness.* We have an adjunction

$$(\iota \dashv \mathsf{Core}) \colon \quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, bend left, "\iota"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Core}"] \end{tikzcd}$$

[15] *Slogan:* The groupoid $\mathsf{Core}(\mathcal{C})$ is the maximal subgroupoid of $\mathcal{C}$.

witnessed by a bijection of sets

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) \cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the functor K_0 of Item 1 of Proposition 8.3.2.1.3, a triple adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core}): \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] \arrow[r, phantom, shift left=2ex, "\perp"] \arrow[r, phantom, shift right=2ex, "\perp"] & \mathsf{Grpd}, \arrow[l, hook', "\iota" description] \end{tikzcd}$$

witnessed by bijections of sets

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{G}), \\ \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{G}, \mathcal{D}) &\cong \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

00ZR 4. *2-Adjointness.* We have an adjunction

$$(\iota \dashv \mathsf{Core}): \quad \begin{tikzcd} \mathsf{Grpd} \arrow[r, hook, bend left, "\iota"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Core}"] \end{tikzcd}$$

witnessed by an isomorphism of categories

$$\mathsf{Fun}(\mathcal{G}, \mathcal{D}) \cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})),$$

natural in $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$ and $\mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, forming, together with the 2-functor K_0 of Item 2 of Proposition 8.3.2.1.3, a triple 2-adjunction

$$(\mathrm{K}_0 \dashv \iota \dashv \mathsf{Core}): \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left=50, "\mathrm{K}_0"] \arrow[r, bend right=50, "\mathsf{Core}"'] \arrow[r, phantom, shift left=2ex, "\perp_2"] \arrow[r, phantom, shift right=2ex, "\perp_2"] & \mathsf{Grpd}, \arrow[l, hook', "\iota" description] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\begin{aligned} \mathsf{Fun}(\mathrm{K}_0(\mathcal{C}), \mathcal{G}) &\cong \mathsf{Fun}(\mathcal{C}, \mathcal{G}), \\ \mathsf{Fun}(\mathcal{G}, \mathcal{D}) &\cong \mathsf{Fun}(\mathcal{G}, \mathsf{Core}(\mathcal{D})), \end{aligned}$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$ and $\mathcal{G} \in \mathrm{Obj}(\mathsf{Grpd})$.

00ZS 5. *Symmetric Strong Monoidality With Respect to Products.* The core functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathsf{Core}, \mathsf{Core}^{\times}, \mathsf{Core}^{\times}_{\mathbb{1}}\right) : (\mathsf{Cats}, \times, \mathrm{pt}) \to (\mathsf{Grpd}, \times, \mathrm{pt})$$

being equipped with isomorphisms

$$\mathsf{Core}^{\times}_{\mathcal{C},\mathcal{D}} : \mathsf{Core}(\mathcal{C}) \times \mathsf{Core}(\mathcal{D}) \xrightarrow{\cong} \mathsf{Core}(\mathcal{C} \times \mathcal{D}),$$
$$\mathsf{Core}^{\times}_{\mathbb{1}} : \mathrm{pt} \xrightarrow{\cong} \mathsf{Core}(\mathrm{pt}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

00ZT 6. *Symmetric Strong Monoidality With Respect to Coproducts.* The core functor of Item 1 has a symmetric strong monoidal structure

$$\left(\mathsf{Core}, \mathsf{Core}^{\coprod}, \mathsf{Core}^{\coprod}_{\mathbb{1}}\right) : \left(\mathsf{Cats}, \coprod, \emptyset_{\mathsf{cat}}\right) \to \left(\mathsf{Grpd}, \coprod, \emptyset_{\mathsf{cat}}\right)$$

being equipped with isomorphisms

$$\mathsf{Core}^{\coprod}_{\mathcal{C},\mathcal{D}} : \mathsf{Core}(\mathcal{C}) \coprod \mathsf{Core}(\mathcal{D}) \xrightarrow{\cong} \mathsf{Core}(\mathcal{C} \coprod \mathcal{D}),$$
$$\mathsf{Core}^{\coprod}_{\mathbb{1}} : \emptyset_{\mathsf{cat}} \xrightarrow{\cong} \mathsf{Core}(\emptyset_{\mathsf{cat}}),$$

natural in $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$.

Proof. Item 1, Functoriality: Omitted.
Item 2, 2-Functoriality: Omitted.
Item 3, Adjointness: Omitted.
Item 4, 2-Adjointness: Omitted.
Item 5, Symmetric Strong Monoidality With Respect to Products: Omitted.
Item 6, Symmetric Strong Monoidality With Respect to Coproducts: Omitted. □

00ZU 8.4 Functors

00ZV 8.4.1 Foundations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

00ZW **Definition 8.4.1.1.1.** A **functor $F: \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$**[16] consists of:

[16] *Further Terminology:* Also called a **covariant functor**.

1. *Action on Objects.* A map of sets
$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}),$$
called the **action on objects of F**.

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map
$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(A),F(B)),$$
called the **action on morphisms of F at (A,B)**[17].

satisfying the following conditions:

1. *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram
$$\begin{tikzcd} \mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F(A)}"] & \\ \mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "F_{A,A}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A),F(A)) \end{tikzcd}$$
commutes, i.e. we have
$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}.$$

2. *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram
$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r, "\circ^{\mathcal{C}}_{A,B,C}"] \arrow[d, "F_{B,C} \times F_{A,B}"'] & \mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[d, "F_{A,C}"] \\ \mathrm{Hom}_{\mathcal{D}}(F(B),F(C)) \times \mathrm{Hom}_{\mathcal{D}}(F(A),F(B)) \arrow[r, "\circ^{\mathcal{D}}_{F(A),F(B),F(C)}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A),F(C)) \end{tikzcd}$$
commutes, i.e. for each composable pair (g,f) of morphisms of $\mathcal{C}$, we have
$$F(g \circ f) = F(g) \circ F(f).$$

00ZX **Notation 8.4.1.1.2.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and write $\mathcal{C}^{\mathsf{op}}$ for the opposite category of $\mathcal{C}$ of **??**.

00ZY 1. Given a functor
$$F\colon \mathcal{C} \to \mathcal{D},$$
we also write F_A for $F(A)$.

[17] *Further Terminology:* Also called **action on Hom-sets of F at (A,B)**.

00ZZ 2. Given a functor

$$F\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D},$$

we also write F^A for $F(A)$.

0100 3. Given a functor

$$F\colon \mathcal{C} \times \mathcal{C} \to \mathcal{D},$$

we also write $F_{A,B}$ for $F(A,B)$.

0101 4. Given a functor

$$F\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D},$$

we also write F^A_B for $F(A,B)$.

We employ a similar notation for morphisms, writing e.g. F_f for $F(f)$ given a functor $F\colon \mathcal{C} \to \mathcal{D}$.

0102 **Notation 8.4.1.1.3.** Following the notation $[\![x \mapsto f(x)]\!]$ for a function $f\colon X \to Y$ introduced in Notation 1.1.1.1.2, we will sometimes denote a functor $F\colon \mathcal{C} \to \mathcal{D}$ by

$$F \stackrel{\text{def}}{=} [\![A \mapsto F(A)]\!],$$

specially when the action on morphisms of F is clear from its action on objects.

0103 **Example 8.4.1.1.4.** The **identity functor** of a category $\mathcal{C}$ is the functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ where

1. *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$\mathrm{id}_{\mathcal{C}}(A) \stackrel{\text{def}}{=} A.$$

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$(\mathrm{id}_{\mathcal{C}})_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \underbrace{\mathrm{Hom}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}}(A), \mathrm{id}_{\mathcal{C}}(B))}_{\stackrel{\text{def}}{=}\mathrm{Hom}_{\mathcal{C}}(A,B)}$$

of $\mathrm{id}_{\mathcal{C}}$ at (A,B) is defined by

$$(\mathrm{id}_{\mathcal{C}})_{A,B} \stackrel{\text{def}}{=} \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(A,B)}.$$

Proof. Preservation of Identities: We have $\mathrm{id}_{\mathcal{C}}(\mathrm{id}_A) \stackrel{\text{def}}{=} \mathrm{id}_A$ for each $A \in \mathrm{Obj}(\mathcal{C})$ by definition.

Preservation of Compositions: For each composable pair $A \xrightarrow{f} B \xrightarrow{g} B$ of morphisms of $\mathcal{C}$, we have

$$\begin{aligned}\mathrm{id}_{\mathcal{C}}(g \circ f) &\stackrel{\text{def}}{=} g \circ f\\ &\stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{C}}(g) \circ \mathrm{id}_{\mathcal{C}}(f).\end{aligned}$$

This finishes the proof. □

0104 **Definition 8.4.1.1.5.** The **composition** of two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ is the functor $G \circ F$ where

- *Action on Objects.* For each $A \in \mathrm{Obj}(\mathcal{C})$, we have
$$[G \circ F](A) \stackrel{\text{def}}{=} G(F(A)).$$
- *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms
$$(G \circ F)_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{E}}(G_{F_A}, G_{F_B})$$
of $G \circ F$ at (A, B) is defined by
$$[G \circ F](f) \stackrel{\text{def}}{=} G(F(f)).$$

Proof. Preservation of Identities: For each $A \in \mathrm{Obj}(\mathcal{C})$, we have
$$\begin{aligned} G_{F_{\mathrm{id}_A}} &= G_{\mathrm{id}_{F_A}} && \text{(functoriality of } F\text{)} \\ &= \mathrm{id}_{G_{F_A}}. && \text{(functoriality of } G\text{)} \end{aligned}$$

Preservation of Composition: For each composable pair (g, f) of morphisms of $\mathcal{C}$, we have
$$\begin{aligned} G_{F_{g\circ f}} &= G_{F_g \circ F_f} && \text{(functoriality of } F\text{)} \\ &= G_{F_g} \circ G_{F_f}. && \text{(functoriality of } G\text{)} \end{aligned}$$

This finishes the proof. □

0105 **Proposition 8.4.1.1.6.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

0106 1. *Preservation of Isomorphisms.* If f is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.[18]

Proof. Item 1, Preservation of Isomorphisms: Indeed, we have
$$\begin{aligned} F(f)^{-1} \circ F(f) &= F\left(f^{-1} \circ f\right) \\ &= F(\mathrm{id}_A) \\ &= \mathrm{id}_{F(A)} \end{aligned}$$
and
$$\begin{aligned} F(f) \circ F(f)^{-1} &= F\left(f \circ f^{-1}\right) \\ &= F(\mathrm{id}_B) \\ &= \mathrm{id}_{F(B)}, \end{aligned}$$
showing $F(f)$ to be an isomorphism. □

[18] When the converse holds, we call F *conservative*, see Definition 8.5.4.1.1.

0107 8.4.2 Contravariant Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\mathcal{C}^{\mathsf{op}}$ denote the opposite category of $\mathcal{C}$ of ??.

0108 **Definition 8.4.2.1.1.** A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathsf{op}}$ to $\mathcal{D}$.

0109 **Remark 8.4.2.1.2.** In detail, a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ consists of:

1. *Action on Objects.* A map of sets
$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}),$$
called the **action on objects of** F.

2. *Action on Morphisms.* For each $A, B \in \mathrm{Obj}(\mathcal{C})$, a map
$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(B),F(A)),$$
called the **action on morphisms of** F **at** (A,B).

satisfying the following conditions:

1. *Preservation of Identities.* For each $A \in \mathrm{Obj}(\mathcal{C})$, the diagram
$$\begin{tikzcd}
\mathrm{pt} \arrow[d, "\mathbb{1}^{\mathcal{C}}_{A}"'] \arrow[dr, "\mathbb{1}^{\mathcal{D}}_{F(A)}"] & \\
\mathrm{Hom}_{\mathcal{C}}(A,A) \arrow[r, "F_{A,A}"'] & \mathrm{Hom}_{\mathcal{D}}(F(A),F(A))
\end{tikzcd}$$
commutes, i.e. we have
$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}.$$

2. *Preservation of Composition.* For each $A, B, C \in \mathrm{Obj}(\mathcal{C})$, the diagram
$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathcal{D}}(F(C),F(B)) \times \mathrm{Hom}_{\mathcal{D}}(F(B),F(A)) \arrow[dr, dashed, "\sigma^{\mathsf{Sets}}_{\mathrm{Hom}_{\mathcal{D}}(F(C),F(B)),\mathrm{Hom}_{\mathcal{D}}(F(B),F(A))}", "\sim"'] & \\
\mathrm{Hom}_{\mathcal{C}}(B,C) \times \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[ur, "F_{B,C}\times F_{A,B}"] \arrow[d, "\circ^{\mathcal{C}}_{A,B,C}"'] & & \mathrm{Hom}_{\mathcal{D}}(F(B),F(A)) \times \mathrm{Hom}_{\mathcal{D}}(F(C),F(B)) \arrow[d, "\circ^{\mathcal{D}}_{F(C),F(B),F(A)}"] \\
\mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[rr, "F_{A,C}"'] & & \mathrm{Hom}_{\mathcal{D}}(F(C),F(A))
\end{tikzcd}$$

commutes, i.e. for each composable pair (g, f) of morphisms of $\mathcal{C}$, we have

$$F(g \circ f) = F(f) \circ F(g).$$

010A **Remark 8.4.2.1.3.** Throughout this work we will not use the term "contravariant" functor, speaking instead simply of functors $F\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$. We will usually, however, write

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$$

for the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

of F, as well as write $F(g \circ f) = F(f) \circ F(g)$.

010B 8.4.3 Forgetful Functors

010C **Definition 8.4.3.1.1.** There isn't a precise definition of a **forgetful functor**.

010D **Remark 8.4.3.1.2.** Despite there not being a formal or precise definition of a forgetful functor, the term is often very useful in practice, similarly to the word "canonical". The idea is that a "forgetful functor" is a functor that forgets structure or properties, and is best explained through examples, such as the ones below (see Examples 8.4.3.1.3 and 8.4.3.1.4).

010E **Example 8.4.3.1.3.** Examples of forgetful functors that forget structure include:

010F 1. *Forgetting Group Structures.* The functor $\mathsf{Grp} \to \mathsf{Sets}$ sending a group (G, μ_G, η_G) to its underlying set G, forgetting the multiplication and unit maps μ_G and η_G of G.

010G 2. *Forgetting Topologies.* The functor $\mathsf{Top} \to \mathsf{Sets}$ sending a topological space $(X, \mathcal{T}_X)$ to its underlying set X, forgetting the topology $\mathcal{T}_X$.

010H 3. *Forgetting Fibrations.* The functor $\mathsf{FibSets}(K) \to \mathsf{Sets}$ sending a K-fibred set $\phi_X\colon X \to K$ to the set X, forgetting the map ϕ_X and the base set K.

010J **Example 8.4.3.1.4.** Examples of forgetful functors that forget properties include:

010K 1. *Forgetting Commutativity.* The inclusion functor $\iota\colon \mathsf{CMon} \hookrightarrow \mathsf{Mon}$ which forgets the property of being commutative.

010L 2. *Forgetting Inverses.* The inclusion functor $\iota\colon \mathsf{Grp} \hookrightarrow \mathsf{Mon}$ which forgets the property of having inverses.

010M **Notation 8.4.3.1.5.** Throughout this work, we will denote forgetful functors that forget structure by 忘, e.g. as in

$$忘\colon \mathsf{Grp} \to \mathsf{Sets}.$$

The symbol 忘, pronounced *wasureru* (see Item 1 of Remark 8.4.3.1.6 below), means *to forget*, and is a kanji found in the following words in Japanese and Chinese:

010N 1. 忘れる, transcribed as *wasureru*, meaning *to forget*.

2. 忘却関手, transcribed as *boukyaku kanshu*, meaning *forgetful func-*
010P *tor*.

010Q 3. 忘记 or 忘記, transcribed as *wàngjì*, meaning *to forget*.

4. 遗忘函子 or 遺忘函子, transcribed as *yíwàng hánzǐ*, meaning *for-*
010R *getful functor*.

010S **Remark 8.4.3.1.6.** Here we collect the pronunciation of the words in Notation 8.4.3.1.5 for accuracy and completeness.

010T 1. Pronunciation of 忘れる:

   - Audio: see https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-01.mp3
   - IPA broad transcription: [wäsɯɾe̞ɾɯ].
   - IPA narrow transcription: [ɰ$^{\beta}$äsɨ$^{\beta}$ɾe̞ɾɯ̟$^{\beta}$].

010U 2. Pronunciation of 忘却関手: Pronunciation:

   - Audio: see https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-02.mp3
   - IPA broad transcription: [bo̞ːk^{j}äkɯ kä̃ɰ̃ɕɯ].
   - IPA narrow transcription: [bo̞ːk^{j}äkɯ̟$^{\beta}$ kä̃ɰ̃ɕɯ$^{\beta}$].

010V 3. Pronunciation of 忘记:

   - Audio: see https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-03.ogg
   - Broad IPA transcription: [wɑŋtɕi].
   - Sinological IPA transcription: [wɑŋ$^{51-53}$t͡ɕi^{51}].

010W 4. Pronunciation of 遗忘函子:

- Audio: see https://topological-modular-forms.github.io/the-clowder-project/static/sounds/wasureru-04.mp3
- Broad IPA transcription: [iwɑŋ xänt͡ʂʐɨ].
- Sinological IPA transcription: [i^{35}wɑŋ^{51} xän^{35}t͡ʂʐ̩$^{214-21(4)}$].

010X 8.4.4 The Natural Transformation Associated to a Functor

010Y **Definition 8.4.4.1.1.** Every functor $F\colon \mathcal{C} \to \mathcal{D}$ defines a natural transformation[19]

$$F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}} \circ (F^{\mathsf{op}} \times F), \qquad \begin{tikzcd} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \arrow[rr, "F^{\mathsf{op}} \times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \arrow[d, phantom, "\overset{F^{\dagger}}{\Longrightarrow}"] & \mathcal{D}^{\mathsf{op}} \times \mathcal{D} \arrow[dl, "\mathrm{Hom}_{\mathcal{D}}"] \\ & \mathsf{Sets}, & \end{tikzcd}$$

called the **natural transformation associated to** F, consisting of the collection

$$\left\{F^{\dagger}_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)\right\}_{(A,B)\in\mathrm{Obj}(\mathcal{C}^{\mathsf{op}}\times\mathcal{C})}$$

with

$$F^{\dagger}_{A,B} \overset{\text{def}}{=} F_{A,B}.$$

Proof. The naturality condition for $F^{\dagger}$ is the requirement that for each morphism

$$(\phi,\psi)\colon (X,Y) \to (A,B)$$

of $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "\phi^{*}\circ\psi_{*}=\psi_{*}\circ\phi^{*}"] \arrow[d, "F_{X,Y}"'] & \mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[d, "F_{A,B}"] \\ \mathrm{Hom}_{\mathcal{D}}(F_X,F_Y) \arrow[r, "F(\phi)^{*}\circ F(\psi)_{*}=F(\psi)_{*}\circ F(\phi)^{*}"'] & \mathrm{Hom}_{\mathcal{D}}(F_A,F_B), \end{tikzcd}$$

acting on elements as

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & \psi \circ f \circ \phi \arrow[d, mapsto] \\ F(f) \arrow[r, mapsto] & F(\psi)\circ F(f)\circ F(\psi) = F(\psi\circ f\circ\phi) \end{tikzcd}$$

commutes, which follows from the functoriality of F. □

[19]This is the 1-categorical version of Item 1 of Proposition 2.4.1.1.3.

