SAPIENZA
UNIVERSITÀ DI ROMA

Solving the homogeneous Bethe-Salpeter equation for a fermion-scalar system in Minkowski space

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Anno Accademico 2016/2017

Abstract

The homogeneous Bethe-Salpeter equation for a bound state composed by a fermion and a scalar, interacting through the exchange of a massive scalar particle, is solved in the physical Minkowski space within the ladder approximation. The method of solution, based on i) the Nakanishi integral representation of the Bethe-Salpeter amplitude and ii) the projection of the equation onto the null hyperplane $x^+ = 0$, previously applied to bound states of two identical scalars or fermions, is generalized to the case of unequal constituent spins and masses.

The binding-energy/coupling-constant functional dependence and the valence light-front wave functions of the bound state are obtained. The low-binding solutions behave non-relativistically, while the high-binding ones exhibit a new phenomenon of binding saturation. A physical discussion is presented and some perspectives for future work are outlined.

Riassunto

L'equazione omogenea di Bethe-Salpeter per uno stato legato composto da un fermione ed uno scalare, interagenti tramite lo scambio di una particella scalare massiva, viene risolta nello spazio fisico di Minkowski nell'ambito dell'approssimazione ladder. Il metodo di soluzione, basato su i) la rappresentazione integrale di Nakanishi dell'ampiezza di Bethe-Salpeter e ii) la proiezione dell'equazione sull'iperpiano $x^+ = 0$, applicato in precedenza a stati legati di due scalari o fermioni identici, viene esteso al caso di costituenti con spin e masse distinte.

Vengono ricavate la dipendenza funzionale energia di legame/costante di accoppiamento e le funzioni d'onda light-front di valenza dello stato legato. Le soluzioni per basse energie di legame esibiscono un comportamento non relativistico, mentre ad alte energie si manifesta un fenomeno inatteso di saturazione del legame. Si presenta una discussione fisica e si delineano alcune prospettive per lavori futuri.

Acknowledgements

I wish to thank Professor Giovanni Salmè, for having introduced me to the beautiful subject of bound states in quantum field theory, and for dedicating so much of his time into this work. I am grateful to my friend and colleague Jorge Henrique de Alvarenga Nogueira, which worked by my side from the first steps of my Thesis, and was a precious guide for the numerical part of this work. The numerical solution of the Wick-rotated BSE is in the large part due to him.

I am indebted to the collaborators of Prof. Salmè and, in particular, to my colleague Damaso Colasante (whom I did not get the luck to meet), which performed an essential part of the formal work for the numerical solution of the Minkowski space BSE. Moreover, I am happy to thank Professor Emanuele Pace for patiently checking the calculations for the Minkowski space BSE, and for pointing out several mistakes and possible improvements in the presentation of the formulas, and Professor Tobias Frederico, for the useful discussions and insights, which eventually enabled us to get things working.

Ringrazio i miei genitori, Margherita e Guido, per aver reso possibile tutto ciò. Ringrazio Giovanna, Matteo e Sara, compagni di viaggio in questi ultimi due anni. E infine, ringrazio il piccolo Sfera, il mio unico contatto con la realtà nel caldissimo mese di agosto 2017.

Grazie,

Valerio.

Contents

Chapter 1

Introduction

It is not an overstatement to say that, after ninety years since the birth of quantum field theory, our current understanding of bound states in the relativistic domain is still far from complete. Indeed, while one can easily grasp, on the formal side, the implications of the existence of a bound state in the particle spectrum (i.e. the presence of poles in the Green's functions of the theory), the calculation of the corresponding physical observables has to confront difficulties both from the theoretical and computational point of view. This is a quite unpleasant situation if one considers that, e.g., according to our present knowledge of strong interactions, the entire hadronic spectrum is composed of bound states. Also in condensed matter physics the issue of realistically describing bound systems, yet taking properly into account the relativistic effects, is nowadays an alive topic, as shown by the intense research on graphene.

The main answer to such a problem has been the path integral method, that in high-energy physics has achieved a very refined standard in order to deal with the highly non linear behavior of the Quantum Chromo-Dynamics. Nonetheless, the interest in elaborating tools for reaching a full-glory description of the bound system dynamics in the physical space is still present.

To understand the root of the difficulties, one must realize that a great deal of our current insights into quantum field theory comes from the perturbative approach. When the coupling constants are small, the perturbative framework is particularly suitable for describing scattering processes. On the other hand, bound states are by nature a totally non-perturbative phenomenon, as clearly indicated by the need of reconstructing a pole in the relevant Green's function. This qualitative difference might be understood as follows: while scattering particles interact for a small amount of time, during which they may exchange only a few virtual mediators, the constituents of a bound state interact indefinitely. In the former case, the scattering cross section can be calculated from a few relevant Feynman diagrams. In the latter case, one needs to consider, loosely speaking, a diagram with an infinite number of virtual exchanges, which results on the formal level in the bound state amplitude satisfying an homogeneous integral equation.

In 1951, Salpeter and Bethe [1] and Gell-Mann and Low [2] set the basis for a rigorous field theoretical description of bound states, with what has come to be known as the Bethe-Salpeter equation (BSE). As Salpeter put it in the late 2008 [3]:

> «The Bethe-Salpeter equation was an attempt to put bound states between two or more elementary particles on a fully covariant relativistic footing.»

This was achieved by writing down a four-dimensional homogeneous integral equation for a quantity, the Bethe-Salpeter (BS) amplitude, which Bethe and Salpeter suggested to be the relativistic analogue of the Schrödinger wave-function of a bound system. The solution of the BSE allowed one to calculate the mass of the bound state for a fixed coupling constant (or vice-versa), and was applied in Bethe-Salpeter's original paper to obtain the mass/coupling-constant relation for the deuteron's ground state.

The field-theoretical definition of the BS amplitude was immediately clarified by Gell-Mann and Low [2], which gave its modern definition in terms of an elementary correlation function. Subsequently, the BS formalism was greatly extended to cope with the calculation of bound-state dynamical observables and S-matrix elements, due, in particular, to the work of Mandelstam [4] and Zimmermann [5][1].

As one can realize from Salpeter's historical note [3], the computational difficulty posed by the BSE was immediately understood by the authors of the equation themselves. To cope with one of the hardest issues, namely how to deal with the singularity structure of both the BS amplitude and the interaction kernel, Wick proposed an analytic continuation of the BS amplitude [7], which became famous as the Wick-rotation. The Wick-rotated BSE was a far more manageable problem, to the point that it was even possible to exhibit analytic solutions for a very simplified version of the equation, the so-called Wick-Cutkosky model, after Refs. [7, 8]. This model was elaborated for describing a bound state composed by two massive scalars exchanging a massless scalar, involving the interaction kernel in ladder approximation, i.e. when the exchanged quanta do not cross each other. In spite of its very simplified nature, the analytic solution of Wick and Cutkosky still provides a precious guide in the investigation of the BSE within more refined models.

Unfortunately, the Wick-rotation of the BS amplitude had also serious drawbacks. First of all, the analytic continuation in the whole BSE was rigorously shown to be possible only for a ladder kernel. More importantly, the Wick-rotated BS amplitude was not directly applicable to the calculation of dynamical quantities, such as form factors, so that the four-dimensional BSE was downgraded to a tool to compute the bound-state mass spectrum.

Shortly, one should also mention the attempts based on suitable three-dimensional reductions, the first of which was proposed by Salpeter himself [9]. These reductions enabled to overcome the difficulties preventing the direct treatment in Minkowski space, but also had undesired features (such as the loss of covariance in Salpeter's equation, which involved an instantaneous interaction kernel).

During approximately the same years, the Japanese physicist Noboru Nakanishi was developing an integral representation for general Feynman diagrams [10], called in what follows the Nakanishi Integral Representation (NIR), which allowed to formally sum up the infinite set of diagrams contributing to a given n-leg transition amplitude. The author himself proposed to extend his representation to the BS amplitude, for which he rigorously proved its validity in the case of a scalar ladder kernel [11]. Nakanishi also wrote an authoritative review on the BSE [12], in which however the NIR did not play a central role.

Apparently, Nakanishi's work on integral representations did not have much impact on the field of the BSE, until 1995, when the NIR was resumed by K. Kusaka and A. G. Williams [13] and applied to the numerical solution of the scalar ladder BSE *in Minkowski space,* i.e. without resorting to the Wick-rotation. The use of NIR allowed to transform the four-dimensional momentum space BSE into an equivalent two-dimensional integral equation for a weight function of two real variables, which could be solved numerically by standard methods. Some years later, the technique was refined by J. Carbonell and V. A. Karmanov [14, 15] within the context of light-front (LF) quantum field theory. The same authors applied their NIR solution to the calculation of electromagnetic form factors [16], thereby clarifying some issues related to the Wick-rotation of the BS amplitude, and performed a first exploration of the BSE for a two-fermions bound state [17].