010Z **Proposition 8.4.4.1.2.** Let $F\colon \mathcal{C}\to\mathcal{D}$ and $G\colon \mathcal{D}\to\mathcal{E}$ be functors.

0110 1. *Interaction With Natural Isomorphisms.* The following conditions are equivalent:

   (a) The natural transformation $F^{\dagger}\colon \mathrm{Hom}_{\mathcal{C}} \Longrightarrow \mathrm{Hom}_{\mathcal{D}}\circ(F^{\mathsf{op}}\times F)$
0111 associated to F is a natural isomorphism.

0112 (b) The functor F is fully faithful.

0113 2. *Interaction With Composition.* We have an equality of pasting diagrams

$$
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}}\times\mathcal{C} \arrow[r, "F^{\mathsf{op}}\times F"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] & \mathcal{D}^{\mathsf{op}}\times\mathcal{D} \arrow[r, "G^{\mathsf{op}}\times G"] \arrow[d, "\mathrm{Hom}_{\mathcal{D}}"] & \mathcal{E}^{\mathsf{op}}\times\mathcal{E} \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \\
 & \mathsf{Sets} &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C}^{\mathsf{op}}\times\mathcal{C} \arrow[rr, "(G\circ F)^{\mathsf{op}}\times(G\circ F)"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}"'] \arrow[dr, phantom, shift left=6, "\overset{(G\circ F)^{\dagger}}{\Longrightarrow}"] & & \mathcal{E}^{\mathsf{op}}\times\mathcal{E}, \arrow[dl, "\mathrm{Hom}_{\mathcal{E}}"] \\
 & \mathsf{Sets} &
\end{tikzcd}
$$

in Cats_{2}, i.e. we have

$$(G\circ F)^{\dagger} = \left(G^{\dagger}\star \mathrm{id}_{F^{\mathsf{op}}\times F}\right)\circ F^{\dagger}.$$

0114 3. *Interaction With Identities.* We have

$$\mathrm{id}_{\mathcal{C}}^{\dagger} = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(-_1,-_2)},$$

i.e. the natural transformation associated to $\mathrm{id}_{\mathcal{C}}$ is the identity natural transformation of the functor $\mathrm{Hom}_{\mathcal{C}}(-_1,-_2)$.

Proof. *Item 1*, *Interaction With Natural Isomorphisms*: Clear.
Item 2, *Interaction With Composition*: Clear.
Item 3, *Interaction With Identities*: Clear. □

0115 8.5 Conditions on Functors

0116 8.5.1 Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

0117 **Definition 8.5.1.1.1.** A functor $F\colon \mathcal{C}\to\mathcal{D}$ is **faithful** if, for each $A,B\in\mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A,B)\to\mathrm{Hom}_{\mathcal{D}}(F_A,F_B)$$

of F at (A,B) is injective.

0118 **Proposition 8.5.1.1.2.** Let $F\colon \mathcal{C}\to\mathcal{D}$ be a functor.

0119 1. *Interaction With Postcomposition.* The following conditions are equivalent:

011A (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is faithful.

011B (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is faithful.

(c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably faithful morphism
011C in Cats_2 in the sense of Definition 9.1.1.1.1.

011D 2. *Interaction With Precomposition I.* Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

011E (a) If F is faithful, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

can fail to be faithful.

011F (b) Conversely, if the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is faithful, then F *can fail* to be faithful.

011G 3. *Interaction With Precomposition II.* If F is essentially surjective, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is faithful.

011H 4. *Interaction With Precomposition III.* The following conditions are equivalent:

011J (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is faithful.

011K (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is conservative.

011L (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is monadic.

(d) The functor $F \colon \mathcal{C} \to \mathcal{D}$ is a corepresentably faithful morphism
011M in Cats_2 in the sense of Definition 9.2.1.1.1.

011N (e) The components

$$\eta_G \colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

of the unit

$$\eta \colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all monomorphisms.

011P (f) The components

$$\epsilon_G \colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon \colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all epimorphisms.

(g) The functor F is dominant (Definition 8.6.1.1.1), i.e. every
011Q object of $\mathcal{D}$ is a retract of some object in $\mathrm{Im}(F)$:

($\star$) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:
- An object A of $\mathcal{C}$;
- A morphism $s \colon B \to F(A)$ of $\mathcal{D}$;
- A morphism $r \colon F(A) \to B$ of $\mathcal{D}$;

such that $r \circ s = \mathrm{id}_B$.

Proof. *Item 1, Interaction With Postcomposition*: Omitted.
Item 2, Interaction With Precomposition I: See [MSE 733163] for Item 2a. Item 2b follows from Item 3 and the fact that there are essentially surjective functors that are not faithful.
Item 3, Interaction With Precomposition II: Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.
Item 4, Interaction With Precomposition III: We claim Items 4a to 4g are equivalent:

- *Items 4a and 4d Are Equivalent:* This is true by the definition of corepresentably faithful morphism; see Definition 9.2.1.1.1.

- *Items 4a to 4c and 4g Are Equivalent:* See [Adá+01, Proposition 4.1] or alternatively [Fre09, Lemmas 3.1 and 3.2] for the equivalence between Items 4a and 4g.

- *Items 4a, 4e and 4f Are Equivalent:* See **??** of **??**.

This finishes the proof. □

011R 8.5.2 Full Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

011S **Definition 8.5.2.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **full** if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is surjective.

011T **Proposition 8.5.2.1.2.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

011U 1. *Interaction With Postcomposition.* The following conditions are equivalent:

011V (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is full.

011W (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is full.

(c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably full morphism in
011X Cats_2 in the sense of Definition 9.1.2.1.1.

011Y 2. *Interaction With Precomposition I.* If F is full, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

can fail to be full.

011Z 3. *Interaction With Precomposition II.* If the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is full, then F *can fail* to be full.

0120 4. *Interaction With Precomposition III.* If F is essentially surjective and full, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

is full (and also faithful by Item 3 of Proposition 8.5.1.1.2).

0121 5. *Interaction With Precomposition IV.* The following conditions are equivalent:

0122 (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

is full.

(b) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a corepresentably full morphism in
0123 Cats_2 in the sense of Definition 9.2.1.1.1.

0124 (c) The components

$$\eta_G\colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

of the unit

$$\eta\colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all retractions/split epimorphisms.

0125 (d) The components

$$\epsilon_G\colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon\colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all sections/split monomorphisms.

0126 (e) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A_B of $\mathcal{C}$;
- A morphism $s_B\colon B \to F(A_B)$ of $\mathcal{D}$;
- A morphism $r_B\colon F(A_B) \to B$ of $\mathcal{D}$;

satisfying the following condition:

($\star$) For each $A \in \mathrm{Obj}(\mathcal{C})$ and each pair of morphisms

$$\begin{aligned} r&\colon F(A) \to B, \\ s&\colon B \to F(A) \end{aligned}$$

of $\mathcal{D}$, we have

$$[(A_B, s_B, r_B)] = [(A, s, r \circ s_B \circ r_B)]$$

in $\int^{A\in\mathcal{C}} h^{B'}_{F_A} \times h^{F_A}_{B}$.

Proof. *Item 1, Interaction With Postcomposition*: Omitted.
Item 2, Interaction With Precomposition I: Omitted.
Item 3, Interaction With Precomposition II: See [BS10, p. 47].
Item 4, Interaction With Precomposition III: Omitted, but see `https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html` for a formalised proof.
Item 5, Interaction With Precomposition IV: We claim Items 5a to 5e are equivalent:

- *Items 5a and 5b Are Equivalent:* This is true by the definition of corepresentably full morphism; see Definition 9.2.2.1.1.
- *Items 5a, 5c and 5d Are Equivalent:* See **??** of **??**.
- *Items 5a and 5e Are Equivalent:* See [Adá+01, Item (b) of Remark 4.3].

This finishes the proof. □

0127 **Question 8.5.2.1.3.** Item 5 of Proposition 8.5.2.1.2 gives a characterisation of the functors F for which F^* is full, but the characterisations given there are really messy. Are there better ones?
This question also appears as [MO 468121b].

0128 8.5.3 Fully Faithful Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

0129 **Definition 8.5.3.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **fully faithful** if F is full and faithful, i.e. if, for each $A, B \in \mathrm{Obj}(\mathcal{C})$, the action on morphisms

$$F_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F_A, F_B)$$

of F at (A, B) is bijective.

012A **Proposition 8.5.3.1.2.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

012B 1. *Characterisations.* The following conditions are equivalent:

012C (a) The functor F is fully faithful.

012D (b) We have a pullback square

$$\mathsf{Arr}(\mathcal{C}) \cong (\mathcal{C} \times \mathcal{C}) \times_{\mathcal{D}\times\mathcal{D}} \mathsf{Arr}(\mathcal{D}), \qquad \begin{tikzcd} \mathsf{Arr}(\mathcal{C}) \arrow[r, "\mathsf{Arr}(F)"] \arrow[d, "\mathrm{src}\times\mathrm{tgt}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Arr}(\mathcal{D}) \arrow[d, "\mathrm{src}\times\mathrm{tgt}"] \\ \mathcal{C}\times\mathcal{C} \arrow[r, "F\times F"'] & \mathcal{D}\times\mathcal{D} \end{tikzcd}$$

in Cats.

012E 2. *Conservativity.* If F is fully faithful, then F is conservative.

012F 3. *Essential Injectivity.* If F is fully faithful, then F is essentially injective.

012G 4. *Interaction With Co/Limits.* If F is fully faithful, then F reflects co/limits.

012H 5. *Interaction With Postcomposition.* The following conditions are equivalent:

012J (a) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful.

012K (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is fully faithful.

(c) The functor $F\colon \mathcal{C} \to \mathcal{D}$ is a representably fully faithful mor-
012L phism in Cats_2 in the sense of Definition 9.1.3.1.1.

012M 6. *Interaction With Precomposition I.* If F is fully faithful, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

can fail to be fully faithful.

012N 7. *Interaction With Precomposition II.* If the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is fully faithful, then F *can fail* to be fully faithful (and in fact it can also fail to be either full or faithful).

012P 8. *Interaction With Precomposition III.* If F is essentially surjective and full, then the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is fully faithful.

012Q 9. *Interaction With Precomposition IV.* The following conditions are equivalent:

012R (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is fully faithful.

012S (b) The precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathsf{Sets}) \to \mathsf{Fun}(\mathcal{C}, \mathsf{Sets})$$

is fully faithful.

012T (c) The functor

$$\mathrm{Lan}_F \colon \mathsf{Fun}(\mathcal{C}, \mathsf{Sets}) \to \mathsf{Fun}(\mathcal{D}, \mathsf{Sets})$$

is fully faithful.

012U (d) The functor F is a corepresentably fully faithful morphism in Cats_2 in the sense of Definition 9.2.3.1.1.

012V (e) The functor F is absolutely dense.

012W (f) The components

$$\eta_G \colon G \Longrightarrow \mathrm{Ran}_F(G \circ F)$$

of the unit

$$\eta \colon \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})} \Longrightarrow \mathrm{Ran}_F \circ F^*$$

of the adjunction $F^* \dashv \mathrm{Ran}_F$ are all isomorphisms.

012X (g) The components

$$\epsilon_G \colon \mathrm{Lan}_F(G \circ F) \Longrightarrow G$$

of the counit

$$\epsilon \colon \mathrm{Lan}_F \circ F^* \Longrightarrow \mathrm{id}_{\mathsf{Fun}(\mathcal{D},\mathcal{X})}$$

of the adjunction $\mathrm{Lan}_F \dashv F^*$ are all isomorphisms.

012Y (h) The natural transformation

$$\alpha \colon \mathrm{Lan}_{h_F}\left(h^F\right) \Longrightarrow h$$

with components

$$\alpha_{B',B} \colon \int^{A \in \mathcal{C}} h^{B'}_{F_A} \times h^{F_A}_{B} \to h^{B'}_{B}$$

given by

$$\alpha_{B',B}([(\phi, \psi)]) = \psi \circ \phi$$

is a natural isomorphism.

012Z (i) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A_B of $\mathcal{C}$;

- A morphism $s_B \colon B \to F(A_B)$ of $\mathcal{D}$;
- A morphism $r_B \colon F(A_B) \to B$ of $\mathcal{D}$;

satisfying the following conditions:

i. The triple $(F(A_B), r_B, s_B)$ is a retract of B, i.e. we have
0130 $r_B \circ s_B = \mathrm{id}_B$.
0131 ii. For each morphism $f \colon B' \to B$ of $\mathcal{D}$, we have

$$[(A_B, s_{B'}, f \circ r_{B'})] = [(A_B, s_B \circ f, r_B)]$$

in $\int^{A \in \mathcal{C}} h^{B'}_{F_A} \times h^{F_A}_{B}$.

Proof. Item 1, Characterisations: Omitted.
Item 2, Conservativity: This is a repetition of Item 2 of Proposition 8.5.4.1.2, and is proved there.
Item 3, Essential Injectivity: Omitted.
Item 4, Interaction With Co/Limits: Omitted.
Item 5, Interaction With Postcomposition: This follows from Item 1 of Proposition 8.5.1.1.2 and Item 1 of Proposition 8.5.2.1.2.
Item 6, Interaction With Precomposition I: See [MSE 733161] for an example of a fully faithful functor whose precomposition with which fails to be full.
Item 7, Interaction With Precomposition II: See [MSE 749304, Item 3].
Item 8, Interaction With Precomposition III: Omitted, but see https://unimath.github.io/doc/UniMath/d4de26f//UniMath.CategoryTheory.precomp_fully_faithful.html for a formalised proof.
Item 9, Interaction With Precomposition IV: We claim Items 9a to 9i are equivalent:

- *Items 9a and 9d Are Equivalent:* This is true by the definition of corepresentably fully faithful morphism; see Definition 9.2.3.1.1.
- *Items 9a, 9f and 9g Are Equivalent:* See **??** of **??**.
- *Items 9a to 9c Are Equivalent:* This follows from [Low15, Proposition A.1.5].
- *Items 9a, 9e, 9h and 9i Are Equivalent:* See [Fre09, Theorem 4.1] and [Adá+01, Theorem 1.1].

This finishes the proof. □

0132 8.5.4 Conservative Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

0133 **Definition 8.5.4.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **conservative** if it satisfies the following condition:[20]

($\star$) For each $f \in \mathrm{Mor}(\mathcal{C})$, if $F(f)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.

0134 **Proposition 8.5.4.1.2.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

0135 1. *Characterisations.* The following conditions are equivalent:

0136 (a) The functor F is conservative.

(b) For each $f \in \mathrm{Mor}(\mathcal{C})$, the morphism $F(f)$ is an isomorphism
0137 in $\mathcal{D}$ iff f is an isomorphism in $\mathcal{C}$.

0138 2. *Interaction With Fully Faithfulness.* Every fully faithful functor is conservative.

0139 3. *Interaction With Precomposition.* The following conditions are equivalent:

013A (a) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is conservative.

(b) The equivalent conditions of Item 4 of Proposition 8.5.1.1.2
013B are satisfied.

Proof. *Item 1, Characterisations*: This follows from Item 1 of Proposition 8.4.1.1.6.

Item 2, Interaction With Fully Faithfulness: Let $F\colon \mathcal{C} \to \mathcal{D}$ be a fully faithful functor, let $f\colon A \to B$ be a morphism of $\mathcal{C}$, and suppose that F_f is an isomorphism. We have

$$\begin{aligned} F(\mathrm{id}_B) &= \mathrm{id}_{F(B)} \\ &= F(f) \circ F(f)^{-1} \\ &= F\Big(f \circ f^{-1}\Big). \end{aligned}$$

Similarly, $F(\mathrm{id}_A) = F\big(f^{-1} \circ f\big)$. But since F is fully faithful, we must have

$$\begin{aligned} f \circ f^{-1} &= \mathrm{id}_B, \\ f^{-1} \circ f &= \mathrm{id}_A, \end{aligned}$$

showing f to be an isomorphism. Thus F is conservative. □

[20] *Slogan:* A functor F is **conservative** if it reflects isomorphisms.

013C **Question 8.5.4.1.3.** Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ satisfying the following condition:

(⋆) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is conservative?

This question also appears as [MO 468121a].

013D 8.5.5 Essentially Injective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

013E **Definition 8.5.5.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially injective** if it satisfies the following condition:

(⋆) For each $A, B \in \mathrm{Obj}(\mathcal{C})$, if $F(A) \cong F(B)$, then $A \cong B$.

013F **Question 8.5.5.1.2.** Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

013G 1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is essentially injective, i.e. if $\phi \circ F \cong \psi \circ F$, then $\phi \cong \psi$ for all functors ϕ and ψ?

013H 2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is essentially injective, i.e. if $F \circ \phi \cong F \circ \psi$, then $\phi \cong \psi$?

This question also appears as [MO 468121a].

013J 8.5.6 Essentially Surjective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

013K **Definition 8.5.6.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially surjective**[21] if it satisfies the following condition:

(⋆) For each $D \in \mathrm{Obj}(\mathcal{D})$, there exists some object A of $\mathcal{C}$ such that $F(A) \cong D$.

[21] *Further Terminology:* Also called an **eso** functor, where the name "eso" comes

013L **Question 8.5.6.1.2.** Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

013M 1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is essentially surjective?

013N 2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is essentially surjective?

This question also appears as [MO 468121a].

013P 8.5.7 Equivalences of Categories

013Q **Definition 8.5.7.1.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

1. An **equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ consists of a
013R pair of functors

$$F\colon \mathcal{C} \to \mathcal{D},$$
$$G\colon \mathcal{D} \to \mathcal{C}$$

together with natural isomorphisms

$$\eta\colon \mathrm{id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} G \circ F,$$
$$\epsilon\colon F \circ G \overset{\sim}{\Longrightarrow} \mathrm{id}_{\mathcal{D}}.$$

2. An **adjoint equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$ is an equivalence (F, G, η, ϵ) between $\mathcal{C}$ and $\mathcal{D}$ which is also an adjunc-
013S tion.