The NIR-LF approach was subsequently improved by T. Frederico *et. al*, which confirmed and extended the previous results on the scalar ladder BSE [18, 19], and indicated the correct treatment of singularities in the fermion ladder BSE [20, 21]. The same authors also performed a first exploration of the scattering BSE [22, 23], which is the inhomogeneous analogue of the bound state BSE and allows a non-perturbative computation of the elastic scattering amplitude.

[1] With respect to Zimmermann's work, the thorough analysis of asymptotic conditions performed by Haag [6] is also relevant, although Haag did not make direct use of Bethe-Salpeter amplitudes in his work.

At this point, it is worth to emphasize that the state of art in the solution of the Minkowski space BSE is still at an infant stage. In fact, adopting free propagators and retaining only the ladder contribution to the interaction kernel for getting numerical solutions, as discussed in detail in what follows, still prevents the Minkowskian BSE from being a tool for wide investigations in the phenomenological realm. Nonetheless, as is well known, the solution of models with simple but non trivial ingredients provides valuable insights into a field which is, to the present date, relatively unexplored. This is the spirit which guided the investigations to be described in what follows.

In this Thesis, we shall present the results of a quantitative study on the ladder BSE in Minkowski space for a bound system composed by a fermion and a scalar, freely propagating inside the system but interacting through the exchange of a massive scalar.

The interest in the fermion-scalar problem is two-fold. In the first place, this is a natural extension of the previous investigations, with the additional feature that the constituents are necessarily distinguishable particles. As it will become clear in the following, the spoiling of the particle-permutation symmetry makes the actual solution of the BSE somewhat trickier. On the other hand, this is an important step to be performed, before being able to move to more realistic situations. In this respect, it is worth to mention that we have solved the BSE through NIR for generally unequal masses, which has never been done before. Indeed, such a new degree of freedom allows one to better understand the interplay between different dynamical regimes (e.g. low binding/strong binding). In second instance, the fermion-scalar system might be regarded as a primitive version of a quark-diquark baryon model, so that this kind of investigations could turn out useful, allowing to increase our physical intuition on the issue, for driving the elaboration of more refined descriptions, where the self-energy of the constituents will play a fundamental role.

The Thesis is organized as follows. In Chapter 2 we review the basic formalism of the Bethe-Salpeter equation. In Chapter 3 we introduce the main ingredients of our numerical approach to the BSE, namely: (i) the NIR of the BS amplitude and (ii) the light-front projection of the BSE. Moreover, we quote some first relevant results. In Chapter 4 we present our formal elaboration for the fermion-scalar BSE. In Chapter 5 we present our numerical results. Finally, in Chapter 6 we draw our conclusions and discuss the perspectives for future work.

In order to avoid too lengthy and obscure discussions in Chaps. 4 and 5 but, at the same time, to present our method in the most transparent way, we have chosen to report our full calculations in Appendix C. The three other appendices contain: (A) a formal derivation of the NIR of the BS amplitude, starting from the one of the bound state vertex, (B) a discussion of some technicalities related to the Poincaré group in the LF formalism, which are necessary in order to deal with a system with spin and (D) a brief review of the non-relativistic Yukawa potential, whose consideration allowed to give a more organized presentation of our results.

Chapter 2

The Bethe-Salpeter Equation

In this Chapter we introduce the standard formalism of Bethe-Salpeter amplitudes for the description of bound states in quantum field theory. In Sec. 2.1 we study the analytic structure of the four-points Green's function and derive the Bethe-Salpeter equation for the bound state amplitude. In Sec. 2.2 we discuss some developments of the formalism, including the Wick-rotation and the calculation of form factors from the BS amplitude.

For the sake of simplicity, in what follows we shall consider only scalar field theories. The necessary generalizations for the fermion-scalar case will be discussed in Chapter 4.

Throughout this Thesis, we shall set $\hbar = c = 1$ and adopt the signature $(+,-,-,-)$ for the Minkowski metric.

2.1 Bound states in quantum field theory

The existence of a bound state in a quantum field theory is revealed by the presence of poles in the Fourier transform of the relevant Green's functions. In particular, for a two particles bound state we find poles in the four-points Green's function of the constituent fields, with residues determined by the Bethe-Salpeter amplitudes (BS amplitudes) of the bound state. These amplitudes satisfy a four-dimensional homogeneous integral equation, the Bethe-Salpeter equation (BSE) [1], which is the subject of this Thesis.

In the following Subsections, we shall review the field-theoretic derivation [2, 4, 24] of the celebrated result of Salpeter and Bethe.

2.1.1 The BS amplitude and the four-points Green's function

Following Refs. [12, 24], we consider the four-points Green's function of two interacting scalar fields[1]:

$$\begin{aligned} G(x_1,x_2;y_1,y_2) &\equiv \langle 0|\mathrm{T}\{\phi_1(x_1)\phi_2(x_2)\phi_1^\dagger(y_1)\phi_2^\dagger(y_2)\}|0\rangle \\ &= \theta(\min(x_1,x_2)-\max(y_1,y_2))\langle 0|\mathrm{T}\{\phi_1(x_1)\phi_2(x_2)\}\mathrm{T}\{\phi_1^\dagger(y_1)\phi_2^\dagger(y_2)\}|0\rangle + \\ &+ \text{other orderings}, \end{aligned} \tag{2.1.1}$$

where the omitted part is irrelevant to our present discussion, as we shall clarify below.

To study the analytic structure of G, we proceed in a manner similar to the Källén-Lehmann decomposition of the propagator. We insert a complete set of eigenstates of the four-momentum

$$P^\mu|n\rangle = P_n^\mu|n\rangle, \qquad \sum_n |n\rangle\langle n| = 1 \tag{2.1.2}$$

[1] In what follows, unless otherwise stated, field operators are always understood in the Heisenberg picture.

between the two time-ordered products in the second line of Eq. (2.1.1):

$$G(x_1,x_2;y_1,y_2)=\theta(\min(x_1,x_2)-\max(y_1,y_2))\sum_n \chi_n(x_1,x_2)\overline{\chi}_n(y_1,y_2)+\cdots, \tag{2.1.3}$$

where we have defined:

$$\chi_n(x_1,x_2)\equiv\langle 0|\mathrm{T}\{\phi_1(x_1)\phi_2(x_2)\}|n\rangle, \tag{2.1.4}$$

$$\overline{\chi}_n(y_1,y_2)\equiv\langle n|\mathrm{T}\{\phi_1^\dagger(y_1)\phi_2^\dagger(y_2)\}|0\rangle. \tag{2.1.5}$$

It is convenient to introduce global and relative coordinates:

$$X\equiv\eta_1 x_1+\eta_2 x_2,\qquad x\equiv x_1-x_2, \tag{2.1.6}$$

where $\eta_{1,2}$ are real numbers satisfying $\eta_1+\eta_2=1$. The inverse relations are:

$$x_1=X+\eta_2 x,\qquad x_2=X-\eta_1 x. \tag{2.1.7}$$

From the translational invariance of the theory, it follows that:

$$\chi_n(x_1,x_2)=e^{-iP_nX}\chi_n(x), \tag{2.1.8}$$

$$\chi_n(x)\equiv\langle 0|\mathrm{T}\{\phi_1(\eta_2 x)\phi_2(-\eta_1 x)\}|n\rangle, \tag{2.1.9}$$

and similarly:

$$\overline{\chi}_n(y_1,y_2)=e^{iP_nY}\overline{\chi}_n(y), \tag{2.1.10}$$

$$\overline{\chi}_n(y)\equiv\langle n|\mathrm{T}\{\phi_1^\dagger(\eta_2 y)\phi_2^\dagger(-\eta_1 y)\}|0\rangle. \tag{2.1.11}$$

For the Green's function G, we have:

$$G(x_1,x_2;y_1,y_2)=G(x_1+a,x_2+a;y_1+a,y_2+a), \tag{2.1.12}$$

which allows to write its Fourier transform as:

$$G(x_1,x_2;y_1,y_2)=\int\frac{\mathrm{d}^4P}{(2\pi)^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4q}{(2\pi)^4}\,e^{-iP(X-Y)-ikx+iqy}G(k,q;P), \tag{2.1.13}$$

$$G(k,q;P)\equiv\int\mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4R\,e^{ikx-iqy+iPR}G(R+\eta_2 x,R-\eta_1 x;\eta_2 y,-\eta_1 y). \tag{2.1.14}$$

We summarize in Table 2.1.1 our conventions for the transformations to and from global and relative variables. The ordinary four-variables Fourier transform of G is related to $G(k,q;P)$ by:

$$\begin{aligned}G(k_1,k_2;q_1,q_2)&\equiv\int\mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4y_1\mathrm{d}^4y_2e^{ik_1x_1+ik_2x_2-iq_1y_1-iq_2y_2}G(x_1,x_2;y_1,y_2)\\&=(2\pi)^4\delta^4(k_1+k_2-q_1-q_2)G(k,q;P),\end{aligned} \tag{2.1.15}$$

where $P=k_1+k_2$. Notice that the same symbol is used for the Green's function in both coordinates and momentum spaces, without ambiguity, given the dependence upon the proper variables.

Coordinates	Conjugate momenta
$x=x_1-x_2$	$p=\eta_2 p_1-\eta_1 p_2$
$X=\eta_1 x_1+\eta_2 x_2$	$P=p_1+p_2$
$x_1=X+\eta_2 x$	$p_1=p+\eta_1 P$
$x_2=X-\eta_1 x$	$p_2=-p+\eta_2 P$

Table 2.1.1. Transformation to and from global and relative variables. Notice that $p_1x_1+p_2x_2=PX+px$.

Let us now consider the contribution of a bound state to the decomposition (2.1.3). We assume, for simplicity, that there exists a single bound state $|b;P\rangle$, with mass $P^2 = M^2$ and spin zero, such that the *BS amplitudes*:

$$\chi(x_1,x_2;P) \equiv \langle 0|\mathrm{T}\{\phi_1(x_1)\phi_2(x_2)\}|b;P\rangle, \tag{2.1.16}$$

$$\overline{\chi}(y_1,y_2;P) \equiv \langle b;P|\mathrm{T}\{\phi_1^\dagger(y_1)\phi_2^\dagger(y_2)\}|0\rangle, \tag{2.1.17}$$

are non-zero (the case of spin degeneracy is treated in detail in Subsec. 2.2.3). This state contributes to the sum in Eq. (2.1.3) as[2]:

$$G_b(x_1,x_2;y_1,y_2) \equiv \theta(\min(x_1,x_2)-\max(y_1,y_2))\times \\ \times \int \frac{\mathrm{d}^3\mathbf{P}}{(2\pi)^3 2E(\mathbf{P})} e^{-iE(\mathbf{P})(X^0-Y^0)+i\mathbf{P}\cdot(\mathbf{X}-\mathbf{Y})}\chi(x;E(\mathbf{P}),\mathbf{P})\overline{\chi}(y;E(\mathbf{P}),\mathbf{P}), \tag{2.1.18}$$

where $E(\mathbf{P}) = \sqrt{M^2+\mathbf{P}^2}$. Using the Fourier transform of the step function:

$$\theta(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\frac{ie^{-i\omega t}}{\omega+i\epsilon}, \tag{2.1.19}$$

and noticing that:

$$\min(x_1^0,x_2^0)-\max(y_1^0,y_2^0) = X^0-Y^0-\frac{|x^0|-(\eta_2-\eta_1)x^0}{2}+\frac{|y^0|-(\eta_2-\eta_1)y^0}{2}$$

we obtain:

$$G_b(x_1,x_2;y_1,y_2) = \int \frac{\mathrm{d}^4P}{(2\pi)^4} e^{-iP(X-Y)}\frac{i\xi(x;P)\overline{\xi}(y;P)}{2E(\mathbf{P})(P^0-E(\mathbf{P})+i\varepsilon)}, \tag{2.1.20}$$

where we have made the change of variables $\omega = P^0 - E(\mathbf{P})$ and defined the auxiliary quantities:

$$\xi(x;P) \equiv \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ikx}\xi(k;P) \equiv \exp\left[+i(P^0-E(\mathbf{P}))(\frac{|x^0|}{2}+\frac{\eta_1-\eta_2}{2}x^0)\right]\chi(x;E(\mathbf{P}),\mathbf{P}), \tag{2.1.21}$$

$$\overline{\xi}(y;P) \equiv \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{+iqx}\overline{\xi}(q;P) \equiv \exp\left[-i(P^0-E(\mathbf{P}))(\frac{|y^0|}{2}+\frac{\eta_1-\eta_2}{2}y^0)\right]\overline{\chi}(y;E(\mathbf{P}),\mathbf{P}). \tag{2.1.22}$$

Comparing Eq. (2.1.14) with (2.1.18), we obtain:

$$G(k,q;P) = \frac{i\xi(k;P)\overline{\xi}(q;P)}{2E(\mathbf{P})(P^0-E(\mathbf{P})+i\varepsilon)} + \text{regular terms}. \tag{2.1.23}$$

We adopt the following conventions for the Fourier transforms of the BS amplitude and its conjugate:

$$\chi(k;P) \equiv \int \mathrm{d}^4x\, e^{ikx}\chi(x;P), \tag{2.1.24}$$

$$\overline{\chi}(q;P) \equiv \int \mathrm{d}^4x\, e^{-iqy}\overline{\chi}(y;P). \tag{2.1.25}$$

[2]We adopt throughout this Thesis the covariant normalization of states:

$$\langle b;P'|b;P\rangle = (2\pi)^3 2E(\mathbf{P})\delta^3(\mathbf{P}-\mathbf{P}').$$

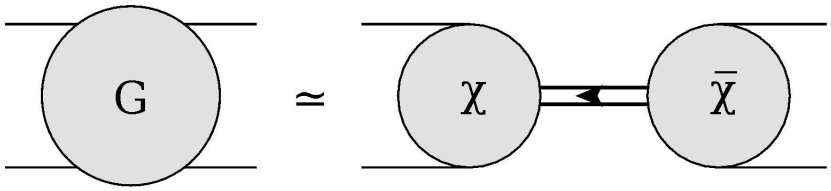

Figure 2.1.1. Diagrammatic representation of the four-points Green's function near the pole $P^0 \simeq E(\mathbf{P})$ (see Eq. (2.1.28)).

Since [24]:

$$\lim_{P^0 \to E(\mathbf{P})} \xi(k;P) = \chi(k;E(\mathbf{P}),\mathbf{P}), \tag{2.1.26}$$

$$\lim_{P^0 \to E(\mathbf{P})} \overline{\xi}(q;P) = \overline{\chi}(q;E(\mathbf{P}),\mathbf{P}), \tag{2.1.27}$$

we see from Eq. (2.1.23) that the four-points Green's function has a pole at $P^0 = E(\mathbf{P})$, with residue given by:

$$\lim_{P^0 \to E(\mathbf{P})} \frac{P^2 - M^2}{i} G(k,q;P) = \chi(k;E(\mathbf{P}),\mathbf{P})\overline{\chi}(q;E(\mathbf{P}),\mathbf{P}), \tag{2.1.28}$$

which is the sought relation between the BS amplitude and the four-points Green's function, depicted in Figure 2.1.1.

It is worth to emphasize that the appearance of a pole in the Green's function G is a purely non-perturbative phenomenon. Indeed, the only poles which can occur from a finite sum of Feynman diagrams are those corresponding to the masses of the elementary fields in the theory. This makes apparent why the bound state problem is a very challenging one.

We conclude this Subsection with a few remarks:

1. We have considered only the contributions coming from the first time-ordering in Eq. (2.1.1). It is not difficult to realize that contributions from other orderings can never give rise to a pole at $P^0 \to E(\mathbf{P})$, as in Eq. (2.1.28). For instance, the contributions from the $\mathcal{CPT}$ conjugated state of $|b\rangle$, which has a non-vanishing scalar product with $\langle 0|\mathrm{T}\{\phi_1^\dagger(y_1)\phi_2^\dagger(y_2)\}$ (and could as well correspond to $|b\rangle$), give rise to a pole at $P^0 \to -E(\mathbf{P})$.

2. The BS amplitude does not have a direct probabilistic interpretation, as is revealed by the presence of a "relative time" x^0 in the reduced amplitude $\chi(x;P)$. On the other hand, the bilocal operator $\phi_1(x_1)\phi_2(x_2)$ plays, in the BS formalism, a somewhat analogous role to the interpolating fields of LSZ scattering theory, as we shall illustrate in Subsec. 2.2.2. Moreover, a probabilistic content can be recovered from the BS amplitude by explicitly fixing a Fock-basis in the Hilbert space of the theory. This point will be illustrated in Subsec. 3.2.3, in the context of the light-front formulation of QFT.

2.1.2 Derivation of the BSE

From the relation between the BS amplitude and the four-points Green's function, Eq. (2.1.28), we can immediately derive the homogeneous Bethe-Salpeter equation (BSE) satisfied by χ [1, 2]. To this end, we need to introduce the concept of *irreducible Feynman diagrams*.

Bethe and Salpeter call a diagram irreducible if it can not be disconnected by removing exactly two internal lines corresponding to a ϕ_1 and a ϕ_2 constituent respectively. Some examples of reducible and irreducible diagrams are shown in Fig. 2.1.2. Loosely speaking, a diagram is irreducible if it does not contain any internal two-particle state. A reducible diagram is evidently built of several irreducible blocks, linked together by two-particle propagations.

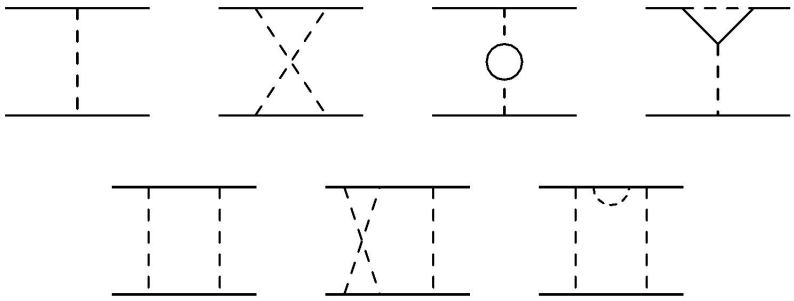

Figure 2.1.2. Examples of irreducible (upper row) and reducible (lower row) Feynman diagrams.