013T **Proposition 8.5.7.1.2.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

013U 1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small[22], then the following conditions are equivalent:[23]

---

from *essentially surjective on objects*.

[22]Otherwise there will be size issues. One can also work with large categories and universes, or require F to be *constructively* essentially surjective; see [MSE 1465107].

[23]In ZFC, the equivalence between Item 1a and Item 1b is equivalent to the axiom of choice; see [MO 119454].

In Univalent Foundations, this is true without requiring neither the axiom of choice nor the law of excluded middle.

013V (a) The functor F is an equivalence of categories.

013W (b) The functor F is fully faithful and essentially surjective.

013X (c) The induced functor

$$F|_{\mathsf{Sk}(\mathcal{C})} \colon \mathsf{Sk}(\mathcal{C}) \to \mathsf{Sk}(\mathcal{D})$$

is an *isomorphism* of categories.

013Y (d) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^* \colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is an equivalence of categories.

013Z (e) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is an equivalence of categories.

0140 2. *Two-Out-of-Three.* Let

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "G \circ F"] \arrow[dr, "F"'] & & \mathcal{E} \\ & \mathcal{D} \arrow[ur, "G"'] & \end{tikzcd}$$

be a diagram in Cats. If two out of the three functors among F, G, and $G \circ F$ are equivalences of categories, then so is the third.

0141 3. *Stability Under Composition.* Let

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \arrow[r, shift left, "F'"] & \mathcal{E} \arrow[l, shift left, "G'"] \end{tikzcd}$$

be a diagram in Cats. If (F, G) and (F', G') are equivalences of categories, then so is their composite $(F' \circ F, G' \circ G)$.

0142 4. *Equivalences vs.Adjoint Equivalences.* Every equivalence of categories can be promoted to an adjoint equivalence.[24]

0143 5. *Interaction With Groupoids.* If $\mathcal{C}$ and $\mathcal{D}$ are groupoids, then the following conditions are equivalent:

0144 (a) The functor F is an equivalence of groupoids.

0145 (b) The following conditions are satisfied:

[24] More precisely, we can promote an equivalence of categories (F, G, η, ϵ) to adjoint

0146 i. The functor F induces a bijection

$$\pi_0(F)\colon \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$$

of sets.

0147 ii. For each $A \in \mathrm{Obj}(\mathcal{C})$, the induced map

$$F_{x,x}\colon \mathrm{Aut}_{\mathcal{C}}(A) \to \mathrm{Aut}_{\mathcal{D}}(F_A)$$

is an isomorphism of groups.

Proof. *Item 1, Characterisations*: We claim that Items 1a to 1e are indeed equivalent:

1. *Item 1a* $\implies$ *Item 1b:* Clear.

2. *Item 1b* $\implies$ *Item 1a:* Since F is essentially surjective and $\mathcal{C}$ and $\mathcal{D}$ are small, we can choose, using the axiom of choice, for each $B \in \mathrm{Obj}(\mathcal{D})$, an object j_B of $\mathcal{C}$ and an isomorphism $i_B\colon B \to F_{j_B}$ of $\mathcal{D}$.

   Since F is fully faithful, we can extend the assignment $B \mapsto j_B$ to a *unique* functor $j\colon \mathcal{D} \to \mathcal{C}$ such that the isomorphisms $i_B\colon B \to F_{j_B}$ assemble into a natural isomorphism $\eta\colon \mathrm{id}_{\mathcal{D}} \overset{\sim}{\Longrightarrow} F \circ j$, with a similar natural isomorphism $\epsilon\colon \mathrm{id}_{\mathcal{C}} \overset{\sim}{\Longrightarrow} j \circ F$. Hence F is an equivalence.

3. *Item 1a* $\implies$ *Item 1c:* This follows from Item 4 of Proposition 8.1.5.1.3.

4. *Item 1c* $\implies$ *Item 1a:* Omitted.

5. *Items 1a, 1d and 1e Are Equivalent:* This follows from **??**.

This finishes the proof of Item 1.
Item 2, Two-Out-of-Three: Omitted.
Item 3, Stability Under Composition: Clear.
Item 4, Equivalences vs.Adjoint Equivalences: See [Rie17, Proposition 4.4.5].
Item 5, Interaction With Groupoids: See [nLa24, Proposition 4.4]. □

0148 8.5.8 Isomorphisms of Categories

0149 **Definition 8.5.8.1.1.** An **isomorphism of categories** is a pair of functors

$$F\colon \mathcal{C} \to \mathcal{D},$$
$$G\colon \mathcal{D} \to \mathcal{C}$$

such that we have

$$G \circ F = \mathrm{id}_{\mathcal{C}},$$

$$F \circ G = \mathrm{id}_{\mathcal{D}}.$$

014A **Example 8.5.8.1.2.** Categories can be equivalent but non-isomorphic. For example, the category consisting of two isomorphic objects is equivalent to pt, but not isomorphic to it.

014B **Proposition 8.5.8.1.3.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

014C 1. *Characterisations.* If $\mathcal{C}$ and $\mathcal{D}$ are small, then the following conditions are equivalent:

014D (a) The functor F is an isomorphism of categories.

014E (b) The functor F is fully faithful and bijective on objects.

014F (c) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is an isomorphism of categories.

014G (d) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is an isomorphism of categories.

Proof. *Item 1, Characterisations*: We claim that Items 1a to 1d are indeed equivalent:

1. *Items 1a and 1b Are Equivalent:* Omitted, but similar to Item 1 of Proposition 8.5.7.1.2.

2. *Items 1a, 1c and 1d Are Equivalent:* This follows from **??**.

This finishes the proof. □

014H 8.6 More Conditions on Functors

014J 8.6.1 Dominant Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

014K **Definition 8.6.1.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **dominant** if every object of $\mathcal{D}$ is a retract of some object in $\mathrm{Im}(F)$, i.e.:

($\star$) For each $B \in \mathrm{Obj}(\mathcal{D})$, there exist:

- An object A of $\mathcal{C}$;
- A morphism $r\colon F(A) \to B$ of $\mathcal{D}$;
- A morphism $s\colon B \to F(A)$ of $\mathcal{D}$;

such that we have

$$r \circ s = \mathrm{id}_B, \qquad \begin{tikzcd} B \arrow[r, "s"] \arrow[dr, "\mathrm{id}_B"'] & F(A) \arrow[d, "r"] \\ & B. \end{tikzcd}$$

014L **Proposition 8.6.1.1.2.** Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors and let $I\colon \mathcal{X} \to \mathcal{C}$ be a functor.

014M 1. *Interaction With Right Whiskering.* If I is full and dominant, then the map

$$- \star \mathrm{id}_I\colon \mathrm{Nat}(F, G) \to \mathrm{Nat}(F \circ I, G \circ I)$$

is a bijection.

014N 2. *Interaction With Adjunctions.* Let $(F, G)\colon \mathcal{C} \rightleftarrows \mathcal{D}$ be an adjunction.

014P (a) If F is dominant, then G is faithful.

014Q (b) The following conditions are equivalent:

014R i. The functor G is full.

014S ii. The restriction

$$G|_{\mathrm{Im}_F}\colon \mathrm{Im}(F) \to \mathcal{C}$$

of G to $\mathrm{Im}(F)$ is full.

Proof. *Item 1*, *Interaction With Right Whiskering*: See [DFH75, Proposition 1.4].
Item 2, *Interaction With Adjunctions*: See [DFH75, Proposition 1.7]. □

014T **Question 8.6.1.1.3.** Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

equivalences (F, G, η', ϵ) and (F, G, η, ϵ').

014U 1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

is dominant?

014V 2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X},\mathcal{C}) \to \mathsf{Fun}(\mathcal{X},\mathcal{D})$$

is dominant?

This question also appears as [MO 468121a].

014W 8.6.2 Monomorphisms of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

014X **Definition 8.6.2.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **monomorphism of categories** if it is a monomorphism in Cats (see **??**).

014Y **Proposition 8.6.2.1.2.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

014Z 1. *Characterisations.* The following conditions are equivalent:

0150 (a) The functor F is a monomorphism of categories.

(b) The functor F is injective on objects and morphisms, i.e. F is
0151 injective on objects and the map

$$F\colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$$

is injective.

Proof. *Item 1*, *Characterisations*: Omitted. ☐

0152 **Question 8.6.2.1.3.** Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

0153 1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D},\mathcal{X}) \to \mathsf{Fun}(\mathcal{C},\mathcal{X})$$

is a monomorphism of categories?

0154 2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X},\mathcal{C}) \to \mathsf{Fun}(\mathcal{X},\mathcal{D})$$

is a monomorphism of categories?

This question also appears as [MO 468121a].

0155 8.6.3 Epimorphisms of Categories

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

0156 **Definition 8.6.3.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **epimorphism of categories** if it is a epimorphism in Cats (see **??**).

0157 **Proposition 8.6.3.1.2.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

0158 1. *Characterisations.* The following conditions are equivalent:[25]

0159 (a) The functor F is a epimorphism of categories.

015A (b) For each morphism $f\colon A \to B$ of $\mathcal{D}$, we have a diagram

$$\begin{tikzcd}
 & & & & A \arrow[dllll, "\phi_1"'] \arrow[dll, "\phi_2"'] \arrow[d, "\phi_3"] \arrow[drr, "\phi_m"] \arrow[drrrr, dashed, "\mathrm{id}_A"] & & & & \\
X_1 \arrow[dr, dashed, "\alpha_1"] \arrow[ddrrrr, dashed, bend right, "\alpha_0"'] & & X_2 \arrow[dl, dashed, "\alpha_2"'] \arrow[dr, dashed, "\alpha_3"] & & X_3 \arrow[dl, dashed, "\alpha_4"'] \arrow[dr, dashed, "\alpha_6"] & \cdots & X_m \arrow[dl, dashed, "\alpha_{2m-2}"'] \arrow[dr, dashed, "\alpha_{2m-1}"] & & A \arrow[dl, dashed, "\alpha_{2m}"'] \\
 & Y_1 \arrow[drrr, "\psi_1"] & & Y_2 \arrow[dr, "\psi_2"] & & \cdots \arrow[dl, "\psi_3"'] & & Y_m \arrow[dlll, "\psi_m"'] & \\
 & & & & B & & & &
\end{tikzcd}$$

in $\mathcal{D}$ satisfying the following conditions:

015B i. We have $f = \alpha_0 \circ \phi_1$.

015C ii. We have $f = \psi_m \circ \alpha_{2m}$.

015D iii. For each $0 \leq i \leq 2m$, we have $\alpha_i \in \mathrm{Mor}(\mathrm{Im}(F))$.

015E 2. *Surjectivity on Objects.* If F is an epimorphism of categories, then F is surjective on objects.

Proof. *Item 1*, *Characterisations*: See [Isb68].
Item 2, *Surjectivity on Objects*: Omitted. □

015F **Question 8.6.3.1.3.** Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

015G 1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is an epimorphism of categories?

[25] *Further Terminology:* This statement is known as **Isbell's zigzag theorem**.

015H 2. For each $\mathcal{X} \in \text{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_* \colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is an epimorphism of categories?

This question also appears as [MO 468121a].

015J 8.6.4 Pseudomonic Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

015K **Definition 8.6.4.1.1.** A functor $F \colon \mathcal{C} \to \mathcal{D}$ is **pseudomonic** if it satisfies the following conditions:

015L 1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

if we have

$$\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta,$$

then $\alpha = \beta$.

015M 2. For each $\mathcal{X} \in \text{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta \colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

015N **Proposition 8.6.4.1.2.** Let $F\colon\mathcal{C}\to\mathcal{D}$ be a functor.

015P 1. *Characterisations.* The following conditions are equivalent:

015Q (a) The functor F is pseudomonic.

015R (b) The functor F satisfies the following conditions:

i. The functor F is faithful, i.e. for each $A,B\in\mathrm{Obj}(\mathcal{C})$, the

015S action on morphisms

$$F_{A,B}\colon\mathrm{Hom}_{\mathcal{C}}(A,B)\to\mathrm{Hom}_{\mathcal{D}}(F_A,F_B)$$

of F at (A,B) is injective.

015T ii. For each $A,B\in\mathrm{Obj}(\mathcal{C})$, the restriction

$$F^{\mathrm{iso}}_{A,B}\colon\mathrm{Iso}_{\mathcal{C}}(A,B)\to\mathrm{Iso}_{\mathcal{D}}(F_A,F_B)$$

of the action on morphisms of F at (A,B) to isomorphisms is surjective.

015U (c) We have an isocomma square of the form

$$\mathcal{C}\overset{\text{eq.}}{\cong}\mathcal{C}\overset{\leftrightarrow}{\times}_{\mathcal{D}}\mathcal{C},\qquad \begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{id}_{\mathcal{C}}"] \arrow[d, "\mathrm{id}_{\mathcal{C}}"'] & \mathcal{C} \arrow[d, "F"] \arrow[dl, Leftrightarrow, dashed] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd}$$

in Cats_2 up to equivalence.

015V (d) We have an isocomma square of the form

$$\mathcal{C}\overset{\text{eq.}}{\cong}\mathcal{C}\overset{\leftrightarrow}{\times}_{\mathsf{Arr}(\mathcal{D})}\mathcal{D},\qquad \begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, "F"'] & \mathsf{Arr}(\mathcal{C}) \arrow[d, "\mathsf{Arr}(F)"] \arrow[dl, Leftrightarrow, dashed] \\ \mathcal{D} \arrow[r, hook] & \mathsf{Arr}(\mathcal{D}) \end{tikzcd}$$

in Cats_2 up to equivalence.

015W (e) For each $\mathcal{X}\in\mathrm{Obj}(\mathsf{Cats})$, the postcomposition [26] functor

$$F_*\colon\mathsf{Fun}(\mathcal{X},\mathcal{C})\to\mathsf{Fun}(\mathcal{X},\mathcal{D})$$

is pseudomonic.

[26] Asking the precomposition functors

$$F^*\colon\mathsf{Fun}(\mathcal{D},\mathcal{X})\to\mathsf{Fun}(\mathcal{C},\mathcal{X})$$

015X 2. *Conservativity.* If F is pseudomonic, then F is conservative.

015Y 3. *Essential Injectivity.* If F is pseudomonic, then F is essentially injective.

Proof. *Item 1*, *Characterisations*: Omitted.
Item 2, *Conservativity*: Omitted.
Item 3, *Essential Injectivity*: Omitted. □

015Z 8.6.5 Pseudoepic Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

0160 **Definition 8.6.5.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **pseudoepic** if it satisfies the following conditions:

0161 1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha \Downarrow \Downarrow \beta"] & \mathcal{X}, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

then $\alpha = \beta$.

0162 2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X} \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X} \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F.$$

0163 **Proposition 8.6.5.1.2.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

0164 1. *Characterisations.* The following conditions are equivalent:

0165 (a) The functor F is pseudoepic.

0166 (b) For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

given by precomposition by F is pseudomonic.

0167 (c) We have an isococomma square of the form

$$\mathcal{D} \overset{\text{eq.}}{\cong} \mathcal{D} \overset{\leftrightarrow}{\textstyle\coprod}_{\mathcal{C}} \mathcal{D}, \qquad \begin{tikzcd} \mathcal{D} & \mathcal{D} \arrow[l, "\mathrm{id}_{\mathcal{D}}"'] \\ \mathcal{D} \arrow[u, "\mathrm{id}_{\mathcal{D}}"] \arrow[ur, phantom, "\Leftrightarrow"] & \mathcal{C} \arrow[l, "F"] \arrow[u, "F"'] \end{tikzcd}$$

in Cats_2 up to equivalence.

0168 2. *Dominance.* If F is pseudoepic, then F is dominant (Definition 8.6.1.1.1).

Proof. *Item 1, Characterisations*: Omitted.
Item 2, Dominance: If F is pseudoepic, then

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is pseudomonic for all $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, and thus in particular faithful. By Item 4g of Item 4 of Proposition 8.5.1.1.2, this is equivalent to requiring F to be dominant. $\square$

0169 **Question 8.6.5.1.3.** Is there a nice characterisation of the pseudoepic functors, similarly to the characterisaiton of pseudomonic functors given in Item 1b of Item 1 of Proposition 8.6.4.1.2?
This question also appears as [`MO 321971`].

016A **Question 8.6.5.1.4.** A pseudomonic and pseudoepic functor is dominant, faithful, essentially injective, and full on isomorphisms. Is it necessarily an equivalence of categories? If not, how bad can this fail, i.e. how far can a pseudomonic and pseudoepic functor be from an equivalence of categories?
This question also appears as [`MO 468334`].

---

to be pseudomonic leads to pseudoepic functors; see Item 1b of Item 1 of Proposition 8.6.5.1.2.

016B **Question 8.6.5.1.5.** Is there a characterisation of functors $F\colon \mathcal{C} \to \mathcal{D}$ such that:

016C 1. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the precomposition functor

$$F^*\colon \mathsf{Fun}(\mathcal{D}, \mathcal{X}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{X})$$

is pseudoepic?

016D 2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition functor

$$F_*\colon \mathsf{Fun}(\mathcal{X}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{X}, \mathcal{D})$$

is pseudoepic?

This question also appears as [MO 468121a].

016E 8.7 Even More Conditions on Functors

016F 8.7.1 Injective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

016G **Definition 8.7.1.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **injective on objects** if the action on objects

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is injective.

016H **Proposition 8.7.1.1.2.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

016J 1. *Characterisations.* The following conditions are equivalent:

016K (a) The functor F is injective on objects.

016L (b) The functor F is an isocofibration in Cats_2.

Proof. Item 1, Characterisations: Omitted. □

016M 8.7.2 Surjective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

016N **Definition 8.7.2.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **surjective on objects** if the action on objects

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is surjective.

016P 8.7.3 Bijective on Objects Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

016Q **Definition 8.7.3.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **bijective on objects**[27] if the action on objects

$$F\colon \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$$

of F is a bijection.

016R 8.7.4 Functors Representably Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

016S **Definition 8.7.4.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is faithful.

016T **Remark 8.7.4.1.2.** In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably faithful on cores** if, given a diagram of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

if α and β are natural isomorphisms and we have

$$\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta,$$

then $\alpha = \beta$.

016U **Question 8.7.4.1.3.** Is there a characterisation of functors representably faithful on cores?

016V 8.7.5 Functors Representably Full on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

016W **Definition 8.7.5.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably full on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is full.

[27] *Further Terminology:* Also called a **bo** functor.

016X **Remark 8.7.5.1.2.** In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon F \circ \phi \overset{\sim}{\Longrightarrow} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

016Y **Question 8.7.5.1.3.** Is there a characterisation of functors representably full on cores?
This question also appears as [MO 468121a].

016Z 8.7.6 Functors Representably Fully Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

0170 **Definition 8.7.6.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably fully faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is fully faithful.

0171 **Remark 8.7.6.1.2.** In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **representably fully faithful on cores** if it satisfies the conditions in Remarks 8.7.4.1.2 and 8.7.5.1.2, i.e.:

0172 1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D}, \end{tikzcd}$$

with α and β natural isomorphisms, if we have $\mathrm{id}_F \star \alpha = \mathrm{id}_F \star \beta$, then $\alpha = \beta$.

0173 2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta \colon F \circ \phi \xRightarrow{\sim} F \circ \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of $\mathcal{C}$, there exists a natural isomorphism

$$\alpha \colon \phi \xRightarrow{\sim} \psi, \qquad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \mathrm{id}_F \star \alpha.$$

0174 **Question 8.7.6.1.3.** Is there a characterisation of functors representably fully faithful on cores?

0175 8.7.7 Functors Corepresentably Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

0176 **Definition 8.7.7.1.1.** A functor $F \colon \mathcal{C} \to \mathcal{D}$ is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_* \colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is faithful.

0177 **Remark 8.7.7.1.2.** In detail, a functor $F \colon \mathcal{C} \to \mathcal{D}$ is **corepresentably faithful on cores** if, given a diagram of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$

if α and β are natural isomorphisms and we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

then $\alpha = \beta$.

0178 **Question 8.7.7.1.3.** Is there a characterisation of functors corepresentably faithful on cores?

0179 8.7.8 Functors Corepresentably Full on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

017A **Definition 8.7.8.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is full.

017B **Remark 8.7.8.1.2.** In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably full on cores** if, for each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X}, \end{tikzcd}$$

there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F.$$

017C **Question 8.7.8.1.3.** Is there a characterisation of functors corepresentably full on cores?
This question also appears as [MO 468121a].

017D 8.7.9 Functors Corepresentably Fully Faithful on Cores

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

017E **Definition 8.7.9.1.1.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably fully faithful on cores** if, for each $X \in \mathrm{Obj}(\mathsf{Cats})$, the postcomposition by F functor

$$F_*\colon \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{C})) \to \mathsf{Core}(\mathsf{Fun}(\mathcal{X}, \mathcal{D}))$$

is fully faithful.

017F **Remark 8.7.9.1.2.** In detail, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is **corepresentably fully faithful on cores** if it satisfies the conditions in Remarks 8.7.7.1.2 and 8.7.8.1.2, i.e.:

017G 1. For all diagrams of the form

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & \mathcal{X}, \end{tikzcd}$$

   if α and β are natural isomorphisms and we have

$$\alpha \star \mathrm{id}_F = \beta \star \mathrm{id}_F,$$

   then $\alpha = \beta$.

017H 2. For each $\mathcal{X} \in \mathrm{Obj}(\mathsf{Cats})$ and each natural isomorphism

$$\beta\colon \phi \circ F \overset{\sim}{\Longrightarrow} \psi \circ F, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\phi\circ F"] \arrow[r, bend right, "\psi\circ F"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{X}, \end{tikzcd}$$

   there exists a natural isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{X} \end{tikzcd}$$

   such that we have an equality

$$\begin{tikzcd} \mathcal{X} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{X} \arrow[r, bend left, "F\circ\phi"] \arrow[r, bend right, "F\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

   of pasting diagrams in Cats_2, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_F.$$

017J **Question 8.7.9.1.3.** Is there a characterisation of functors corepresentably fully faithful on cores?

017K 8.8 Natural Transformations

017L 8.8.1 Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

017M **Definition 8.8.1.1.1.** A **transformation**[28] $\alpha\colon F \Rightarrow G$ **from** F **to** G is a collection

$$\{\alpha_A\colon F(A) \to G(A)\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

of morphisms of $\mathcal{D}$.

017N **Notation 8.8.1.1.2.** We write $\mathrm{Trans}(F, G)$ for the set of transformations from F to G.

017P 8.8.2 Natural Transformations

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

017Q **Definition 8.8.2.1.1.** A **natural transformation** $\alpha\colon F \Longrightarrow G$ **from** F **to** G is a transformation

$$\{\alpha_A\colon F(A) \to G(A)\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

from F to G such that, for each morphism $f\colon A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\
G(A) \arrow[r, "G(f)"'] & G(B)
\end{tikzcd}$$

commutes.[29]

017R **Remark 8.8.2.1.2.** We denote natural transformations in diagrams as

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D}.
\end{tikzcd}$$

017S **Notation 8.8.2.1.3.** We write $\mathrm{Nat}(F, G)$ for the set of natural transformations from F to G.

017T **Example 8.8.2.1.4.** The **identity natural transformation** $\mathrm{id}_F\colon F \Longrightarrow F$ **of** F is the natural transformation consisting of the collection

$$\left\{\mathrm{id}_{F(A)}\colon F(A) \to F(A)\right\}_{A \in \mathrm{Obj}(\mathcal{C})}.$$

Proof. The naturality condition for id_F is the requirement that, for each

[28] *Further Terminology:* Also called an **unnatural transformation** for emphasis.
[29] *Further Terminology:* The morphism $\alpha_A\colon F_A \to G_A$ is called the **component of** α **at** A.

morphism $f\colon A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\mathrm{id}_{F(A)}"'] & F(B) \arrow[d, "\mathrm{id}_{F(B)}"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

commutes, which follows from unitality of the composition of $\mathcal{C}$. □

017U **Definition 8.8.2.1.5.** Two natural transformations $\alpha, \beta\colon F \Longrightarrow G$ are **equal** if we have

$$\alpha_A = \beta_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

017V 8.8.3 Vertical Composition of Natural Transformations

017W **Definition 8.8.3.1.1.** The **vertical composition** of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon G \Longrightarrow H$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, "G"{name=M, description}] \arrow[r, bend right=50, "H"'{name=D}] & \mathcal{D} \arrow[from=U, to=M, Rightarrow, "\alpha"'] \arrow[from=M, to=D, Rightarrow, "\beta"'] \end{tikzcd}$$

is the natural transformation $\beta \circ \alpha\colon F \Longrightarrow H$ consisting of the collection

$$\{(\beta \circ \alpha)_A\colon F(A) \to H(A)\}_{A \in \mathrm{Obj}(\mathcal{C})}$$

with

$$(\beta \circ \alpha)_A \overset{\mathrm{def}}{=} \beta_A \circ \alpha_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$.

Proof. The naturality condition for $\beta \circ \alpha$ is the requirement that the boundary of the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(1)"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)" description] \arrow[d, "\beta_A"'] \arrow[dr, phantom, "(2)"] & G(B) \arrow[d, "\beta_B"] \\ H(A) \arrow[r, "H(f)"'] & H(B) \end{tikzcd}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of α.
2. Subdiagram (2) commutes by the naturality of β.

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation. $\square$

017X **Proposition 8.8.3.1.2.** Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

017Y 1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \circ \alpha$ defines a function

$$\circ_{F,G,H} \colon \mathrm{Nat}(G, H) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(F, H).$$

017Z 2. *Associativity.* Let $F, G, H, K \colon \mathcal{C} \overset{\rightrightarrows}{\rightrightarrows} \mathcal{D}$ be functors. The diagram

$$\begin{tikzcd}
& \mathrm{Nat}(H,K) \times (\mathrm{Nat}(G,H) \times \mathrm{Nat}(F,G)) \arrow[dr, "\mathrm{id}_{\mathrm{Nat}(H,K)} \times \circ_{F,G,H}"] & \\
(\mathrm{Nat}(H,K) \times \mathrm{Nat}(G,H)) \times \mathrm{Nat}(F,G) \arrow[ur, dashed, "\alpha^{\mathsf{Sets}}_{\mathrm{Nat}(H,K),\mathrm{Nat}(G,H),\mathrm{Nat}(F,G)}", "\sim"'] \arrow[d, "\circ_{G,H,K} \times \mathrm{id}_{\mathrm{Nat}(F,G)}"'] & & \mathrm{Nat}(H,K) \times \mathrm{Nat}(F,H) \arrow[d, "\circ_{F,H,K}"] \\
\mathrm{Nat}(G,K) \times \mathrm{Nat}(F,G) \arrow[rr, "\circ_{F,G,K}"'] & & \mathrm{Nat}(F,K)
\end{tikzcd}$$

commutes, i.e. given natural transformations

$$F \overset{\alpha}{\Longrightarrow} G \overset{\beta}{\Longrightarrow} H \overset{\gamma}{\Longrightarrow} K,$$

we have

$$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha).$$

0180 3. *Unitality.* Let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

   (a) *Left Unitality.* The diagram

$$\begin{tikzcd}
\mathrm{pt} \times \mathrm{Nat}(F,G) \arrow[d, "{[\mathrm{id}_G] \times \mathrm{id}_{\mathrm{Nat}(F,G)}}"'] \arrow[dr, dashed, "\lambda^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\
\mathrm{Nat}(G,G) \times \mathrm{Nat}(F,G) \arrow[r, "\circ_{F,G,G}"'] & \mathrm{Nat}(F,G)
\end{tikzcd}$$

   commutes, i.e. given a natural transformation $\alpha \colon F \Longrightarrow G$, we have

$$\mathrm{id}_G \circ \alpha = \alpha.$$

(b) *Right Unitality.* The diagram

$$
\begin{tikzcd}
\mathrm{Nat}(F,G) \times \mathrm{pt} \arrow[d, "\mathrm{id}_{\mathrm{Nat}(F,G)} \times [\mathrm{id}_F]"'] \arrow[dr, dashed, "\rho^{\mathsf{Sets}}_{\mathrm{Nat}(F,G)}", "\sim"'] & \\
\mathrm{Nat}(F,G) \times \mathrm{Nat}(F,F) \arrow[r, "\circ^{\mathcal{C}}_{F,F,G}"'] & \mathrm{Nat}(F,G)
\end{tikzcd}
$$

commutes, i.e. given a natural transformation $\alpha \colon F \Longrightarrow G$, we have

$$\alpha \circ \mathrm{id}_F = \alpha.$$

0181 4. *Middle Four Exchange.* Let $F_1, F_2, F_3 \colon \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3 \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$
\begin{tikzcd}
(\mathrm{Nat}(G_2,G_3) \times \mathrm{Nat}(G_1,G_2)) \times (\mathrm{Nat}(F_2,F_3) \times \mathrm{Nat}(F_1,F_2)) \arrow[rr, "\mu_4", "\sim"'] \arrow[d, "\circ_{G_1,G_2,G_3} \times \circ_{F_1,F_2,F_3}"'] & & (\mathrm{Nat}(G_2,G_3) \times \mathrm{Nat}(F_2,F_3)) \times (\mathrm{Nat}(G_1,G_2) \times \mathrm{Nat}(F_1,F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3} \times \star_{F_1,F_2,G_1,G_2}"] \\
\mathrm{Nat}(G_1,G_3) \times \mathrm{Nat}(F_1,F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathrm{Nat}(G_2 \circ F_2, G_3 \circ F_3) \times \mathrm{Nat}(G_1 \circ F_1, G_2 \circ F_2) \arrow[dl, "\circ_{G_1 \circ F_1, G_2 \circ F_2, G_3 \circ F_3}"] \\
& \mathrm{Nat}(G_1 \circ F_1, G_3 \circ F_3) &
\end{tikzcd}
$$

commutes, i.e. given a diagram

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F_1"{name=U1}] \arrow[r, "F_2"{description, name=M1}] \arrow[r, bend right=60, "F_3"'{name=D1}] & \mathcal{D} \arrow[r, bend left=60, "G_1"{name=U2}] \arrow[r, "G_2"{description, name=M2}] \arrow[r, bend right=60, "G_3"'{name=D2}] & \mathcal{E}
\arrow[from=U1, to=M1, Rightarrow, "\alpha"']
\arrow[from=M1, to=D1, Rightarrow, "\alpha'"']
\arrow[from=U2, to=M2, Rightarrow, "\beta"']
\arrow[from=M2, to=D2, Rightarrow, "\beta'"']
\end{tikzcd}
$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof. *Item 1, Functionality*: Clear.
Item 2, Associativity: Indeed, we have

$$
\begin{aligned}
((\gamma \circ \beta) \circ \alpha)_A &\overset{\text{def}}{=} (\gamma \circ \beta)_A \circ \alpha_A \\
&\overset{\text{def}}{=} (\gamma_A \circ \beta_A) \circ \alpha_A \\
&= \gamma_A \circ (\beta_A \circ \alpha_A) \\
&\overset{\text{def}}{=} \gamma_A \circ (\beta \circ \alpha)_A \\
&\overset{\text{def}}{=} (\gamma \circ (\beta \circ \alpha))_A
\end{aligned}
$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.
Item 3, Unitality: We have

$$\begin{aligned}(\mathrm{id}_G \circ \alpha)_A &= \mathrm{id}_G \circ \alpha_A\\ &= \alpha_A,\\ (\alpha \circ \mathrm{id}_F)_A &= \alpha_A \circ \mathrm{id}_F\\ &= \alpha_A\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.
Item 4, Middle Four Exchange: This is proved in Item 4 of Proposition 8.8.4.1.3. □

0182 8.8.4 Horizontal Composition of Natural Transformations

0183 **Definition 8.8.4.1.1.** The **horizontal composition**[30,31] of two natural transformations $\alpha\colon F \Longrightarrow G$ and $\beta\colon H \Longrightarrow K$ as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\beta\Downarrow"] & \mathcal{E} \end{tikzcd}$$

of α and β is the natural transformation

$$\beta \star \alpha\colon (H \circ F) \Longrightarrow (K \circ G),$$

as in the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "H\circ F"] \arrow[r, bend right, "K\circ G"'] \arrow[r, phantom, "\Downarrow \beta\star\alpha"] & \mathcal{E}, \end{tikzcd}$$

consisting of the collection

$$\{(\beta \star \alpha)_A\colon H(F(A)) \to K(G(A))\}_{A \in \mathrm{Obj}(\mathcal{C})},$$

of morphisms of $\mathcal{E}$ with

$$\begin{aligned}(\beta \star \alpha)_A &\overset{\mathrm{def}}{=} \beta_{G(A)} \circ H(\alpha_A)\\ &= K(\alpha_A) \circ \beta_{F(A)},\end{aligned} \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

[30] *Further Terminology:* Also called the **Godement product** of α and β.
[31] Horizontal composition forms a map

$$\star_{(F,H),(G,K)}\colon \mathrm{Nat}(H,K) \times \mathrm{Nat}(F,G) \to \mathrm{Nat}(H \circ F, K \circ G).$$

Proof. First, we claim that we indeed have

$$\beta_{G(A)} \circ H(\alpha_A) = K(\alpha_A) \circ \beta_{F(A)}, \qquad \begin{tikzcd} H(F(A)) \arrow[r, "H(\alpha_A)"] \arrow[d, "\beta_{F(A)}"'] & H(G(A)) \arrow[d, "\beta_{G(A)}"] \\ K(F(A)) \arrow[r, "K(\alpha_A)"'] & K(G(A)). \end{tikzcd}$$

This is, however, simply the naturality square for β applied to the morphism $\alpha_A \colon F(A) \to G(A)$. Next, we check the naturality condition for $\beta \star \alpha$, which is the requirement that the boundary of the diagram

$$\begin{tikzcd} H(F(A)) \arrow[rr, "H(F(f))"] \arrow[dd, "H(\alpha_A)"'] & & H(F(B)) \arrow[dd, "H(\alpha_B)"] \\ & (1) & \\ H(G(A)) \arrow[rr, "H(G(f))" description] \arrow[dd, "\beta_{G(A)}"'] & & H(G(B)) \arrow[dd, "\beta_{G(B)}"] \\ & (2) & \\ K(G(A)) \arrow[rr, "K(G(f))"'] & & K(G(B)) \end{tikzcd}$$

commutes. Since

1. Subdiagram (1) commutes by the naturality of α.
2. Subdiagram (2) commutes by the naturality of β.

so does the boundary diagram. Hence $\beta \circ \alpha$ is a natural transformation.[32] □

0184 **Definition 8.8.4.1.2.** Let

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F"] & \mathcal{C} \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathcal{D} \arrow[r, "G"] & \mathcal{Y} \end{tikzcd}$$

be a diagram in Cats_2.

[32] *Reference:* [Bor94, Proposition 1.3.4].

0185 1. The **left whiskering of α with G** is the natural transformation [33]

$$\mathrm{id}_G \star \alpha \colon G \circ \phi \Longrightarrow G \circ \psi.$$

2. The **right whiskering of α with F** is the natural transforma-
0186 tion [34]

$$\alpha \star \mathrm{id}_F \colon \phi \circ F \Longrightarrow \psi \circ F.$$

0187 **Proposition 8.8.4.1.3.** Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories.