By removing propagators (together with self-energy corrections) from the external lines of an irreducible diagram, we obtain the general term of the so-called *irreducible kernel* iK (see Fig. 2.1.3). From the above discussion, it follows that the Green's function G, which may be formally regarded as the sum of all Feynman diagrams with four external legs, satisfies the integral equation:

$$G(x_1,x_2;y_1,y_2) = i\Delta_1'(x_1-y_1)i\Delta_2'(x_2-y_2)+ \\ + \int \mathrm{d}^4w_1\mathrm{d}^4w_2\mathrm{d}^4z_1\mathrm{d}^4z_2 i\Delta_1'(x_1-w_1)i\Delta_2'(x_2-w_2)iK(w_1,w_2;z_1,z_2)G(z_1,z_2;y_1,y_2), \tag{2.1.29}$$

where $i\Delta_{1,2}'$ denote the fully dressed propagators:

$$i\Delta_i'(x) = \langle 0|\mathrm{T}\{\phi_i(x)\phi_i^\dagger(0)\}|0\rangle. \tag{2.1.30}$$

Strictly speaking, Eq. (2.1.29) should be regarded as the non-perturbative definition of the irreducible kernel iK (see also Eq. (2.1.32) below). The purpose of the previous remarks is to emphasize the diagrammatic content of the equation, which is clarified in Fig. 2.1.4. Indeed, the key observation in the work of Salpeter and Bethe [1] is that the integral equation (2.1.29) generates, by iteration, the infinite set of Feynman diagrams contributing to G.

If we regard G, $G_0 \equiv i\Delta_1' i\Delta_2'$ and K as the matrices $G^{x_1x_2}_{y_1y_2} \equiv G(x_1,x_2;y_1,y_2)$ etc., we may rewrite Eq. (2.1.29) symbolically as:

$$G = G_0 + G_0(iK)G, \tag{2.1.31}$$

which we may formally solve for G or iK:

$$G = (G_0^{-1} - iK)^{-1}, \qquad iK = G_0^{-1} - G^{-1}. \tag{2.1.32}$$

The translational invariance of the vacuum implies that G and G_0, as well as their functional inverses, satisfy Eq. (2.1.12) and, by Eq. (2.1.32), the same property is shared by the irreducible kernel iK. We may thus write the analogues of the Fourier transforms (2.1.13)-(2.1.14) for G_0 and iK. With these definitions, we may transform the convolution integral in coordinate space of Eq. (2.1.29) in the momentum space integral:

$$G(k,q;P) = G_0(k,q;P) + \int \frac{\mathrm{d}^4k''}{(2\pi)^4}\frac{\mathrm{d}^4k'}{(2\pi)^4} G_0(k,k'';P)iK(k'',k';P)G(k',q;P). \tag{2.1.33}$$

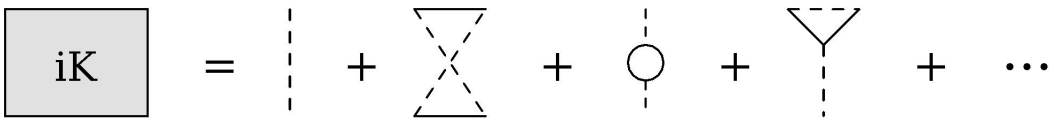

Figure 2.1.3. Perturbative expansion of the irreducible kernel iK.

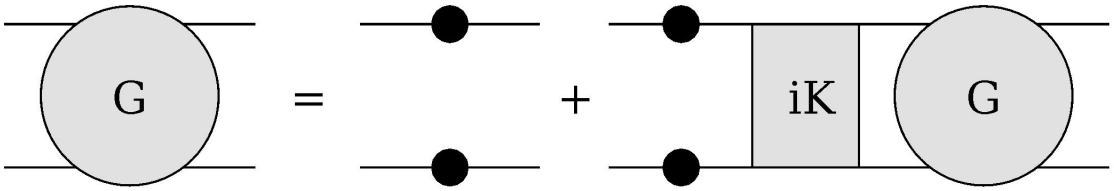

Figure 2.1.4. The inhomogeneous equation (2.1.29) satisfied by the four-points Green's function, which defines the irreducible kernel iK. The black dots denote the fully dressed propagators (2.1.30).

Also, noticing that the disconnected Green's function G_0 conserves the relative momentum, i.e.:

$$G_0(k,q;P) = (2\pi)^4\delta^4(k-q)G_0(k;P), \tag{2.1.34}$$
$$G_0(k;P) \equiv i\Delta_1'(\eta_1 P + k)i\Delta_2'(\eta_2 P - k), \tag{2.1.35}$$

we obtain the result:

$$G(k,q;P) = G_0(k;P)\left[(2\pi)^4\delta^4(k-q) + \int \frac{\mathrm{d}^4k'}{(2\pi)^4} iK(k,k';P)G(k',q;P)\right], \tag{2.1.36}$$

which is the inhomogeneous equation satisfied by the four-point Green's function in momentum space. The symbolic form (2.1.31) is equally valid in momentum space, if we observe that $(2\pi)^4\delta^4(k-k')$ is the identity with respect to the "product" defined by the integration over $\int \frac{\mathrm{d}^4k'}{(2\pi)^4}$.

By taking the residue at the pole $P^2 = M^2$ in both sides of Eq. (2.1.28), we can immediately derive the homogeneous equation satisfied by the bound state amplitude χ (see Fig. 2.1.5):

$$\chi(k;P) = G_0(k;P)\int \frac{\mathrm{d}^4k'}{(2\pi)^4} iK(k,k';P)\chi(k';P), \tag{2.1.37}$$

or, in symbolic form:

$$\chi = G_0 iK \chi. \tag{2.1.38}$$

Equation (2.1.37) is the celebrated result of Bethe and Salpeter. Its intuitive content can be understood by recalling the definition of iK, as the sum of all irreducible Feynman diagrams. Indeed, iteration of Eq. (2.1.37) reproduces all the possible virtual exchanges which can occur between the constituent during the (infinite) life-time of the bound state. This immediately brings to light one of the difficulties at the core of the BS formalism, namely the absence of a closed form for the irreducible kernel iK. This means that if, on one side, we are able to take into account the infinite exchange produced by replicating a primitive Feynman diagram (this is the virtue of the integral equation), on the other side one should consider, in principle, all the possible irreducible diagrams, in order to reproduce the exact BS amplitude.

The BSE (2.1.37) is an homogeneous equation, contrary to the equation (2.1.36) satisfied by the Green's function G, which makes clear the non-perturbative character of the BS amplitude. Furthermore, we see that the BSE is a quite singular equation, because of the presence of the double

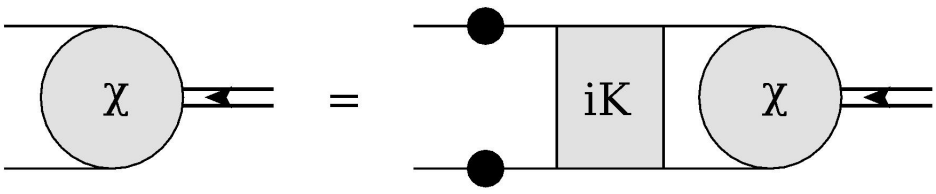

Figure 2.1.5. The homogeneous Bethe-Salpeter equation (cf. Eq. (2.1.37)).

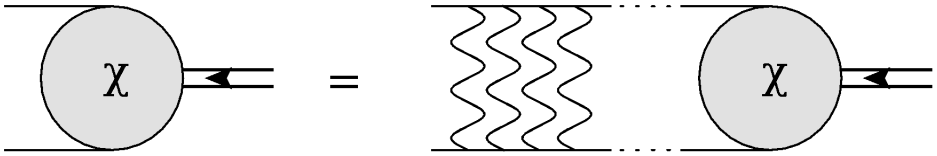

Figure 2.1.6. A ladder kernel BSE (see Eq. (2.1.39)).

propagator G_0 and also of the singular kernel iK. These two features make the solution of the BSE in Minkowski space a quite formidable task.

Due to these difficulties, most of the present studies of the BSE in Minkowski space are limited to so-called ladder kernels (see Fig. 2.1.6), and employ free propagators for the constituents. In these approximations, the BSE (2.1.37) reads:

$$\chi(k;P) = \frac{1}{[(\eta_1 P + k)^2 - m_1^2 + i\varepsilon]\,[(\eta_2 P - k)^2 - m_2^2 + i\varepsilon]} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{i\lambda_1\lambda_2}{(k-k')^2 - \mu^2 + i\varepsilon} \chi(k';P), \tag{2.1.39}$$

where μ is the mass of the exchanged scalar and $\lambda_{1,2}$ are the couplings between the exchanged and the constituent fields. Note that, once the mass M of the bound state is fixed, the ladder BSE (2.1.39) becomes an eigenvalue problem for the coupling constant $\lambda^2 \equiv \lambda_1\lambda_2$. It is thus customary, in order to study the functional relation $M = M(\lambda^2)$, e.g., for the ground state, to fix M and solve numerically for λ^2, rather than the opposite choice, which could appear more natural.