0188 1. *Functionality.* The assignment $(\beta, \alpha) \mapsto \beta \star \alpha$ defines a function

$$\star_{(F,G),(H,K)} \colon \mathrm{Nat}(H, K) \times \mathrm{Nat}(F, G) \to \mathrm{Nat}(H \circ F, K \circ G).$$

0189 2. *Associativity.* Let

$$\mathcal{C} \underset{G_1}{\overset{F_1}{\rightrightarrows}} \mathcal{D} \underset{G_2}{\overset{F_2}{\rightrightarrows}} \mathcal{E} \underset{G_3}{\overset{F_3}{\rightrightarrows}} \mathcal{F}$$

be a diagram in Cats_2. The diagram

$$\begin{tikzcd} \mathrm{Nat}(F_3, G_3) \times \mathrm{Nat}(F_2, G_2) \times \mathrm{Nat}(F_1, G_1) \arrow[r, "\star_{(F_2,G_2),(F_3,G_3)} \times \mathsf{id}"] \arrow[d, "\mathsf{id} \times \star_{(F_1,G_1),(F_2,G_2)}"'] & \mathrm{Nat}(F_3 \circ F_2, G_3 \circ G_2) \times \mathrm{Nat}(F_1, G_1) \arrow[d, "\star_{(F_3 \circ F_2),(G_3 \circ G_2, F_1, G_1)}"] \\ \mathrm{Nat}(F_3, G_3) \times \mathrm{Nat}(F_2 \circ F_1, G_2 \circ G_1) \arrow[r, "\star_{(F_2 \circ F_1),(G_2 \circ G_1, F_3, G_3)}"'] & \mathrm{Nat}(F_3 \circ F_2 \circ F_1, G_3 \circ G_2 \circ G_1) \end{tikzcd}$$

commutes, i.e. given natural transformations

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\alpha \Downarrow"] & \mathcal{D} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\beta \Downarrow"] & \mathcal{E} \arrow[r, bend left, "F_3"] \arrow[r, bend right, "G_3"'] \arrow[r, phantom, "\gamma \Downarrow"] & \mathcal{F}, \end{tikzcd}$$

we have

$$(\gamma \star \beta) \star \alpha = \gamma \star (\beta \star \alpha).$$

018A 3. *Interaction With Identities.* Let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd} \mathrm{pt} \times \mathrm{pt} \arrow[r, "{[\mathrm{id}_G] \times [\mathrm{id}_F]}"] & \mathrm{Nat}(G, G) \times \mathrm{Nat}(F, F) \arrow[d, "\star_{(F,F),(G,G)}"] \\ \mathrm{pt} \arrow[u, dashed, "\wr"] \arrow[r, "{[\mathrm{id}_{G \circ F}]}"'] & \mathrm{Nat}(G \circ F, G \circ F) \end{tikzcd}$$

[33] *Further Notation:* Also written $G\alpha$ or $G \star \alpha$, although we won't use either of these notations in this work.

[34] *Further Notation:* Also written αF or $\alpha \star F$, although we won't use either of these

commutes, i.e. we have

$$\mathrm{id}_G \star \mathrm{id}_F = \mathrm{id}_{G\circ F}.$$

018B 4. *Middle Four Exchange.* Let $F_1, F_2, F_3 : \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3 : \mathcal{D} \to \mathcal{E}$ be functors. The diagram

$$\begin{tikzcd}
(\mathrm{Nat}(G_2,G_3)\times\mathrm{Nat}(G_1,G_2))\times(\mathrm{Nat}(F_2,F_3)\times\mathrm{Nat}(F_1,F_2)) \arrow[rr, "\mu_4", "\sim"'] \arrow[d, "\circ_{G_1,G_2,G_3}\times\circ_{F_1,F_2,F_3}"'] & & (\mathrm{Nat}(G_2,G_3)\times\mathrm{Nat}(F_2,F_3))\times(\mathrm{Nat}(G_1,G_2)\times\mathrm{Nat}(F_1,F_2)) \arrow[d, "\star_{F_2,F_3,G_2,G_3}\times\star_{F_1,F_2,G_1,G_2}"] \\
\mathrm{Nat}(G_1,G_3)\times\mathrm{Nat}(F_1,F_3) \arrow[dr, "\star_{F_1,F_3,G_1,G_3}"'] & & \mathrm{Nat}(G_2\circ F_2,G_3\circ F_3)\times\mathrm{Nat}(G_1\circ F_1,G_2\circ F_2) \arrow[dl, "\circ_{G_1\circ F_1,G_2\circ F_2,G_3\circ F_3}"] \\
& \mathrm{Nat}(G_1\circ F_1,G_3\circ F_3) &
\end{tikzcd}$$

commutes, i.e. given a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F_1"{name=U}] \arrow[r, "F_2"{name=M, description}] \arrow[r, bend right=60, "F_3"'{name=L}] & \mathcal{D} \arrow[r, bend left=60, "G_1"{name=U2}] \arrow[r, "G_2"{name=M2, description}] \arrow[r, bend right=60, "G_3"'{name=L2}] & \mathcal{E}
\arrow[Rightarrow, from=U, to=M, "\alpha"']
\arrow[Rightarrow, from=M, to=L, "\alpha'"']
\arrow[Rightarrow, from=U2, to=M2, "\beta"']
\arrow[Rightarrow, from=M2, to=L2, "\beta'"']
\end{tikzcd}$$

in Cats_2, we have

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

Proof. *Item 1*, *Functionality*: Clear.
Item 2, *Associativity*: Omitted.
Item 3, *Interaction With Identities*: We have

$$\begin{aligned}
(\mathrm{id}_G \star \mathrm{id}_F)_A &\overset{\text{def}}{=} (\mathrm{id}_G)_{F_A} \circ G_{(\mathrm{id}_F)_A} \\
&\overset{\text{def}}{=} \mathrm{id}_{G_{F_A}} \circ G_{\mathrm{id}_{F_A}} \\
&= \mathrm{id}_{G_{F_A}} \circ \mathrm{id}_{G_{F_A}} \\
&= \mathrm{id}_{G_{F_A}} \\
&\overset{\text{def}}{=} (\mathrm{id}_{G\circ F})_A
\end{aligned}$$

for each $A \in \mathrm{Obj}(\mathcal{C})$, showing the desired equality.

notations in this work.

Item 4, *Middle Four Exchange*: Let $A \in \mathrm{Obj}(\mathcal{C})$ and consider the diagram

$$\begin{tikzcd}
 & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1\left(\alpha'_A\right)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
 & & G_2(F_2(A)) \arrow[ur, "G_2\left(\alpha'_A\right)"'] & &
\end{tikzcd}$$

The top composition

$$\begin{tikzcd}
 & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1\left(\alpha'_A\right)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
 & & G_2(F_2(A)) \arrow[ur, "G_2\left(\alpha'_A\right)"'] & &
\end{tikzcd}$$

is given by $((\beta' \circ \beta) \star (\alpha' \circ \alpha))_A$, while the bottom composition

$$\begin{tikzcd}
 & & G_1(F_3(A)) \arrow[dr, "\beta_{F_3(A)}"] & & \\
G_1(F_1(A)) \arrow[r, "G_1(\alpha_A)"] & G_1(F_2(A)) \arrow[ur, "G_1\left(\alpha'_A\right)"] \arrow[dr, "\beta_{F_2(A)}"'] & (1) & G_2(F_3(A)) \arrow[r, "\beta'_{F_3(A)}"] & G_3(F_3(A)). \\
 & & G_2(F_2(A)) \arrow[ur, "G_2\left(\alpha'_A\right)"'] & &
\end{tikzcd}$$

is given by $((\beta' \star \alpha') \circ (\beta \star \alpha))_A$. Now, Subdiagram (1) corresponds to the naturality condition

$$G_2(\alpha'_A) \circ \beta_{F_2(A)} = \beta_{F_3}(A) \circ G_1(\alpha'_A), \qquad
\begin{tikzcd}
G_1(F_2(A)) \arrow[r, "G_1\left(\alpha'_A\right)"] \arrow[d, "\beta_{F_2(A)}"'] & G_1(F_3(A)) \arrow[d, "\beta_{F_3(A)}"] \\
G_2(F_2(A)) \arrow[r, "G_2\left(\alpha'_A\right)"'] & G_2(F_3(A))
\end{tikzcd}$$

for $\beta\colon G_1 \Longrightarrow G_2$ at $\alpha'_A\colon F_2(A) \to F_3(A)$, and thus commutes. Thus we have

$$\left((\beta' \circ \beta) \star (\alpha' \circ \alpha)\right)_A = \left((\beta' \star \alpha') \circ (\beta \star \alpha)\right)_A$$

for each $A \in \mathrm{Obj}(\mathcal{C})$ and therefore

$$(\beta' \star \alpha') \circ (\beta \star \alpha) = (\beta' \circ \beta) \star (\alpha' \circ \alpha).$$

This finishes the proof. □

018C 8.8.5 Properties of Natural Transformations

018D **Proposition 8.8.5.1.1.** Let $F, G\colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors. The following data are equivalent:[35]

018E 1. A natural transformation $\alpha\colon F \Longrightarrow G$.

018F 2. A functor $[\alpha]\colon \mathcal{C} \to \mathcal{D}^{\mathbb{1}}$ filling the diagram

$$\begin{tikzcd} & \mathcal{D} \\ \mathcal{C} \arrow[ur, "F"] \arrow[r, "{[\alpha]}"] \arrow[dr, "G"'] & \mathcal{D}^{\mathbb{1}}. \arrow[u, two heads, "\mathrm{ev}_0"'] \arrow[d, two heads, "\mathrm{ev}_1"] \\ & \mathcal{D} \end{tikzcd}$$

018G 3. A functor $[\alpha]\colon \mathcal{C} \times \mathbb{1} \to \mathcal{D}$ filling the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[dr, "F"] & \\ \mathcal{C} \times \mathbb{1} \arrow[u, two heads, "\mathrm{ev}_0"] \arrow[d, two heads, "\mathrm{ev}_1"'] \arrow[r, "{[\alpha]}"] & \mathcal{D}. \\ \mathcal{C} \arrow[ur, "G"'] & \end{tikzcd}$$

Proof. From Item 1 to Item 2 and Back: We may identify $\mathcal{D}^{\mathbb{1}}$ with

[35] Taken from [MO 64365].

$\mathsf{Arr}(\mathcal{D})$. Given a natural transformation $\alpha \colon F \Longrightarrow G$, we have a functor

$$
\begin{aligned}
[\alpha] \colon \mathcal{C} &\longrightarrow \mathcal{D}^{\mathbb{1}} \\
A &\longmapsto \alpha_A \\
(f \colon A \to B) &\longmapsto \left(
\begin{tikzcd}
F_A \arrow[r, "F_f"] \arrow[d, "\alpha_A"'] & F_B \arrow[d, "\alpha_B"] \\
G_A \arrow[r, "G_f"'] & G_B
\end{tikzcd}
\right)
\end{aligned}
$$

making the diagram in Item 2 commute. Conversely, every such functor gives rise to a natural transformation from F to G, and these constructions are inverse to each other.

From Item 2 to Item 3 and Back: This follows from Item 3 of Proposition 8.9.1.1.2. □

018H
8.8.6 Natural Isomorphisms

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F, G \colon \mathcal{C} \rightrightarrows \mathcal{D}$ be functors.

018J **Definition 8.8.6.1.1.** A natural transformation $\alpha \colon F \Longrightarrow G$ is a **natural isomorphism** if there exists a natural transformation $\alpha^{-1} \colon G \Longrightarrow F$ such that

$$
\begin{aligned}
\alpha^{-1} \circ \alpha &= \mathrm{id}_F, \\
\alpha \circ \alpha^{-1} &= \mathrm{id}_G.
\end{aligned}
$$

018K **Proposition 8.8.6.1.2.** Let $\alpha \colon F \Longrightarrow G$ be a natural transformation.

018L 1. *Characterisations.* The following conditions are equivalent:

018M (a) The natural transformation α is a natural isomorphism.

(b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism $\alpha_A \colon F_A \to G_A$ is an
018N isomorphism.

2. *Componentwise Inverses of Natural Transformations Assemble Into
018P Natural Transformations.* Let $\alpha^{-1} \colon G \Longrightarrow F$ be a transformation such that, for each $A \in \mathrm{Obj}(\mathcal{C})$, we have

$$
\begin{aligned}
\alpha_A^{-1} \circ \alpha_A &= \mathrm{id}_{F(A)}, \\
\alpha_A \circ \alpha_A^{-1} &= \mathrm{id}_{G(A)}.
\end{aligned}
$$

Then α^{-1} is a natural transformation.

Proof. *Item 1, Characterisations*: The implication Item 1a $\Longrightarrow$ Item 1b is clear, whereas the implication Item 1b $\Longrightarrow$ Item 1a follows from Item 2. *Item 2, Componentwise Inverses of Natural Transformations Assemble Into Natural Transformations*: The naturality condition for α^{-1} corresponds to the commutativity of the diagram

$$\begin{tikzcd} G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] & G(B) \arrow[d, "\alpha_B^{-1}"] \\ F(A) \arrow[r, "F(f)"'] & F(B) \end{tikzcd}$$

for each $A, B \in \mathrm{Obj}(\mathcal{C})$ and each $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Considering the diagram

$$\begin{tikzcd} G(A) \arrow[r, "G(f)"] \arrow[d, "\alpha_A^{-1}"'] \arrow[dr, phantom, "(1)"] & G(B) \arrow[d, "\alpha_B^{-1}"] \\ F(A) \arrow[r, dashed, "F(f)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(2)"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B), \end{tikzcd}$$

where the boundary diagram as well as Subdiagram (2) commute, we have

$$\begin{aligned} G(f) &= G(f) \circ \mathrm{id}_{G(A)} \\ &= G(f) \circ \alpha_A \circ \alpha_A^{-1} \\ &= \alpha_B \circ F(f) \circ \alpha_A^{-1}. \end{aligned}$$

Postcomposing both sides with α_B^{-1}, we get

$$\begin{aligned} \alpha_B^{-1} \circ G(f) &= \alpha_B^{-1} \circ \alpha_B \circ F(f) \circ \alpha_A^{-1} \\ &= \mathrm{id}_{F(B)} \circ F(f) \circ \alpha_A^{-1} \\ &= F(f) \circ \alpha_A^{-1}, \end{aligned}$$

which is the naturality condition we wanted to show. Thus α^{-1} is a natural transformation. □

018Q 8.9 Categories of Categories

018R 8.9.1 Functor Categories

Let $\mathcal{C}$ be a category and $\mathcal{D}$ be a small category.

018S **Definition 8.9.1.1.1.** The **category of functors from $\mathcal{C}$ to $\mathcal{D}$**[36] is the category $\mathsf{Fun}(\mathcal{C},\mathcal{D})$[37] where

- *Objects.* The objects of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ are functors from $\mathcal{C}$ to $\mathcal{D}$.
- *Morphisms.* For each $F,G \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C},\mathcal{D}))$, we have
$$\mathrm{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{D})}(F,G) \stackrel{\text{def}}{=} \mathrm{Nat}(F,G).$$
- *Identities.* For each $F \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C},\mathcal{D}))$, the unit map
$$\mathbb{1}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_F \colon \mathrm{pt} \to \mathrm{Nat}(F,F)$$
of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ at F is given by
$$\mathrm{id}^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_F \stackrel{\text{def}}{=} \mathrm{id}_F,$$
where $\mathrm{id}_F \colon F \Longrightarrow F$ is the identity natural transformation of F of Example 8.8.2.1.4.
- *Composition.* For each $F,G,H \in \mathrm{Obj}(\mathsf{Fun}(\mathcal{C},\mathcal{D}))$, the composition map
$$\circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \colon \mathrm{Nat}(G,H) \times \mathrm{Nat}(F,G) \to \mathrm{Nat}(F,H)$$
of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ at (F,G,H) is given by
$$\beta \circ^{\mathsf{Fun}(\mathcal{C},\mathcal{D})}_{F,G,H} \alpha \stackrel{\text{def}}{=} \beta \circ \alpha,$$
where $\beta \circ \alpha$ is the vertical composition of α and β of Item 1 of Proposition 8.8.3.1.2.

018T **Proposition 8.9.1.1.2.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.

018U 1. *Functoriality.* The assignments $\mathcal{C},\mathcal{D},(\mathcal{C},\mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C},\mathcal{D})$ define functors
$$\begin{aligned}\mathsf{Fun}(\mathcal{C},-_2) &\colon \mathsf{Cats} \to \mathsf{Cats},\\ \mathsf{Fun}(-_1,\mathcal{D}) &\colon \mathsf{Cats}^{\mathsf{op}} \to \mathsf{Cats},\\ \mathsf{Fun}(-_1,-_2) &\colon \mathsf{Cats}^{\mathsf{op}} \times \mathsf{Cats} \to \mathsf{Cats}.\end{aligned}$$

018V 2. *2-Functoriality.* The assignments $\mathcal{C},\mathcal{D},(\mathcal{C},\mathcal{D}) \mapsto \mathsf{Fun}(\mathcal{C},\mathcal{D})$ define 2-functors
$$\begin{aligned}\mathsf{Fun}(\mathcal{C},-_2) &\colon \mathsf{Cats}_2 \to \mathsf{Cats}_2,\\ \mathsf{Fun}(-_1,\mathcal{D}) &\colon \mathsf{Cats}^{\mathsf{op}}_2 \to \mathsf{Cats}_2,\\ \mathsf{Fun}(-_1,-_2) &\colon \mathsf{Cats}^{\mathsf{op}}_2 \times \mathsf{Cats}_2 \to \mathsf{Cats}_2.\end{aligned}$$

[36] *Further Terminology:* Also called the **functor category** $\mathsf{Fun}(\mathcal{C},\mathcal{D})$.
[37] *Further Notation:* Also written $\mathcal{D}^{\mathcal{C}}$ and $[\mathcal{C},\mathcal{D}]$.

018W 3. *Adjointness.* We have adjunctions

$$(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -))\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "\mathcal{C}\times-"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"] \end{tikzcd}$$

$$(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -))\colon \quad \begin{tikzcd} \mathsf{Cats} \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp"] & \mathsf{Cats}, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"] \end{tikzcd}$$

witnessed by bijections of sets

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})),$$
$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})),$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$.

018X 4. *2-Adjointness.* We have 2-adjunctions

$$(\mathcal{C} \times - \dashv \mathsf{Fun}(\mathcal{C}, -))\colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "\mathcal{C}\times-"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{C},-)"] \end{tikzcd}$$

$$(- \times \mathcal{D} \dashv \mathsf{Fun}(\mathcal{D}, -))\colon \quad \begin{tikzcd} \mathsf{Cats}_2 \arrow[r, bend left, "-\times\mathcal{D}"] \arrow[r, phantom, "\perp_2"] & \mathsf{Cats}_2, \arrow[l, bend left, "\mathsf{Fun}(\mathcal{D},-)"] \end{tikzcd}$$

witnessed by isomorphisms of categories

$$\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathsf{Fun}(\mathcal{D}, \mathsf{Fun}(\mathcal{C}, \mathcal{E})),$$
$$\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E})),$$

natural in $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$.

018Y 5. *Interaction With Punctual Categories.* We have a canonical isomorphism of categories

$$\mathsf{Fun}(\mathsf{pt}, \mathcal{C}) \cong \mathcal{C},$$

natural in $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$.

018Z 6. *Objectwise Computation of Co/Limits.* Let

$$D\colon \mathcal{I} \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

be a diagram in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. We have isomorphisms

$$\lim(D)_A \cong \lim_{i\in\mathcal{I}}(D_i(A)),$$
$$\mathrm{colim}(D)_A \cong \operatorname*{colim}_{i\in\mathcal{I}}(D_i(A)),$$

naturally in $A \in \mathrm{Obj}(\mathcal{C})$.

0190 7. *Interaction With Co/Completeness.* If $\mathcal{E}$ is co/complete, then so is $\mathsf{Fun}(\mathcal{C}, \mathcal{E})$.

0191 8. *Monomorphisms and Epimorphisms.* Let $\alpha\colon F \Longrightarrow G$ be a morphism of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. The following conditions are equivalent:

0192 (a) The natural transformation

$$\alpha\colon F \Longrightarrow G$$

is a monomorphism (resp. epimorphism) in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

0193 (b) For each $A \in \mathrm{Obj}(\mathcal{C})$, the morphism

$$\alpha_A\colon F_A \to G_A$$

is a monomorphism (resp. epimorphism) in $\mathcal{D}$.

Proof. *Item 1, Functoriality*: Omitted.
Item 2, 2-Functoriality: Omitted.
Item 3, Adjointness: Omitted.
Item 4, 2-Adjointness: Omitted.
Item 5, Interaction With Punctual Categories: Omitted.
Item 6, Objectwise Computation of Co/Limits: Omitted.
Item 7, Interaction With Co/Completeness: This follows from **??**.
Item 8, Monomorphisms and Epimorphisms: Omitted. □

0194 8.9.2 The Category of Categories and Functors

0195 **Definition 8.9.2.1.1.** The **category of** (**small**) **categories and functors** is the category Cats where

- *Objects.* The objects of Cats are small categories.
- *Morphisms.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats})$, we have

$$\mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathrm{Obj}(\mathsf{Fun}(\mathcal{C}, \mathcal{D})).$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats})$, the unit map

$$\mathbb{1}^{\mathsf{Cats}}_{\mathcal{C}}\colon \mathrm{pt} \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{C})$$

of Cats at $\mathcal{C}$ is defined by

$$\mathrm{id}^{\mathsf{Cats}}_{\mathcal{C}} \stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{C}},$$

where $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ is the identity functor of $\mathcal{C}$ of Example 8.4.1.1.4.

- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats})$, the composition map

  $$\circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathsf{Cats}}(\mathcal{C}, \mathcal{E})$$

  of Cats at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is given by

  $$G \circ^{\mathsf{Cats}}_{\mathcal{C},\mathcal{D},\mathcal{E}} F \stackrel{\text{def}}{=} G \circ F,$$

  where $G \circ F \colon \mathcal{C} \to \mathcal{E}$ is the composition of F and G of Definition 8.4.1.1.5.

0196 **Proposition 8.9.2.1.2.** Let $\mathcal{C}$ be a category.

0197 1. *Co/Completeness.* The category Cats is complete and cocomplete.

0198 2. *Cartesian Monoidal Structure.* The quadruple $(\mathsf{Cats}, \times, \mathsf{pt}, \mathsf{Fun})$ is a Cartesian closed monoidal category.

Proof. *Item 1, Co/Completeness*: Omitted.
Item 2, Cartesian Monoidal Structure: Omitted. □

0199 ### 8.9.3 The 2-Category of Categories, Functors, and Natural Transformations

019A **Definition 8.9.3.1.1.** The **2-category of (small) categories, functors, and natural transformations** is the 2-category Cats_2 where

- *Objects.* The objects of Cats_2 are small categories.

- Hom-*Categories.* For each $\mathcal{C}, \mathcal{D} \in \mathrm{Obj}(\mathsf{Cats}_2)$, we have

  $$\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \mathsf{Fun}(\mathcal{C}, \mathcal{D}).$$

- *Identities.* For each $\mathcal{C} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the unit functor

  $$\mathbb{1}^{\mathsf{Cats}_2}_{\mathcal{C}} \colon \mathsf{pt} \to \mathsf{Fun}(\mathcal{C}, \mathcal{C})$$

  of Cats_2 at $\mathcal{C}$ is the functor picking the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ of $\mathcal{C}$.

- *Composition.* For each $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Obj}(\mathsf{Cats}_2)$, the composition bifunctor

  $$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}} \colon \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}) \to \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{E})$$

  of Cats_2 at $(\mathcal{C}, \mathcal{D}, \mathcal{E})$ is the functor where

  - *Action on Objects.* For each object $(G, F) \in \mathsf{Obj}(\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D}))$, we have

    $$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(G, F) \stackrel{\text{def}}{=} G \circ F.$$

– *Action on Morphisms.* For each morphism $(\beta, \alpha) \colon (K, H) \Longrightarrow (G, F)$ of $\mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{D}, \mathcal{E}) \times \mathsf{Hom}_{\mathsf{Cats}_2}(\mathcal{C}, \mathcal{D})$, we have

$$\circ^{\mathsf{Cats}_2}_{\mathcal{C},\mathcal{D},\mathcal{E}}(\beta, \alpha) \stackrel{\text{def}}{=} \beta \star \alpha,$$

where $\beta \star \alpha$ is the horizontal composition of α and β of Definition 8.8.4.1.1.

019B **Proposition 8.9.3.1.2.** Let $\mathcal{C}$ be a category.

019C 1. *2-Categorical Co/Completeness.* The 2-category Cats_2 is complete and cocomplete as a 2-category, having all 2-categorical and bicategorical co/limits.

Proof. *Item 1, Co/Completeness*: Omitted. □

019D
8.9.4 The Category of Groupoids

019E **Definition 8.9.4.1.1.** The **category of (small) groupoids** is the full subcategory Grpd of Cats spanned by the groupoids.

019F
8.9.5 The 2-Category of Groupoids

019G **Definition 8.9.5.1.1.** The **2-category of (small) groupoids** is the full sub-2-category Grpd_2 of Cats_2 spanned by the groupoids.

Appendices

8.A Other Chapters

Part IV

Bicategories

Chapter 9

Types of Morphisms in Bicategories

019H In this chapter, we study special kinds of morphisms in bicategories:

1. *Monomorphisms and Epimorphisms in Bicategories (Sections 9.1 and 9.2).* There is a large number of different notions capturing the idea of a "monomorphism" or of an "epimorphism" in a bicategory.

   Arguably, the notion that best captures these concepts is that of a *pseudomonic morphism* (Definition 9.1.10.1.1) and of a *pseudoepic morphism* (Definition 9.2.10.1.1), although the other notions introduced in Sections 9.1 and 9.2 are also interesting on their own.

Contents

019J 9.1 Monomorphisms in Bicategories

019K 9.1.1 Representably Faithful Morphisms

Let $\mathcal{C}$ be a bicategory.

019L **Definition 9.1.1.1.1.** A 1-morphism $f \colon A \to B$ of $\mathcal{C}$ is **representably faithful**[1] if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_* \colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is faithful.

019M **Remark 9.1.1.1.2.** In detail, f is representably faithful if, for all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha \Downarrow \Downarrow \beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

019N **Example 9.1.1.1.3.** Here are some examples of representably faithful morphisms.

[1] *Further Terminology:* Also called simply a **faithful morphism**, based on Item 1 of Example 9.1.1.1.3.

019P 1. *Representably Faithful Morphisms in* Cats_2. The representably faithful morphisms in Cats_2 are precisely the faithful functors; see Item 1 of Proposition 8.5.1.1.2.

019Q 2. *Representably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is representably faithful; see Item 1 of Proposition 5.3.8.1.1.

019R 9.1.2 Representably Full Morphisms

Let $\mathcal{C}$ be a bicategory.

019S **Definition 9.1.2.1.1.** A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably full**[2] if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$

given by postcomposition by f is full.

019T **Remark 9.1.2.1.2.** In detail, f is representably full if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

019U **Example 9.1.2.1.3.** Here are some examples of representably full morphisms.

019V 1. *Representably Full Morphisms in* Cats_2. The representably full morphisms in Cats_2 are precisely the full functors; see Item 1 of Proposition 8.5.2.1.2.

019W 2. *Representably Full Morphisms in* **Rel**. The representably full morphisms in **Rel** are characterised in Item 2 of Proposition 5.3.8.1.1.

[2] *Further Terminology:* Also called simply a **full morphism**, based on Item 1 of

019X 9.1.3 Representably Fully Faithful Morphisms

Let $\mathcal{C}$ be a bicategory.

019Y **Definition 9.1.3.1.1.** A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably fully faithful**[3] if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful (Definition 9.1.1.1.1)
019Z and representably full (Definition 9.1.2.1.1).
01A0 2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor
$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$
given by postcomposition by f is fully faithful.

01A1 **Remark 9.1.3.1.2.** In detail, f is representably fully faithful if the conditions in Remark 9.1.1.1.2 and Remark 9.1.2.1.2 hold:

1. For all diagrams in $\mathcal{C}$ of the form
$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$
if we have
$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$
then $\alpha = \beta$.
2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism
$$\beta\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$
of $\mathcal{C}$, there exists a 2-morphism
$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$
of $\mathcal{C}$ such that we have an equality
$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

Example 9.1.2.1.3.

[3] *Further Terminology:* Also called simply a **fully faithful morphism**, based on

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01A2 **Example 9.1.3.1.3.** Here are some examples of representably fully faithful morphisms.

01A3 1. *Representably Fully Faithful Morphisms in* Cats_2. The representably fully faithful morphisms in Cats_2 are precisely the fully faithful functors; see Item 5 of Proposition 8.5.3.1.2.

01A4 2. *Representably Fully Faithful Morphisms in* **Rel**. The representably fully faithful morphisms of **Rel** coincide (Item 3 of Proposition 5.3.8.1.1) with the representably full morphisms in **Rel**, which are characterised in Item 2 of Proposition 5.3.8.1.1.

01A5 9.1.4 Morphisms Representably Faithful on Cores

Let C be a bicategory.

01A6 **Definition 9.1.4.1.1.** A 1-morphism $f \colon A \to B$ of C is **representably faithful on cores** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_* \colon \mathsf{Core}(\mathsf{Hom}_C(X, A)) \to \mathsf{Core}(\mathsf{Hom}_C(X, B))$$

given by postcomposition by f is faithful.

01A7 **Remark 9.1.4.1.2.** In detail, f is representably faithful on cores if, for all diagrams in C of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$

then $\alpha = \beta$.

01A8 9.1.5 Morphisms Representably Full on Cores

Let C be a bicategory.

Item 1 of Example 9.1.3.1.3.

01A9 **Definition 9.1.5.1.1.** A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably full on cores** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, A)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by f is full.

01AA **Remark 9.1.5.1.2.** In detail, f is representably full on cores if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, shorten=2pt, "\beta"'] \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[from=U, to=D, Rightarrow, shorten=2pt, "\alpha"'] \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[r, "f"] \arrow[from=U, to=D, Rightarrow, shorten=2pt, "\alpha"'] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] & B \arrow[from=U, to=D, Rightarrow, shorten=2pt, "\beta"'] \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01AB
9.1.6 Morphisms Representably Fully Faithful on Cores

Let $\mathcal{C}$ be a bicategory.

01AC **Definition 9.1.6.1.1.** A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **representably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is representably faithful on cores (Definition 9.1.5.1.1)
01AD and representably full on cores (Definition 9.1.4.1.1).

01AE 2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, A)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(X, B))$$

given by postcomposition by f is fully faithful.

01AF **Remark 9.1.6.1.2.** In detail, f is representably fully faithful on cores if the conditions in Remark 9.1.4.1.2 and Remark 9.1.5.1.2 hold:

1. For all diagrams in $\mathcal{C}$ of the form
$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$
if α and β are 2-isomorphisms and we have
$$\mathrm{id}_f \star \alpha = \mathrm{id}_f \star \beta,$$
then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism
$$\beta \colon f \circ \phi \overset{\sim}{\Longrightarrow} f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$
of $\mathcal{C}$, there exists a 2-isomorphism
$$\alpha \colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$
of $\mathcal{C}$ such that we have an equality
$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$
of pasting diagrams in $\mathcal{C}$, i.e. such that we have
$$\beta = \mathrm{id}_f \star \alpha.$$

01AG 9.1.7 Representably Essentially Injective Morphisms

Let $\mathcal{C}$ be a bicategory.

01AH **Definition 9.1.7.1.1.** A 1-morphism $f \colon A \to B$ of $\mathcal{C}$ is **representably essentially injective** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor
$$f_* \colon \mathsf{Hom}_{\mathcal{C}}(X, A) \to \mathsf{Hom}_{\mathcal{C}}(X, B)$$
given by postcomposition by f is essentially injective.

01AJ **Remark 9.1.7.1.2.** In detail, f is representably essentially injective if, for each pair of morphisms $\phi, \psi \colon X \rightrightarrows A$ of $\mathcal{C}$, the following condition is satisfied:

($\star$) If $f \circ \phi \cong f \circ \psi$, then $\phi \cong \psi$.

01AK 9.1.8 Representably Conservative Morphisms

Let C be a bicategory.

01AL **Definition 9.1.8.1.1.** A 1-morphism $f\colon A \to B$ of C is **representably conservative** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is conservative.

01AM **Remark 9.1.8.1.2.** In detail, f is representably conservative if, for each pair of morphisms $\phi, \psi\colon X \rightrightarrows A$ and each 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"{name=U}] \arrow[r, bend right, "\psi"'{name=D}] & A \arrow[from=U, to=D, phantom, "\alpha\Downarrow"] \end{tikzcd}$$

of C, if the 2-morphism

$$\mathrm{id}_f \star \alpha\colon f \circ \phi \Longrightarrow f \circ \psi, \qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"{name=U}] \arrow[r, bend right, "f\circ\psi"'{name=D}] & B \arrow[from=U, to=D, phantom, "\Downarrow \mathrm{id}_f\star\alpha"] \end{tikzcd}$$

is a 2-isomorphism, then so is α.

01AN 9.1.9 Strict Monomorphisms

Let C be a bicategory.

01AP **Definition 9.1.9.1.1.** A 1-morphism $f\colon A \to B$ of C is a **strict monomorphism** if, for each $X \in \mathrm{Obj}(C)$, the functor

$$f_*\colon \mathsf{Hom}_C(X, A) \to \mathsf{Hom}_C(X, B)$$

given by postcomposition by f is injective on objects, i.e. its action on objects

$$f_*\colon \mathrm{Obj}(\mathsf{Hom}_C(X, A)) \to \mathrm{Obj}(\mathsf{Hom}_C(X, B))$$

is injective.

01AQ **Remark 9.1.9.1.2.** In detail, f is a strict monomorphism in C if, for each diagram in C of the form

$$X \underset{\psi}{\overset{\phi}{\rightrightarrows}} A \xrightarrow{f} B,$$

if $f \circ \phi = f \circ \psi$, then $\phi = \psi$.

01AR **Example 9.1.9.1.3.** Here are some examples of strict monomorphisms.

01AS 1. *Strict Monomorphisms in* Cats_2. The strict monomorphisms in Cats_2 are precisely the functors which are injective on objects and injective on morphisms; see Item 1 of Proposition 8.6.2.1.2.

01AT 2. *Strict Monomorphisms in* **Rel**. The strict monomorphisms in **Rel** are characterised in Proposition 5.3.7.1.1.

01AU
9.1.10 Pseudomonic Morphisms

Let $\mathcal{C}$ be a bicategory.

01AV **Definition 9.1.10.1.1.** A 1-morphism $f\colon A\to B$ of $\mathcal{C}$ is **pseudomonic** if, for each $X\in\mathrm{Obj}(\mathcal{C})$, the functor

$$f_*\colon \mathsf{Hom}_{\mathcal{C}}(X,A)\to\mathsf{Hom}_{\mathcal{C}}(X,B)$$

given by postcomposition by f is pseudomonic.

01AW **Remark 9.1.10.1.2.** In detail, a 1-morphism $f\colon A\to B$ of $\mathcal{C}$ is pseudomonic if it satisfies the following conditions:

01AX 1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & A \arrow[r, "f"] & B, \end{tikzcd}$$

if we have

$$\mathrm{id}_f\star\alpha=\mathrm{id}_f\star\beta,$$

then $\alpha=\beta$.

01AY 2. For each $X\in\mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon f\circ\phi\overset{\sim}{\Longrightarrow}f\circ\psi,\qquad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi\overset{\sim}{\Longrightarrow}\psi,\qquad \begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \end{tikzcd}$$

of C such that we have an equality

$$\begin{tikzcd} X \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & A \arrow[r, "f"] & B \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r, bend left, "f\circ\phi"] \arrow[r, bend right, "f\circ\psi"'] \arrow[r, phantom, "\beta\Downarrow"] & B \end{tikzcd}$$

of pasting diagrams in C, i.e. such that we have

$$\beta = \mathrm{id}_f \star \alpha.$$

01AZ **Proposition 9.1.10.1.3.** Let $f\colon A \to B$ be a 1-morphism of C.

01B0 1. *Characterisations.* The following conditions are equivalent:

01B1 (a) The morphism f is pseudomonic.

01B2 (b) The morphism f is representably full on cores and representably faithful.

01B3 (c) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_B A, \qquad \begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] & A \arrow[d, "F"] \\ A \arrow[r, "F"'] \arrow[ur, Leftrightarrow, dashed] & B \end{tikzcd}$$

in C up to equivalence.

01B4 2. *Interaction With Cotensors.* If C has cotensors with $\mathbb{1}$, then the following conditions are equivalent:

(a) The morphism f is pseudomonic.

(b) We have an isocomma square of the form

$$A \overset{\text{eq.}}{\cong} A \overset{\leftrightarrow}{\times}_{\mathbb{1}\pitchfork F} B, \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d, "F"'] & \mathbb{1} \pitchfork A \arrow[d, "\mathbb{1}\pitchfork F"] \\ B \arrow[r, hook] \arrow[ur, Leftrightarrow, dashed] & \mathbb{1} \pitchfork B \end{tikzcd}$$

in C up to equivalence.

Proof. *Item 1, Characterisations*: Omitted.
Item 2, Interaction With Cotensors: Omitted. □

01B5 9.2 Epimorphisms in Bicategories

01B6 9.2.1 Corepresentably Faithful Morphisms

Let $\mathcal{C}$ be a bicategory.

01B7 **Definition 9.2.1.1.1.** A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably faithful** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is faithful.

01B8 **Remark 9.2.1.1.2.** In detail, f is corepresentably faithful if, for all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

01B9 **Example 9.2.1.1.3.** Here are some examples of corepresentably faithful morphisms.

01BA 1. *Corepresentably Faithful Morphisms in* Cats_2. The corepresentably faithful morphisms in Cats_2 are characterised in Item 4 of Proposition 8.5.1.1.2.

01BB 2. *Corepresentably Faithful Morphisms in* **Rel**. Every morphism of **Rel** is corepresentably faithful; see Item 1 of Proposition 5.3.10.1.1.