Notice that the ladder kernel:

$$iK_{\mathrm{lad}}(k,k';P) = i\lambda_1 \frac{i}{(k-k')^2 - \mu^2 + i\varepsilon} i\lambda_2, \tag{2.1.40}$$

is actually independent of P. This feature simplifies greatly the discussion of the normalization of the BS amplitude, to which we turn now.

2.1.3 Normalization of the amplitude

Since the BSE for the bound state amplitude is homogeneous, it needs a normalization condition to fix the overall scale of χ. A correct normalization is required, for instance, to obtain the so-called valence light-front wave function (and, in particular, the valence probability) of the bound state, as discussed in Sec. 3.2, as well as for the computation of physical observables, like decay constants and electromagnetic form factors, by the methods outlined in Subsec. 2.2.2.

We may find the normalization condition as follows [25] (see [12] for further discussion). Employing Eq. (2.1.31), we write:

$$G(G_0^{-1} - iK)G = G.$$

By taking the residue at $P^0 \to E(\mathbf{P})$, we obtain:

$$\lim_{P^0 \to E(\mathbf{P})} i\frac{\overline{\chi}(G_0^{-1} - iK)\chi}{P^2 - M^2} = \lim_{P^0 \to E(\mathbf{P})} i\frac{\overline{\chi}(G_0^{-1} - iK)\chi}{(P^0)^2 - E(\mathbf{P})^2} = 1 \tag{2.1.41}$$

or, equivalently :

$$\overline{\chi}\,\frac{i}{2}\frac{\partial(G_0^{-1} - iK)}{\partial P^\mu}\bigg|_{\overline{P}} \chi = \overline{P}_\mu, \tag{2.1.42}$$

where $\overline{P}^2 = M^2$. Explicitly, Eq. (2.1.42) means:

$$\int \frac{\mathrm{d}^4 q''}{(2\pi)^4} \frac{\mathrm{d}^4 q'}{(2\pi)^4} \overline{\chi}(q'';\overline{P}) N_\mu(q'',q';\overline{P}) \chi(q';\overline{P}) = \overline{P}_\mu, \tag{2.1.43}$$

where the normalization kernel N_μ is given by:

$$N_\mu \equiv \frac{i}{2} \left. \frac{\partial (G_0^{-1} - iK)}{\partial P^\mu} \right|_{\overline{P}} . \tag{2.1.44}$$

As a matter of fact, the peculiar structure of the normalization condition (2.1.41) has a strict analogy with the elementary particle amplitude normalization, usually written as:

$$\langle 0|\phi(x)|p\rangle = \sqrt{Z} e^{-ipx}. \tag{2.1.45}$$

The full propagator of the scalar field ϕ is given by:

$$i\Delta'(p) = \frac{i}{p^2 - m^2 + \Sigma(p) + i\varepsilon},$$

or:

$$(i\Delta'(p))^{-1} = (i\Delta(p))^{-1} - i\Sigma(p)$$

where $i\Delta(p) = i(p^2 - m^2 + i\varepsilon)^{-1}$ is the free propagator and $i\Sigma$ is the 1PI self-energy, defined by:

$$i\Delta'(p) = i\Delta(p) + i\Delta(p)\, i\Sigma(p)\, i\Delta'(p).$$

From the Källén-Lehmann decomposition, we know that:

$$\lim_{p^2 \to m^2} \frac{p^2 - m^2}{i} i\Delta'(p) = Z.$$

Hence, the normalization condition (2.1.45) can be rewritten as:

$$\lim_{p^0 \to E(\mathbf{p})} i \frac{\langle p|\phi(0)|0\rangle \left[(i\Delta(p))^{-1} - i\Sigma(p)\right] \langle 0|\phi(0)|p\rangle}{p^2 - m^2} = 1, \tag{2.1.46}$$

We thus see that the condition (2.1.42) has its exact analogue in the one-particle case, where the irreducible kernel iK is replaced by the 1PI self-energy $i\Sigma$.

2.2 Developments of the BS formalism

In this Section, we discuss some developments of the basic formalism outlined in Sec. 2.1. Subsec. 2.2.1 deals with the Wick-rotation, mainly in view of its importance as a cross-check for our numerical solution of the Minkowski space BSE. In Subsec. 2.2.2 we review the Mandelstam formalism for the computation of bound-state dynamical observables. This presentation should clarify the interest in solving the BSE directly in Minkowski space. Finally, in Subsec. 2.2.3, we give a simple derivation of the BSE for the bound-state amplitude in the presence of spin-degeneracy, based on the transformation property of the BS amplitude under the rotation group SO(3).

2.2.1 Wick rotation: pros and cons

A powerful method to solve the BSE is provided by the Wick rotation [7], which is based on the fact that the reduced BS amplitude in momentum space, $\chi(k;P)$, can be analytically continued to the first and third quadrants of the complex k^0 plane [12], provided that the stability condition:

$$M < m_1 + m_2 \tag{2.2.1}$$

is satisfied (which is of course true for a stable bound state, if the constituents appear as physical particles in the spectrum of the theory). If the same operation can be performed in the RHS of the BSE, one obtains the Wick-rotated BSE[3] in the rest frame $P_{\rm CM} = (M, \mathbf{0})$:

$$\hat{G}_0(k; P_{\rm CM})^{-1} \hat{\chi}(k; P_{\rm CM}) = \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \hat{K}(k, k'; P_{\rm CM}) \hat{\chi}(k'; P_{\rm CM}), \tag{2.2.2}$$

where

$$\hat{\chi}(k; P_{\rm CM}) = \chi(ik^4, \mathbf{k}; P_{\rm CM}), \tag{2.2.3}$$

$$\hat{G}_0(k; P_{\rm CM}) = -G_0(ik^4, \mathbf{k}; P_{\rm CM}) \tag{2.2.4}$$

$$\hat{K}(k, k'; P_{\rm CM}) = \hat{K}(ik^4, \mathbf{k}, ik'^4, \mathbf{k}'; P_{\rm CM}). \tag{2.2.5}$$

In particular, for the ladder kernel (2.1.40), we obtain:

$$\hat{G}_0(k; P_{\rm CM})^{-1} \hat{\chi}(k; P_{\rm CM}) = \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{g^2}{(k^4 - k'^4)^2 + (\mathbf{k} - \mathbf{k}')^2 + \mu^2} \hat{\chi}(k'; P_{\rm CM}), \tag{2.2.6}$$

$$\hat{G}_0(k; P_{\rm CM})^{-1} = \left[m_1^2 + \mathbf{k}^2 + (k^4 - i\eta_1 M)^2 \right] \left[m_2^2 + \mathbf{k}^2 + (k^4 + i\eta_2 M)^2 \right]. \tag{2.2.7}$$

Note that the pole $(k - k')^2 = \mu^2$ of the ladder kernel is no longer present in the Wick-rotated kernel of Eq. (2.2.6). Also, if we choose $\eta_{1,2} = \frac{m_{1,2}}{m_1 + m_2}$, the double propagator (2.2.7) is also seen to be free from poles. The possibility to analytically continue the BS amplitude provides a powerful tool for obtaining the bound state mass spectrum (the mass M which appears in Eq. (2.2.2) is the same as in the original BSE). Since Eq. (2.2.2) is free from the singularities arising from denominators in Feynman diagrams, the Wick-rotated BS amplitude is a regular function of $k = (\mathbf{k}, k^4)$ and one can apply mathematical theorems to obtain analytic [12, 25, 26], or numerical [27] solutions of the equation. For instance, exploiting the fact that Eq. (2.2.6) has an Euclidean $O(4)$ symmetry in the limit $M \to 0$, in Ref. [27] a robust numerical algorithm for the solution of the ladder BSE for the entire bound state spectrum was developed .

In spite of these advantages, the analytic continuation has also several drawbacks. In the first place, the possibility to perform the Wick rotation in the RHS integral of the BSE must be investigated case by case, by studying the analyticity properties of the kernel. For kernels more complicated than the ladder one, this can be a difficult (and not necessarily successful) task. Hence, the method is not general.

In the second instance, the Wick-rotated BS amplitude is generally not sufficient for the computation of physical quantities, e.g. electromagnetic form factors [16]. In the latter case, the shortcoming arises from two difficulties: (i) the occurrence of residues, not computable from the Wick-rotated BS amplitude, after rotating the contour of the needed integral for evaluating the form factor (see the next Subsection); (ii) the need of the Wick-rotated BS amplitude for non zero values of the the bound-state tri-momentum $\mathbf{P}$, which in turn requires the knowledge of the BS amplitude for general complex values of its argument k (see Ref. [16] for details). This is clearly an harder task than the original one in Minkowski space.