01BC 9.2.2 Corepresentably Full Morphisms

Let $\mathcal{C}$ be a bicategory.

01BD **Definition 9.2.2.1.1.** A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably full** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is full.

01BE **Remark 9.2.2.1.2.** In detail, f is corepresentably full if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta\colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha\colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

01BF **Example 9.2.2.1.3.** Here are some examples of corepresentably full morphisms.

01BG 1. *Corepresentably Full Morphisms in* Cats_2. The corepresentably full morphisms in Cats_2 are characterised in Item 5 of Proposition 8.5.2.1.2.

01BH 2. *Corepresentably Full Morphisms in* **Rel**. The corepresentably full morphisms in **Rel** are characterised in **??** of Proposition 5.3.8.1.1.

01BJ 9.2.3 Corepresentably Fully Faithful Morphisms

Let $\mathcal{C}$ be a bicategory.

01BK **Definition 9.2.3.1.1.** A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably fully faithful**[4] if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full (Definition 9.2.2.1.1) and
01BL corepresentably faithful (Definition 9.2.1.1.1).

[4] *Further Terminology:* Corepresentably fully faithful morphisms have also been called **lax epimorphisms** in the literature (e.g. in [Adá+01]), though we will always use the name "corepresentably fully faithful morphism" instead in this work.

01BM 2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^* \colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is fully faithful.

01BN **Remark 9.2.3.1.2.** In detail, f is corepresentably fully faithful if the conditions in Remark 9.2.1.1.2 and Remark 9.2.2.1.2 hold:

1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-morphism

$$\beta \colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-morphism

$$\alpha \colon \phi \Longrightarrow \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

01BP **Example 9.2.3.1.3.** Here are some examples of corepresentably fully faithful morphisms.

01BQ 1. *Corepresentably Fully Faithful Morphisms in* Cats_2. The fully faithful epimorphisms in Cats_2 are characterised in Item 9 of Proposition 8.5.3.1.2.

01BR 2. *Corepresentably Fully Faithful Morphisms in* **Rel**. The corepresentably fully faithful morphisms of **Rel** coincide (Item 3 of Proposition 5.3.10.1.1) with the corepresentably full morphisms in **Rel**, which are characterised in Item 2 of Proposition 5.3.10.1.1.

01BS 9.2.4 Morphisms Corepresentably Faithful on Cores

Let $\mathcal{C}$ be a bicategory.

01BT **Definition 9.2.4.1.1.** A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably faithful on cores** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(B, X)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by f is faithful.

01BU **Remark 9.2.4.1.2.** In detail, f is corepresentably faithful on cores if, for all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

01BV 9.2.5 Morphisms Corepresentably Full on Cores

Let $\mathcal{C}$ be a bicategory.

01BW **Definition 9.2.5.1.1.** A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably full on cores** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(B, X)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by f is full.

01BX **Remark 9.2.5.1.2.** In detail, f is corepresentably full on cores if, for each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

01BY 9.2.6 Morphisms Corepresentably Fully Faithful on Cores

Let $\mathcal{C}$ be a bicategory.

01BZ **Definition 9.2.6.1.1.** A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably fully faithful on cores** if the following equivalent conditions are satisfied:

1. The 1-morphism f is corepresentably full on cores (Definition 9.2.5.1.1)
01C0 and corepresentably faithful on cores (Definition 9.2.1.1.1).

01C1 2. For each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(B, X)) \to \mathsf{Core}(\mathsf{Hom}_{\mathcal{C}}(A, X))$$

given by precomposition by f is fully faithful.

01C2 **Remark 9.2.6.1.2.** In detail, f is corepresentably fully faithful on cores if the conditions in Remark 9.2.4.1.2 and Remark 9.2.5.1.2 hold:

1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if α and β are 2-isomorphisms and we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

01C3 9.2.7 Corepresentably Essentially Injective Morphisms

Let $\mathcal{C}$ be a bicategory.

01C4 **Definition 9.2.7.1.1.** A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably essentially injective** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is essentially injective.

01C5 **Remark 9.2.7.1.2.** In detail, f is corepresentably essentially injective if, for each pair of morphisms $\phi, \psi\colon B \rightrightarrows X$ of $\mathcal{C}$, the following condition is satisfied:

($\star$) If $\phi \circ f \cong \psi \circ f$, then $\phi \cong \psi$.

01C6 9.2.8 Corepresentably Conservative Morphisms

Let $\mathcal{C}$ be a bicategory.

01C7 **Definition 9.2.8.1.1.** A 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is **corepresentably conservative** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^*\colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is conservative.

01C8 **Remark 9.2.8.1.2.** In detail, f is corepresentably conservative if, for each pair of morphisms $\phi, \psi\colon B \rightrightarrows X$ and each 2-morphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, if the 2-morphism

$$\alpha \star \mathrm{id}_f \colon \phi \circ f \Longrightarrow \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left=45, "\phi\circ f"{name=U}] \arrow[r, bend right=45, "\psi\circ f"'{name=D}] & X \arrow[from=U, to=D, phantom, "\Downarrow \alpha\star\mathrm{id}_f"] \end{tikzcd}$$

is a 2-isomorphism, then so is α.

01C9 ## 9.2.9 Strict Epimorphisms

Let $\mathcal{C}$ be a bicategory.

01CA **Definition 9.2.9.1.1.** A 1-morphism $f \colon A \to B$ is a **strict epimorphism in $\mathcal{C}$** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^* \colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is injective on objects, i.e. its action on objects

$$f_* \colon \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(B, X)) \to \mathrm{Obj}(\mathsf{Hom}_{\mathcal{C}}(A, X))$$

is injective.

01CB **Remark 9.2.9.1.2.** In detail, f is a strict epimorphism if, for each diagram in $\mathcal{C}$ of the form

$$A \xrightarrow{f} B \underset{\psi}{\overset{\phi}{\rightrightarrows}} X,$$

if $\phi \circ f = \psi \circ f$, then $\phi = \psi$.

01CC **Example 9.2.9.1.3.** Here are some examples of strict epimorphisms.

01CD 1. *Strict Epimorphisms in* Cats_2. The strict epimorphisms in Cats_2 are characterised in Item 1 of Proposition 8.6.3.1.2.

01CE 2. *Strict Epimorphisms in* $\mathbf{Rel}$. The strict epimorphisms in $\mathbf{Rel}$ are characterised in Proposition 5.3.9.1.1.

01CF ## 9.2.10 Pseudoepic Morphisms

Let $\mathcal{C}$ be a bicategory.

01CG **Definition 9.2.10.1.1.** A 1-morphism $f \colon A \to B$ of $\mathcal{C}$ is **pseudoepic** if, for each $X \in \mathrm{Obj}(\mathcal{C})$, the functor

$$f^* \colon \mathsf{Hom}_{\mathcal{C}}(B, X) \to \mathsf{Hom}_{\mathcal{C}}(A, X)$$

given by precomposition by f is pseudomonic.

01CH **Remark 9.2.10.1.2.** In detail, a 1-morphism $f\colon A \to B$ of $\mathcal{C}$ is pseudoepic if it satisfies the following conditions:

01CJ 1. For all diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow\Downarrow\beta"] & X, \end{tikzcd}$$

if we have

$$\alpha \star \mathrm{id}_f = \beta \star \mathrm{id}_f,$$

then $\alpha = \beta$.

01CK 2. For each $X \in \mathrm{Obj}(\mathcal{C})$ and each 2-isomorphism

$$\beta\colon \phi \circ f \overset{\sim}{\Longrightarrow} \psi \circ f, \qquad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$, there exists a 2-isomorphism

$$\alpha\colon \phi \overset{\sim}{\Longrightarrow} \psi, \qquad \begin{tikzcd} B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

of $\mathcal{C}$ such that we have an equality

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, bend left, "\phi\circ f"] \arrow[r, bend right, "\psi\circ f"'] \arrow[r, phantom, "\beta\Downarrow"] & X \end{tikzcd}$$

of pasting diagrams in $\mathcal{C}$, i.e. such that we have

$$\beta = \alpha \star \mathrm{id}_f.$$

01CL **Proposition 9.2.10.1.3.** Let $f\colon A \to B$ be a 1-morphism of $\mathcal{C}$.

01CM 1. *Characterisations.* The following conditions are equivalent:

01CN (a) The morphism f is pseudoepic.

(b) The morphism f is corepresentably full on cores and corepre-
01CP sentably faithful.

01CQ (c) We have an isococomma square of the form

$$
B \overset{\text{eq.}}{\cong} B \overleftrightarrow{\coprod}_A B, \qquad
\begin{tikzcd}
B & B \arrow[l, "\mathrm{id}_B"'] \\
B \arrow[u, "\mathrm{id}_B"] \arrow[ur, dashed, Leftrightarrow] & A \arrow[l, "F"] \arrow[u, "F"']
\end{tikzcd}
$$

in $\mathcal{C}$ up to equivalence.

Proof. *Item 1*, *Characterisations*: Omitted. □

Appendices

9.A Other Chapters

Part V

Extra Part

Chapter 10

Miscellaneous Notes

01CR Contents

01CS 10.1 To Do List

01CT 10.1.1 Omitted Proofs To Add

Не так благотворна истина, как зловредна ее видимость.

Даниил Данковский

Truth does not do as much good in the world as the appearance of truth does evil.

Daniil Dankovsky

There's a very large number of omitted proofs throughout these notes. Here I list them in decreasing order of how nice it would be to add them.

01CU **Remark 10.1.1.1.1.** Proofs that *need* to be added at some point:

1. **??**.
2. **??**.
3. Horizontal composition of natural transformations is associative: **??** of **??**.
4. Fully faithful functors are essentially injective: **??** of **??**.

Proofs that *would be very nice* to be added at some point:

1. Properties of pseudomonic functors: **??**.
2. Characterisation of fully faithful functors: **??** of **??**.

Proofs that *would be nice* to be added at some point:

1. Properties of posetal categories: **??**.
2. The quadruple adjunction between categories and sets: **??**.
3. Properties of groupoid completions: **??**.
4. Properties of cores: **??**.
5. F_* faithful iff F faithful: **??** of **??**.
6. F_* full iff F full: **??** of **??**.
7. Injective on objects functors are precisely the isocofibrations in Cats_2: **??** of **??**.
8. Characterisations of monomorphisms of categories: **??** of **??**.
9. Epimorphisms of categories are surjective on objects: **??** of **??**.
10. Properties of pseudoepic functors: **??**.

01CV 10.1.2 Things To Explore/Add

Here we list things to be explored/added to this work in the future.

01CW **Remark 10.1.2.1.1.** Set theory through a category theory lens:

1. Isbell duality for sets.
2. Density comonads and codensity monads for sets.

Relations:

1. 2-Categorical monomorphisms and epimorphisms in **Rel**.
2. Co/limits in **Rel**.
3. Apartness composition, categorical properties of **Rel** with apartness, and apartness relations.
4. Apartness defines a composition for relations, but its analogue

$$\mathfrak{q} \mathbin{\Box} \mathfrak{p} \overset{\text{def}}{=} \int_{A \in C} \mathfrak{p}_A^{-1} \coprod \mathfrak{q}^A_{-2}$$

fails to be unital for profunctors. Is there a less obvious analogue of apartness composition for profunctors?

5. Codensity monad $\mathrm{Ran}_J(J)$ of a relation (What about $\mathrm{Rift}_J(J)$?)

6. Relative comonads in the 2-category of relations

7. Discrete fibrations and Street fibrations in **Rel**.

8. Consider adding the sections

   - The Monoidal Bicategory of Relations
   - The Monoidal Double Category of Relations

   to Relations.

Spans:

1. Universal property of the bicategory of spans, https://ncatlab.org/nlab/show/span

2. Write about cospans.

Un/Straightening:

1. Write proper sections on straightening for lax functors from sets to Rel or Span (displayed sets)

Categories:

1. Expand **??** and add a proof to it.

2. Sections and retractions; retracts, https://ncatlab.org/nlab/show/retract.

3. Regular categories: https://arxiv.org/pdf/2004.08964.pdf.

4. Are pseudoepic functors those functors whose restricted Yoneda embedding is pseudomonic and Yoneda preserves absolute colimits?

5. Absolutely dense functors enriched over $\mathbb{R}^+$ apparently reduce to topological density

Types of Morphisms in Categories:

1. Behaviour in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$, e.g. pointwise sections vs. sections in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.

2. A faithful functor from balanced category is conservative

Yoneda stuff:

1. Properties of restricted Yoneda embedding, e.g. if the restricted Yoneda embedding is full, then what can we conclude? Related: https://qchu.wordpress.com/2015/05/17/generators/

Adjunctions:

1. Adjunctions, units, counits, and fully faithfulness as in https://mathoverflow.net/questions/100808/properties-of-functors-and-their-adjoints.

2. Morphisms between adjunctions and bicategory $\mathsf{Adj}(\mathcal{C})$.

3. https://ncatlab.org/nlab/show/transformation+of+adjoints

Constructions With Categories:

1. Comparison between pseudopullbacks and isocomma categories: the "evident" functor $\mathcal{C} \times^{\mathsf{ps}}_{\mathcal{E}} \mathcal{D} \to \mathcal{C} \overset{\leftrightarrow}{\times}_{\mathcal{E}} \mathcal{D}$ is essentially surjective and full, but not faithful in general.

Co/limits:

1. Add the characterisations of absolutely dense functors given in **??** to **??**.

2. Absolutely dense functors, https://ncatlab.org/nlab/show/absolutely+dense+functor. Also theorem 1.1 here: http://www.tac.mta.ca/tac/volumes/8/n20/n20.pdf.

3. Dense functors, codense functors, and absolutely codense functors.

Co/ends:

1. Examples of co/ends: https://mathoverflow.net/a/461814

2. Cofinality for co/ends, https://mathoverflow.net/questions/353876

Fibred category theory:

1. Internal **Hom** in categories of co/Cartesian fibrations.

2. *Tensor structures on fibered categories* by Luca Terenzi: https://arxiv.org/abs/2401.13491. Check also the other papers by Luca Terenzi.

3. https://ncatlab.org/nlab/show/cartesian+natural+transformation (this is a cartesian morphism in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ apparently)

4. CoCartesian fibration classifying $\mathsf{Fun}(F, G)$, https://mathoverflow.net/questions/457533/cocartesian-fibration-classifying-mathrmfunf-g

Monoidal categories:

1. Free braided monoidal category with a braided monoid: https://ncatlab.org/nlab/show/vine

Skew monoidal categories:

1. Does the $\mathbb{E}_1$ tensor product of monoids admit a skew monoidal category structure?

2. Is there a (right?) skew monoidal category structure on $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ using right Kan extensions instead of left Kan extensions?

3. Similarly, are there skew monoidal category structures on the subcategory of $\mathbf{Rel}(A, B)$ spanned by the functions using left Kan extensions and left Kan lifts?

Higher categories:

1. Internal adjunctions in Mod as in [JY21, Section 6.3]; see [JY21, Example 6.2.6].

2. Comonads in the bicategory of profunctors.

Monoids:

1. Isbell's zigzag theorem for semigroups: the following conditions are equivalent:

   (a) A morphism $f\colon A \to B$ of semigroups is an epimorphism.

   (b) For each $b \in B$, one of the following conditions is satisfied:

      - We have $f(a) = b$.
      - There exist some $m \in \mathbb{N}_{\geq 1}$ and two factorisations

        $$\begin{aligned} b &= a_0 y_1, \\ b &= x_m a_{2m} \end{aligned}$$

        connected by relations

        $$\begin{aligned} a_0 &= x_1 a_1, \\ a_1 y_1 &= a_2 y_2, \\ x_1 a_2 &= x_2 a_3, \\ a_{2m-1} y_m &= a_{2m} \end{aligned}$$

        such that, for each $1 \leq i \leq m$, we have $a_i \in \mathrm{Im}(f)$.

   Wikipedia says in https://en.wikipedia.org/wiki/Isbell%27s_zigzag_theorem:

For monoids, this theorem can be written more concisely:

Types of morphisms in bicategories:

1. Behaviour in 2-categories of pseudofunctors (or lax functors, etc.), e.g. pointwise pseudoepic morphisms in vs. pseudoepic morphisms in 2-categories of pseudofunctors.
2. Statements like "coequifiers are lax epimorphisms", Item 2 of Examples 2.4 of https://arxiv.org/abs/2109.09836, along with most of the other statements/examples there.
3. Dense, absolutely dense, etc. morphisms in bicategories

Other:

1. https://qchu.wordpress.com/
2. https://aroundtoposes.com/
3. https://ncatlab.org/nlab/show/essentially+surjective+and+full+functor
4. https://mathoverflow.net/questions/415363/objects-whose-representable-presheaf-is-a-fibration
5. https://mathoverflow.net/questions/460146/universal-property-of-isbell-duality
6. http://www.tac.mta.ca/tac/volumes/36/12/36-12abs.html (Isbell conjugacy and the reflexive completion)
7. https://ncatlab.org/nlab/show/enrichment+versus+internalisation
8. The works of Philip Saville, https://philipsaville.co.uk/
9. https://golem.ph.utexas.edu/category/2024/02/from_cartesian_to_symmetric_mo.html
10. https://mathoverflow.net/q/463855 (One-object lax transformations)
11. https://ncatlab.org/nlab/show/analytic+completion+of+a+ring
12. https://en.wikipedia.org/wiki/Quaternionic_analysis
13. https://arxiv.org/abs/2401.15051 (The Norm Functor over Schemes)

14. https://mathoverflow.net/questions/407291/ (Adjunctions with respect to profunctors)
15. https://mathoverflow.net/a/462726 (Prof is free completion of Cats under right extensions)
16. there's some cool stuff in https://arxiv.org/abs/2312.00990 (Polynomial Functors: A Mathematical Theory of Interaction), e.g. on cofunctors.
17. https://ncatlab.org/nlab/show/adjoint+lifting+theorem
18. https://ncatlab.org/nlab/show/Gabriel%E2%80%93Ulmer+duality

Appendices

10.A Other Chapters

Bibliography

[MO 119454] user30818. *Category and the axiom of choice*. MathOverflow. URL: https://mathoverflow.net/q/119454 (cit. on p. 387).

[MO 321971] Ivan Di Liberti. *Characterization of pseudo monomorphisms and pseudo epimorphisms in Cat*. MathOverflow. URL: https://mathoverflow.net/q/321971 (cit. on p. 397).

[MO 350788] Peter LeFanu Lumsdaine. *Epimorphisms of relations*. MathOverflow. URL: https://mathoverflow.net/q/455260 (cit. on p. 264).

[MO 382264] Neil Strickland. *Proof that a cartesian category is monoidal*. MathOverflow. URL: https://mathoverflow.net/q/382264 (cit. on p. 22).

[MO 460656] Emily de Oliveira Santos. *Existence and characterisations of left Kan extensions and liftings in the bicategory of relations I*. MathOverflow. URL: https://mathoverflow.net/q/460656 (cit. on pp. 281, 282).

[MO 461592] Emily de Oliveira Santos. *Existence and characterisations of left Kan extensions and liftings in the bicategory of relations II*. MathOverflow. URL: https://mathoverflow.net/q/461592 (cit. on pp. 281, 282).

[MO 467527] Emily de Oliveira Santos. *What are the 2-categorical mono/epimorphisms in the 2-category of relations?* MathOverflow. URL: https://mathoverflow.net/q/467527 (cit. on pp. 261, 268).

[MO 468121a] Emily de Oliveira Santos. *Characterisations of functors F such that F^* or F_* is [property], e.g. faithful, conservative, etc.* MathOverflow. URL: https://mathoverflow.net/q/468125 (cit. on pp. 386, 387, 392, 394, 398, 400, 402).

[MO 468121b] Emily de Oliveira Santos. *Looking for a nice characterisation of functors F whose precomposition functor F^* is full*. MathOverflow. URL: https://mathoverflow.net/q/468121 (cit. on p. 381).

[MO 468334] Emily de Oliveira Santos. *Is a pseudomonic and pseudoepic functor necessarily an equivalence of categories?* MathOverflow. URL: https://mathoverflow.net/q/468334 (cit. on p. 397).

[MO 64365] Giorgio Mossa. *Natural transformations as categorical homotopies*. MathOverflow. URL: https://mathoverflow.net/q/64365 (cit. on p. 413).

[MSE 1465107] kilian. *Equivalence of categories and axiom of choice*. Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/1465107 (cit. on p. 387).

[MSE 2096272] Akiva Weinberger. *Is composition of two transitive relations transitive? If not, can you give me a counterexample?* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/2096272 (cit. on p. 333).

[MSE 267469] Zhen Lin. *Show that the powerset partial order is a cartesian closed category*. Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/267469 (cit. on p. 53).

[MSE 2855868] Qiaochu Yuan. *Is the category of pointed sets Cartesian closed?* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/2855868 (cit. on p. 105).

[MSE 350788] Qiaochu Yuan. *Mono's and epi's in the category Rel?* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/350788 (cit. on pp. 257, 263).

[MSE 733161] Stefan Hamcke. *Precomposition with a faithful functor*. Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/733161 (cit. on p. 384).

[MSE 733163] Zhen Lin. *Precomposition with a faithful functor*. Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/733163 (cit. on p. 378).

[MSE 749304] Zhen Lin. *If the functor on presheaf categories given by precomposition by F is ff, is F full? faithful?* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/749304 (cit. on p. 384).

[MSE 884460] Martin Brandenburg. *Why are the category of pointed sets and the category of sets and partial functions "essentially the same"?* Mathematics Stack Exchange. URL: https://math.stackexchange.com/q/884460 (cit. on p. 105).

[Adá+01] Jiří Adámek, Robert El Bashir, Manuela Sobral, and Jiří Velebil. "On Functors Which Are Lax Epimorphisms". In: *Theory Appl. Categ.* 8 (2001), pp. 509–521. ISSN: 1201-561X (cit. on pp. 379, 381, 384, 433).

[Bor94] Francis Borceux. *Handbook of Categorical Algebra I.* Vol. 50. Encyclopedia of Mathematics and its Applications. Basic Category Theory. Cambridge University Press, Cambridge, 1994, pp. xvi+345. ISBN: 0-521-44178-1 (cit. on p. 409).

[BS10] John C. Baez and Michael Shulman. "Lectures on n-Categories and Cohomology". In: *Towards higher categories.* Vol. 152. IMA Vol. Math. Appl. Springer, New York, 2010, pp. 1–68. DOI: 10.1007/978-1-4419-1524-5_1. URL: https://doi.org/10.1007/978-1-4419-1524-5_1 (cit. on pp. 4, 381).

[Cie97] Krzysztof Ciesielski. *Set Theory for the Working Mathematician.* Vol. 39. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1997, pp. xii+236. ISBN: 0-521-59441-3; 0-521-59465-0. DOI: 10.1017/CBO9781139173131. URL: https://doi.org/10.1017/CBO9781139173131 (cit. on p. 47).

[DFH75] Aristide Deleanu, Armin Frei, and Peter Hilton. "Idempotent Triples and Completion". In: *Math. Z.* 143 (1975), pp. 91–104. ISSN: 0025-5874,1432-1823. DOI: 10.1007/BF01173053. URL: https://doi.org/10.1007/BF01173053 (cit. on p. 391).

[Fre09] Jonas Frey. *On the 2-Categorical Duals of (Full and) Faithful Functors.* https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=4c289321d622f8fcf947e7a7cfd1bdf75c95ca3 Archived at https://web.archive.org/web/20240331195546/https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=4c289321d622f8fcf947e7a7cfd1bdf75c95ca33. July 2009. URL: https://citeseerx.ist.psu.edu/document?repid=rep1%5C&type=pdf%5C&doi=4c289321d622f8fcf947e7a7cfd1bdf75 (cit. on pp. 379, 384).

[GGN15] David Gepner, Moritz Groth, and Thomas Nikolaus. "Universality of Multiplicative Infinite Loop Space Machines”. In: *Algebr. Geom. Topol.* 15.6 (2015), pp. 3107–3153. ISSN: 1472-2747. DOI: 10.2140/agt.2015.15.3107. URL: https://doi.org/10.2140/agt.2015.15.3107 (cit. on p. 222).

[Isb68] John R. Isbell. “Epimorphisms and Dominions. III”. In: *Amer. J. Math.* 90 (1968), pp. 1025–1030. ISSN: 0002-9327,1080-6377. DOI: 10.2307/2373286. URL: https://doi.org/10.2307/2373286 (cit. on p. 393).

[JY21] Niles Johnson and Donald Yau. *2-Dimensional Categories*. Oxford University Press, Oxford, 2021, pp. xix+615. ISBN: 978-0-19-887138-5; 978-0-19-887137-8. DOI: 10.1093/oso/9780198871378.001.0001. URL: https://doi.org/10.1093/oso/9780198871378.001.0001 (cit. on p. 446).

[Low15] Zhen Lin Low. *Notes on Homotopical Algebra*. Nov. 2015. URL: https://zll22.user.srcf.net/writing/homotopical-algebra/2015-11-10-Main.pdf (cit. on p. 384).

[nLa24] nLab Authors. *Groupoid*. https://ncatlab.org/nlab/show/groupoid. Oct. 2024 (cit. on p. 389).

[nLab23] nLab Authors. *Skeleton*. 2024. URL: https://ncatlab.org/nlab/show/skeleton (cit. on p. 354).

[Pro24a] Proof Wiki Contributors. *Bijection Between $R \times (S \times T)$ and $(R \times S) \times T$ — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Bijection_between_R_x_(S_x_T)_and_(R_x_S)_x_T (cit. on p. 22).

[Pro24b] Proof Wiki Contributors. *Bijection Between $S \times T$ and $T \times S$ — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Bijection_between_S_x_T_and_T_x_S (cit. on p. 22).

[Pro24c] Proof Wiki Contributors. *Cartesian Product Distributes Over Set Difference — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Cartesian_Product_Distributes_over_Set_Difference (cit. on p. 22).

[Pro24d] Proof Wiki Contributors. *Cartesian Product Distributes Over Symmetric Difference — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Cartesian_Product_Distributes_over_Symmetric_Difference (cit. on p. 22).

[Pro24e] Proof Wiki Contributors. *Cartesian Product Distributes Over Union — Proof Wiki*. 2024. URL: https://proofwiki.

org/wiki/Cartesian_Product_Distributes_over_Union (cit. on p. 22).

[Pro24f] Proof Wiki Contributors. *Cartesian Product Is Empty Iff Factor Is Empty — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Cartesian_Product_is_Empty_iff_Factor_is_Empty (cit. on p. 22).

[Pro24g] Proof Wiki Contributors. *Cartesian Product of Intersections — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Cartesian_Product_of_Intersections (cit. on p. 22).

[Pro24h] Proof Wiki Contributors. *Characteristic Function of Intersection — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Characteristic_Function_of_Intersection (cit. on p. 53).

[Pro24i] Proof Wiki Contributors. *Characteristic Function of Set Difference — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Characteristic_Function_of_Set_Difference (cit. on p. 57).

[Pro24j] Proof Wiki Contributors. *Characteristic Function of Symmetric Difference — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Characteristic_Function_of_Symmetric_Difference (cit. on p. 62).

[Pro24k] Proof Wiki Contributors. *Characteristic Function of Union — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Characteristic_Function_of_Union (cit. on p. 50).

[Pro24l] Proof Wiki Contributors. *Complement of Complement — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Complement_of_Complement (cit. on p. 58).

[Pro24m] Proof Wiki Contributors. *Condition For Mapping from Quotient Set To Be A Surjection — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Condition_for_Mapping_from_Quotient_Set_to_be_Surjection (cit. on p. 341).

[Pro24n] Proof Wiki Contributors. *Condition For Mapping From Quotient Set To Be An Injection— Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Condition_for_Mapping_from_Quotient_Set_to_be_Injection (cit. on p. 341).

[Pro24o] Proof Wiki Contributors. *Condition For Mapping From Quotient Set To Be Well-Defined — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Condition_for_Mapping_from_Quotient_Set_to_be_Well-Defined (cit. on p. 341).

[Pro24p] Proof Wiki Contributors. *De Morgan's Laws (Set Theory) — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/De_Morgan%5C%27s_Laws_(Set_Theory) (cit. on pp. 56, 58).

[Pro24q] Proof Wiki Contributors. *De Morgan's Laws (Set Theory)/Set Difference/Difference with Union — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/De_Morgan%5C%27s_Laws_(Set_Theory)/Set_Difference/Difference_with_Union (cit. on p. 56).

[Pro24r] Proof Wiki Contributors. *Equivalence of Definitions of Symmetric Difference — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Equivalence_of_Definitions_of_Symmetric_Difference (cit. on p. 62).

[Pro24s] Proof Wiki Contributors. *Image of Intersection Under Mapping — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Image_of_Intersection_under_Mapping (cit. on p. 87).

[Pro24t] Proof Wiki Contributors. *Image of Union Under Mapping — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Image_of_Union_under_Mapping (cit. on p. 87).

[Pro24u] Proof Wiki Contributors. *Intersection Distributes Over Symmetric Difference — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Intersection_Distributes_over_Symmetric_Difference (cit. on p. 62).

[Pro24v] Proof Wiki Contributors. *Intersection Is Associative — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Intersection_is_Associative (cit. on p. 53).

[Pro24w] Proof Wiki Contributors. *Intersection Is Commutative — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Intersection_is_Commutative (cit. on p. 53).

[Pro24x] Proof Wiki Contributors. *Intersection With Empty Set — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Intersection_with_Empty_Set (cit. on p. 53).

[Pro24y] Proof Wiki Contributors. *Intersection With Set Difference Is Set Difference With Intersection — Proof Wiki*.

2024. URL: https://proofwiki.org/wiki/Intersection_with_Set_Difference_is_Set_Difference_with_Intersection (cit. on p. 56).

[Pro24z] Proof Wiki Contributors. *Intersection With Subset Is Subset — Proof Wiki.* 2024. URL: https://proofwiki.org/wiki/Intersection_with_Subset_is_Subset (cit. on p. 53).

[Pro24aa] Proof Wiki Contributors. *Mapping From Quotient Set When Defined Is Unique — Proof Wiki.* 2024. URL: https://proofwiki.org/wiki/Mapping_from_Quotient_Set_when_Defined_is_Unique (cit. on p. 341).

[Pro24ab] Proof Wiki Contributors. *Preimage of Intersection Under Mapping — Proof Wiki.* 2024. URL: https://proofwiki.org/wiki/Preimage_of_Intersection_under_Mapping (cit. on p. 91).

[Pro24ac] Proof Wiki Contributors. *Preimage of Union Under Mapping — Proof Wiki.* 2024. URL: https://proofwiki.org/wiki/Preimage_of_Union_under_Mapping (cit. on p. 91).

[Pro24ad] Proof Wiki Contributors. *Quotient Mapping Is Coequalizer — Proof Wiki.* 2024. URL: https://proofwiki.org/wiki/Quotient_Mapping_is_Coequalizer (cit. on p. 45).

[Pro24ae] Proof Wiki Contributors. *Set Difference as Intersection With Complement — Proof Wiki.* 2024. URL: https://proofwiki.org/wiki/Set_Difference_as_Intersection_with_Complement (cit. on p. 57).

[Pro24af] Proof Wiki Contributors. *Set Difference as Symmetric Difference With Intersection — Proof Wiki.* 2024. URL: https://proofwiki.org/wiki/Set_Difference_as_Symmetric_Difference_with_Intersection (cit. on p. 57).

[Pro24ag] Proof Wiki Contributors. *Set Difference Is Right Distributive Over Union — Proof Wiki.* 2024. URL: https://proofwiki.org/wiki/Set_Difference_is_Right_Distributive_over_Union (cit. on p. 56).

[Pro24ah] Proof Wiki Contributors. *Set Difference Over Subset — Proof Wiki.* 2024. URL: https://proofwiki.org/wiki/Set_Difference_over_Subset (cit. on p. 56).

[Pro24ai] Proof Wiki Contributors. *Set Difference With Empty Set Is Self — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Difference_with_Empty_Set_is_Self (cit. on p. 57).

[Pro24aj] Proof Wiki Contributors. *Set Difference With Self Is Empty Set — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Difference_with_Self_is_Empty_Set (cit. on p. 57).

[Pro24ak] Proof Wiki Contributors. *Set Difference With Set Difference Is Union of Set Difference With Intersection — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Difference_with_Set_Difference_is_Union_of_Set_Difference_with_Intersection (cit. on p. 57).

[Pro24al] Proof Wiki Contributors. *Set Difference With Subset Is Superset of Set Difference — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Difference_with_Subset_is_Superset_of_Set_Difference (cit. on p. 56).

[Pro24am] Proof Wiki Contributors. *Set Difference With Union — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Difference_with_Union (cit. on p. 56).

[Pro24an] Proof Wiki Contributors. *Set Intersection Distributes Over Union — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Intersection_Distributes_over_Union (cit. on p. 53).

[Pro24ao] Proof Wiki Contributors. *Set Intersection Is Idempotent — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Intersection_is_Idempotent (cit. on p. 53).

[Pro24ap] Proof Wiki Contributors. *Set Intersection Preserves Subsets — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Intersection_Preserves_Subsets (cit. on p. 53).

[Pro24aq] Proof Wiki Contributors. *Set Union Is Idempotent — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Union_is_Idempotent (cit. on p. 50).

[Pro24ar] Proof Wiki Contributors. *Set Union Preserves Subsets — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Set_Union_Preserves_Subsets (cit. on p. 50).

[Pro24as] Proof Wiki Contributors. *Symmetric Difference Is Associative — Proof Wiki*. 2024. URL: https://proofwiki.

org/wiki/Symmetric_Difference_is_Associative (cit. on p. 62).

[Pro24at] Proof Wiki Contributors. *Symmetric Difference Is Commutative — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_is_Commutative (cit. on p. 62).

[Pro24au] Proof Wiki Contributors. *Symmetric Difference of Complements — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_of_Complements (cit. on p. 62).

[Pro24av] Proof Wiki Contributors. *Symmetric Difference on Power Set Forms Abelian Group — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_on_Power_Set_forms_Abelian_Group (cit. on p. 62).

[Pro24aw] Proof Wiki Contributors. *Symmetric Difference With Complement — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Complement (cit. on p. 62).

[Pro24ax] Proof Wiki Contributors. *Symmetric Difference With Empty Set — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Empty_Set (cit. on p. 62).

[Pro24ay] Proof Wiki Contributors. *Symmetric Difference With Intersection Forms Ring — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Intersection_forms_Ring (cit. on p. 62).

[Pro24az] Proof Wiki Contributors. *Symmetric Difference With Self Is Empty Set — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Self_is_Empty_Set (cit. on p. 62).

[Pro24ba] Proof Wiki Contributors. *Symmetric Difference With Union Does Not Form Ring — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Union_does_not_form_Ring (cit. on p. 61).

[Pro24bb] Proof Wiki Contributors. *Symmetric Difference With Universe — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Symmetric_Difference_with_Universe (cit. on p. 62).

[Pro24bc] Proof Wiki Contributors. *Union as Symmetric Difference With Intersection — Proof Wiki*. 2024. URL: https://

proofwiki.org/wiki/Union_as_Symmetric_Difference_with_Intersection (cit. on p. 50).

[Pro24bd] Proof Wiki Contributors. *Union Distributes Over Intersection — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Union_Distributes_over_Intersection (cit. on p. 50).

[Pro24be] Proof Wiki Contributors. *Union Is Associative — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Union_is_Associative (cit. on p. 50).

[Pro24bf] Proof Wiki Contributors. *Union Is Commutative — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Union_is_Commutative (cit. on p. 50).

[Pro24bg] Proof Wiki Contributors. *Union of Symmetric Differences — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Union_of_Symmetric_Differences (cit. on p. 62).

[Pro24bh] Proof Wiki Contributors. *Union With Empty Set — Proof Wiki*. 2024. URL: https://proofwiki.org/wiki/Union_with_Empty_Set (cit. on p. 50).

[PS19] Maximilien Péroux and Brooke Shipley. "Coalgebras in Symmetric Monoidal Categories of Spectra". In: *Homology Homotopy Appl.* 21.1 (2019), pp. 1–18. ISSN: 1532-0073. DOI: 10.4310/HHA.2019.v21.n1.a1. URL: https://doi.org/10.4310/HHA.2019.v21.n1.a1 (cit. on p. 223).

[Rie17] Emily Riehl. *Category Theory in Context*. Vol. 10. Aurora: Dover Modern Math Originals. Courier Dover Publications, 2017, pp. xviii+240. ISBN: 978-0486809038. URL: http://www.math.jhu.edu/~eriehl/context.pdf (cit. on p. 389).

[Wik24] Wikipedia Contributors. *Multivalued Function — Wikipedia, The Free Encyclopedia*. 2024. URL: https://en.wikipedia.org/wiki/Multivalued_function (cit. on p. 235).

Index of Notation

Index of Set Theory

Index of Category Theory

Index of Higher Category Theory