Finally, as we shall discuss in the next Chapter, the Minkowskian BS amplitude can be properly related to the so-called light-front valence wave function of the bound state (see Sec. 3.2.1), but the relation is lost in the Wick rotation.

With all these limitations in mind, we conclude by observing that, even in the pure Minkowskian approach, the Wick-rotated BSE plays an important role, through the comparison of the bound state

[3] We shall avoid the common practice of referring to Eq. (2.2.2) as the "Euclidean" BSE. What we wish to emphasize is that the analytic continuation is performed only on the relative momentum variable k, so that the Wick-rotation does not generally lead to a definite metric in the rotated BSE, as is clear from Eqs. (2.2.6)-(2.2.7) in the ladder case.

masses obtained with the two methods, as well as the comparison of other invariant quantities, such as the so-called transverse amplitudes [19], which can be obtained both from the Wick-rotated and Minkowskian BS amplitude.

2.2.2 Form factors and decay constants in the BS formalism

To illustrate the relevance of the BS formalism as a building block for phenomenological investigations, let us briefly present the work by S. Mandelstam [4], that showed how to use the BS amplitude for calculating matrix elements of time-ordered products of the form:

$$\langle 0|\mathrm{T}\{O_1(z_1)O_2(z_2)\cdots O_n(z_n)\}|b;P\rangle, \qquad \langle b;P'|\mathrm{T}\{O_1(z_1)O_2(z_2)\cdots O_n(z_n)\}|b;P\rangle,$$

where $O_i(z_i)$ are general local operators.

Mandelstam's derivation is essentially a generalization of the analysis for the four-points Green's function of Subsec. 2.1.1. We shall treat for definiteness the second kind of matrix elements, that can be closely related to quantities experimentally observed (e.g. electromagnetic form factors, parton transverse-momentum distributions, generalized parton distributions, etc.). Consider the $(n+4)$-points Green's function:

$$G^{(n+4)}(x_1,x_2|\,\{z_i\}\,|y_1,y_2) = \langle 0|\mathrm{T}\{\phi_1(x_1)\phi_2(x_2)\left[\prod_{i=1}^{n}O_i(z_i)\right]\phi_1^\dagger(y_1)\phi_2^\dagger(y_2)\}|0\rangle. \tag{2.2.8}$$

Inserting the completeness for the composite system, one gets the following pole contribution:

$$\begin{aligned} G_b^{(n+4)}(x_1,x_2|\,\{z_i\}\,|y_1,y_2) \equiv i^2\int\frac{\mathrm{d}^4P'}{(2\pi)^4}\frac{\mathrm{d}^4P}{(2\pi)^4}e^{-iP'X+iPY}\times \\ \times\frac{i\xi(x;P')\langle b;P'|\mathrm{T}\{\prod_i O_i(z_i)\}|b;P\rangle\bar{\xi}(y;P)}{[2E(\mathbf{P}')(P'^0-E(\mathbf{P}'))+i\varepsilon]\,[2E(\mathbf{P})(P^0-E(\mathbf{P}))+i\varepsilon]}, \end{aligned}$$

where the auxiliary amplitudes ξ and $\bar{\xi}$ are defined by (2.1.21) and (2.1.22). By Fourier transforming with respect to the global variables X and Y, we obtain for the full Green's function:

$$\begin{aligned} \int \mathrm{d}^4X\mathrm{d}^4Y\, e^{iP'X-iPY}G^{(n+4)}(x_1,x_2|\,\{z_i\}\,|y_1,y_2) = i^2\frac{\xi(x;P')\langle b;P'|\mathrm{T}\{\prod_i O_i(z_i)\}|b;P\rangle\bar{\xi}(y;P)}{[2E(\mathbf{P}')(P'^0-E(\mathbf{P}'))+i\varepsilon]\,[2E(\mathbf{P})(P^0-E(\mathbf{P}))+i\varepsilon]}+ \\ +\text{ regular terms.} \end{aligned} \tag{2.2.9}$$

On the other hand, we can define a truncated $(n+4)$-points Green's function $G^{(n+4)}_{\mathrm{trunc}}$ (see Figure 2.2.1) by:

$$\begin{aligned} G^{(n+4)}(x_1,x_2|\,\{z_i\}\,|y_1,y_2) = \int \mathrm{d}^4x_3\mathrm{d}^4x_4\mathrm{d}^4y_3\mathrm{d}^4y_4\, G(x_1,x_2;y_3,y_4)\times \\ \times\, G^{(n+4)}_{\mathrm{trunc}}(y_3,y_4|\,\{z_i\}\,|x_3,x_4)\,G(x_3,x_4;y_1,y_2). \end{aligned} \tag{2.2.10}$$

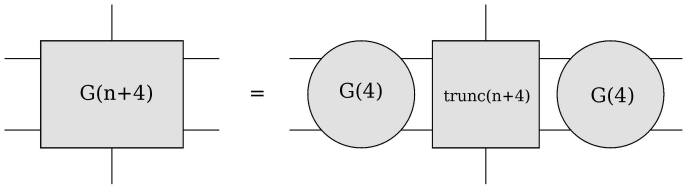

Figure 2.2.1. Definition of the truncated $(n+4)$-points Green's function (cf. Eq. (2.2.10)).

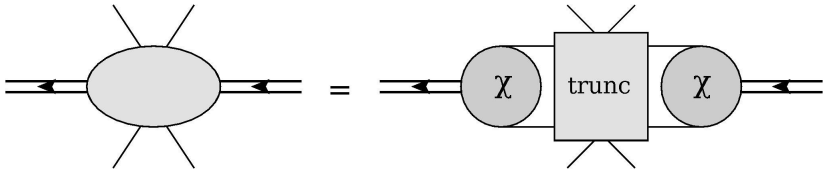

Figure 2.2.2. Matrix elements of time-ordered products in terms of the BS amplitude (cf. Eqs. (2.2.12)-(2.2.13))

Taking into account the pole of G, Eq. (2.1.28), we see that the LHS of (2.2.9) can be expressed as follows:

$$\begin{aligned}\text{LHS} = i^2 \int \mathrm{d}^4y_3\mathrm{d}^4y_4\mathrm{d}^4x_3\mathrm{d}^4x_4 \frac{\chi(x;P')\overline{\chi}(y_3,y_4;P')}{[2E(\mathbf{P}')(P'^0-E(\mathbf{P}'))+i\varepsilon]}\times \\ \times G^{(n+4)}_{\text{trunc}}(y_3,y_4|\,\{z_i\}\,|x_3,x_4)\frac{\chi(x_3,x_4;P)\overline{\chi}(y;P)}{[2E(\mathbf{P})(P^0-E(\mathbf{P}))+i\varepsilon]} + \text{regular terms},\end{aligned} \tag{2.2.11}$$

Comparing Eqs. (2.2.10) and (2.2.11), with the help of the limits (2.1.26) and (2.1.27), one obtains:

$$\langle b;P'|\mathrm{T}\{\prod_i O_i(z_i)\}|b;P\rangle = \int \mathrm{d}^4y_3\mathrm{d}^4y_4\mathrm{d}^4x_3\mathrm{d}^4x_4\,\overline{\chi}(y_3,y_4;P')G^{(n+4)}_{\text{trunc}}(y_3,y_4|\,\{z_i\}\,|x_3,x_4)\chi(x_3,x_4;P), \tag{2.2.12}$$

or, passing to momentum space:

$$\langle b;P'|\mathrm{T}\{\prod_i O_i(z_i)\}|b;P\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4k'}{(2\pi)^4}\overline{\chi}(k';P')G^{(n+4)}_{\text{trunc}}(k',P'|\,\{z_i\}\,|k,P)\chi(k;P), \tag{2.2.13}$$

where the mixed Fourier transform of $G^{(n+4)}_{\text{trunc}}$ is defined by:

$$G^{(n+4)}_{\text{trunc}}(k',P'|\,\{z_i\}\,|k,P) = \int \mathrm{d}^4X\mathrm{d}^4Y\mathrm{d}^4x\mathrm{d}^4y\, e^{-iPX+iP'Y-ikx+ik'x'}G^{(n+4)}_{\text{trunc}}(y_3,y_4|\,\{z_i\}\,|x_3,x_4), \tag{2.2.14}$$

with global and relative coordinates referring to the external ones (x_i, y_j).

A diagrammatic representation of Eqs. (2.2.12) or (2.2.13) is given in Fig. 2.2.2. We remark that the interpretation of Eq. (2.2.10), which defines the truncated Green's function, requires some care if the LHS of Eq. (2.2.10) has contributions from "special" diagrams with some spectator constituent, of the kind depicted in Fig. 2.2.3. As is clear from Fig. 2.1.4, the four-points Green's function G already includes the constituent propagators on its external legs. In order to avoid multiple counting the lower propagator in Fig. 2.2.3, we should ascribe a factor:

$$\frac{(2\pi)^4\delta^4(k_i-k_i')}{i\Delta_i(k_i)}$$

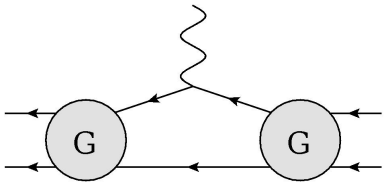

Figure 2.2.3. Example of a diagram with a spectator constituent contributing to the truncated $(n+4)$-points Green's function. The truncated function is obtained by removing the two external four-points functions G (cf. Eq. (2.2.10)).

to the lower line (neglecting self-energy corrections). In case of doubt, one can always refer to Eq. (2.2.10) to find the correct contribution of a particular diagram to the truncated function.

As an application of Eq. (2.2.13), we illustrate the method of calculation of an electromagnetic form factor for the two-scalar system considered in this Chapter [16]. The form factor $F(q^2)$ is defined by:

$$\langle b;P'|j^\mu_{\rm em}(x)|b;P\rangle \equiv (P+P')^\mu F(q^2)e^{-iqx}, \tag{2.2.15}$$

where $q \equiv P-P'$. Fourier transforming in x, we obtain:

$$(2\pi)^4\delta^4(P-P'-q)(P+P')^\mu F(q^2) = (2\pi)^4\delta^4(P-P'-q)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4k'}{(2\pi)^4}\overline{\chi}(k';P')G^\mu_{\rm trunc}(k',P'|k,P)\chi(k;P),$$

where:

$$(2\pi)^4\delta^4(P-P'-q)G^\mu_{\rm trunc}(k';P'|k;P) \equiv \int \mathrm{d}^4X\mathrm{d}^4x\mathrm{d}^4Y\mathrm{d}^4y\mathrm{d}^4z e^{iP'Y+ik'y-ikx-iPX+iqz}G^\mu_{\rm trunc}(y_1,y_2|z|x_1,x_2)$$

Therefore, the form factor is given by:

$$(P+P')^\mu F(q^2) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4k'}{(2\pi)^4}\overline{\chi}(k';P')G^\mu_{\rm trunc}(k',P'|k,P)\chi(k;P). \tag{2.2.16}$$

In the ladder approximation, assuming that only the ϕ_2 particle is charged, the main contributions to the truncated function arise from the diagram in Figure 2.2.4 (this is the so-called impulse approximation, see e.g. [4]), which is of the kind discussed above. Applying the Feynman rules of scalar QED [25], one obtains [16]:

$$(P+P')^\mu F(q^2) = \int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\overline{\chi}(\frac{P'}{2}-\ell;P')(\ell^2-m_1^2)(P+P'-2\ell)^\mu\chi(\frac{P}{2}-\ell;P) \tag{2.2.17}$$

The most appealing feature of Eq. (2.2.17) is that the perturbative and non-perturbative parts of the problem, i.e. the coupling of a photon to an elementary constituent, and the physics of the bound state respectively, are neatly separated in the BS formalism. Thus, once the BS amplitude is known, the computation of a form factor reduces to the evaluation of a few relevant Feynman diagrams.

In a similar fashion, one can derive an exact expression for the decay constant defined by:

$$\langle 0|b(x)|b;P\rangle = if_b e^{-iPx}, \tag{2.2.18}$$

where $b(x)$ is an interpolating field for the bound state b (e.g. $b(x) = (\frac{1}{m_\pi^2}\partial_\mu\bar{d}\gamma^\mu\gamma^5 u)(x)$ for a pion π^+). This can be expressed through the BS amplitude as:

$$if_b e^{-iPx} = \int \mathrm{d}^4x_1\mathrm{d}^4x_2\,\langle 0|\mathrm{T}\{b(x)\phi_1^\dagger(x_1)\phi_2^\dagger(x_2)\}|0\rangle_{\rm trunc}\,\chi(x_1,x_2;P), \tag{2.2.19}$$

where the truncated function is defined by:

$$\langle 0|\mathrm{T}\{b(x)\phi_1^\dagger(x_1)\phi_2^\dagger(x_2)\}|0\rangle_{\rm trunc} = \int \mathrm{d}^4y_1\mathrm{d}^4y_2\langle 0|\mathrm{T}\{b(x)\phi_1^\dagger(y_1)\phi_2^\dagger(y_2)\}|0\rangle G^{-1}(y_1,y_2;x_1,x_2). \tag{2.2.20}$$

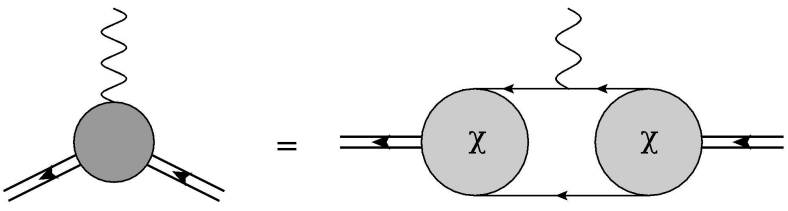

Figure 2.2.4. Calculation of the electromagnetic form factor in terms of the BS amplitude.

2.2.3 BSE for higher-spin bound states

If the composite state has integral spin $j \neq 0$, Eq. (2.1.1) must be generalized to:

$$\lim_{P^0 \to E(\mathbf{P})} \frac{P^2 - M^2}{i} G(k, q; P) = \sum_{s=-j}^{j} \chi_s^{(j)}(k; P) \overline{\chi}_s^{(j)}(q; P), \tag{2.2.21}$$

where s denotes the eigenvalue of the third component of the spin operator and:

$$\chi_s^{(j)}(x; P) \equiv \langle 0 | \mathrm{T}\{\phi_1(\eta_2 x)\phi_2(-\eta_1 x)\} | b^{(j)}; P, s \rangle. \tag{2.2.22}$$

It turns out that all the amplitudes $\chi_s^{(j)}$ satisfy separately the BSE (2.1.37), as well as the generalized normalization condition:

$$\overline{\chi}_{s'}^{(j)} \; \frac{i}{2} \frac{\partial (G_0^{-1} - iK)}{\partial P^\mu}\bigg|_{P^0 = E(\mathbf{P})} \chi_s^{(j)} = \delta_{s's} P_\mu. \tag{2.2.23}$$

We can give a short proof of the first property as follows (a proof of the normalization condition can be given along the same lines).

From Eq. (2.2.21), we derive the equations:

$$\sum_{s=-j}^{j} \left[\chi_s^{(j)}(k; P) - G_0(k; P) \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} iK(k, k'; P) \chi_s^{(j)}(k'; P) \right] \overline{\chi}_s^{(j)}(q; P) = 0. \tag{2.2.24}$$

The conjugated BS amplitude in the rest frame $P_{\mathrm{CM}} = (M, \mathbf{0})$, satisfies (see e.g. [28, Chap. 2]):

$$\overline{\chi}_s^{(j)}(q; P_{\mathrm{CM}}) = \sum_{s'} \mathcal{D}_{s's}^{(j)}(R)^* \overline{\chi}_{s'}^{(j)}(q^0, R\mathbf{q}; P_{\mathrm{CM}}), \tag{2.2.25}$$

where $R \in \mathrm{SO}(3)$ and $\mathcal{D}^{(j)}$ is the spin-j representation of the rotation group. Notice that, apart from a change of notation $j, s \to l, m$, Eq. (2.2.25) is the one satisfied by the conjugated spherical harmonics:

$$Y_l^m(\hat{\boldsymbol{n}})^* = \sum_{m'} \mathcal{D}_{m'm}^{(l)}(R)^* Y_l^{m'}(R\hat{\boldsymbol{n}})^*. \tag{2.2.26}$$

It follows that:

$$\int \mathrm{d}^4 q \, \overline{\chi}_s^{(j)}(q; P_{\mathrm{CM}}) F_l^m(q) \propto \delta_{jl} \delta_{ms}, \tag{2.2.27}$$

if F_l^m is any test function of the form:

$$F_l^m(q) = f(q^0, |\mathbf{q}|) Y_l^m(\hat{\boldsymbol{q}}). \tag{2.2.28}$$

Therefore, in the rest frame of the bound state, any of the terms in square brackets in Eq. (2.2.24) must separately vanish, viz.:

$$\chi_s^{(j)}(k; P_{\mathrm{CM}}) = G_0(k; P_{\mathrm{CM}}) \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} iK(k, k'; P_{\mathrm{CM}}) \chi_s^{(j)}(k'; P_{\mathrm{CM}}). \tag{2.2.29}$$

This is the BSE in the rest frame of the bound state. By Lorentz covariance, it implies the BSE in any frame.

Chapter 3

The NIR-LF approach to the Bethe-Salpeter equation

In this Chapter we introduce the two principal ingredients of our numerical approach to the Minkowski space BSE, namely the Nakanishi Integral Representation (NIR) of the BS amplitude and the light-front (LF) projection onto the null hyperplane.

The theory of NIR and its applications within the context of the BSE are discussed in Sec. 3.1. In Sec. 3.2 we give a brief formal introduction to the description of bound states in LF quantum field theory, focusing on those aspects which have a direct connection with the BS formalism. Finally, in Sec. 3.3, we present the numerical method for solving the BSE, and quote some first relevant results.

As in the previous Chapter, for the sake of simplicity, we consider purely scalar field theories, postponing the due generalizations for the fermion-scalar case to Chapter 4.

3.1 The Nakanishi Integral Representation

The cornerstone of our numerical approach to the Minkowskian BSE is an integral representation of the BS amplitude, known as the Nakanishi Integral Representation (NIR). Nakanishi himself proposed the NIR for the investigation of analytic solutions of the BSE (see e.g. [11]), generalizing in some sense the integral representation proposed in the older works of Wick and Cutkosky [7, 8], where the analytic approach was based on the Wick-rotation.

The first numerical application of the NIR to the BSE for a scalar system can be found in Ref. [13], where the NIR was applied both to the bound-state amplitude and to the ladder kernel, in order to obtain an integral equation involving only the so-called Nakanishi weight function, exploiting its uniqueness stated by a theorem [29]. Later [14], the NIR was elaborated within the light-front framework, to be discussed in the next Sections. This step allowed for a much simpler treatment of the kernel, overcoming the need to invoke the uniqueness theorem.

Before introducing the integral representation of the BS amplitude, it is perhaps useful to briefly recall its history, in order to clarify the status of the NIR in the non-perturbative context of the BSE.

The representation which we adopt for the BS amplitude was investigated by many authors in the fifties-sixties [30, 31], and was formerly known as the DGSI representation (after Deser, Gilbert, Sudarshan and Ida). However, the proofs given by these authors, which were in the context of axiomatic field theory, relied on an integral representation for the double commutator proposed by Dyson [32], which eventually turned out to be incorrect [33].

During the same years, N. Nakanishi obtained, in the context of perturbation theory, an integral representation for general Feynman diagrams [29], which coincided with the DGSI representation for the three-legs amplitude with one leg on the mass-shell. Noteworthy, in connection with the BSE,

the same author gave a rigorous proof of validity of the representation in the case of a ladder kernel interaction in a scalar theory [11].

At this point it must be emphasized that, to the present date, there exists no general proof of the NIR for the solutions of both the homogeneous and inhomogeneous BSE. Its validity has been numerically checked: (i) for a two-scalars bound system with a scalar-exchange ladder kernel [13, 14, 18, 19]; (ii) same as before, but with the cross-ladder kernel [15, 34]; (iii) for a two-fermions bound system with scalar, pseudoscalar and vector-exchanges ladder kernels [17, 20, 21]. Also, the inhomogeneous case has been investigated in Refs. [22, 23]. The first aim of this Thesis is to extend these investigations to the case of fermion-scalar constituents.

3.1.1 NIR of the BS amplitude

In the simplest case of a spinless bound-state of scalar constituents, with masses m_1 and m_2 respectively, the NIR of the BS amplitude reads [11]:

$$\chi(k;P) = i \int_{-1}^{1} \mathrm{d}z \int_{0}^{\infty} \mathrm{d}\gamma \frac{g(\gamma,z)}{\left[(k+z\frac{P}{2})^2 - (1-z^2)\kappa^2 - (z\overline{m}-\Delta)^2 - \gamma + i\varepsilon\right]^3}, \tag{3.1.1}$$

where g is a real function, known as the Nakanishi weight function, and:

$$\overline{m} = \frac{m_1+m_2}{2}, \quad \Delta = \frac{m_2-m_1}{2}, \quad \kappa^2 = \overline{m}^2 - \frac{M^2}{4}. \tag{3.1.2}$$

For the sake of concreteness, in the NIR (3.1.1) it is assumed that the coefficients $\eta_{1,2}$ in the definition of the reduced BS amplitude (see Eqs. (2.1.8)-(2.1.9)) are chosen to be $\eta_1 = \eta_2 = \frac{1}{2}$. We shall assume throughout, whenever employing a NIR, that this choice has been made.

The most appealing feature of the NIR is that it explicitly displays the momentum dependence and the analytic structure of the BS amplitude, thus allowing to perform the needed integrations to make the BSE (2.1.37) a numerically tractable problem. Indeed, as shown in what follows, once the NIR (3.1.1) is assumed, one can plug it into the BSE to obtain an integral equation for the Nakanishi weight function g.

Let us now describe into a bit more detail how Eq. (3.1.1) is formally obtained from the analysis of transition amplitudes within a perturbative framework. In summary, Nakanishi obtained an integral representation for a general connected Feynman diagram with n-external legs[1] (some of them possibly off mass-shell), which is essentially a clever change of variables into the celebrated Feynman's parametric formula [35]. In the NIR, the dependence upon the particular diagram considered appears only in the numerator, while the denominator is common to the whole set of diagrams contributing to the same transition amplitude. Noteworthy, it contains only the independent scalars obtained from the external momenta. Hence, one can formally sum up all such diagrams, achieving an integral representation for the full transition amplitude, with the same denominator pertaining to a single diagram. Such an expression is exact to any finite order of perturbation theory.

To concretize these statements, we consider a particularly relevant example: the NIR of the three-legs vertex $i\Gamma(k;P)$ (see Fig. (3.1.1)). This is given by[2]:

$$i\Gamma(k;P) = i \int_{0}^{1} \mathrm{d}\zeta \int_{0}^{\infty} \mathrm{d}\nu \frac{\varphi(\zeta,\nu)}{\zeta(\frac{P}{2}+k)^2 + (1-\zeta)(\frac{P}{2}-k)^2 - \nu + i\varepsilon}. \tag{3.1.3}$$

[1] Technically, the NIR in Nakanishi's original formulation applies to *amputated* diagrams, i.e. diagrams where the external legs have been removed. This remark is actually relevant to our discussion, since it implies that the NIR of the BS amplitude should be obtained from the one of the bound state vertex (see below).

[2] It is worth noticing that the representation (3.1.3) is equally valid for both the connected vertex and its proper, i.e. 1PI, part (see e.g. [25]).

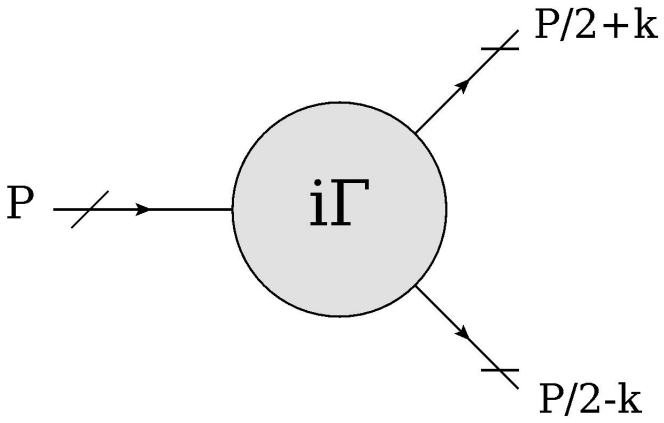

Figure 3.1.1. Three-legs vertex $i\Gamma(k;P)$. The external legs are amputated, i.e. their expression is replaced by a factor of one.

For the scalar theories which we are considering here, the weight function φ is independent from the external momenta. In the general case, φ has a polynomial dependence on the external momenta (see e.g. [35, Chap. 18]; Bjorken & Drell focus on Feynman's parametric formula, but the same remarks apply to the NIR, since the latter is obtained from the former by a simple rewriting of the Feynman parametric integral).

The contribution to $i\Gamma$ from a particular three-legs diagram $\mathscr{G}$ is also given by the same expression (3.1.3), with a weight function $\varphi_{\mathscr{G}}$ which encloses the structure of $\mathscr{G}$ and can be computed, if requested, from perturbation theory (see [10] for details). It is clear that, to any finite order in the coupling constants of the theory, one has:

$$\varphi \equiv \sum_{\mathscr{G}} \varphi_{\mathscr{G}},$$

where the sum extends to all the three-legs diagrams of order less than or equal to the prescribed one.

In the attempt to extend the NIR to the non-perturbative regime, needed for describing a bound state, one can write the BS amplitude in terms of a bound state vertex:

$$\chi(k;P) = i\Delta_1'(k_1)i\Delta_2'(k_2)i\Gamma(k;P), \tag{3.1.4}$$

where $k_{1,2} = \frac{P}{2} \pm k$. The NIR of the BS amplitude, Eq. (3.1.1), can be derived by assuming Eq. (3.1.3) for the bound state vertex. This is a non-trivial assumption, though, since the vertex defined by Eq. (3.1.4) is by nature a non-perturbative object, while Eq. (3.1.3) was obtained from perturbation theory. Of course, the validity of this ansatz is verified when a solution of the BSE in the form (3.1.1) is found. Also, it is worth recalling that Eq. (3.1.1) was rigorously proved by Nakanishi for the solutions of the scalar ladder BSE [11].

Let us make some final considerations on the support of g. Under the simplifying assumption of free propagators, applying the standard Feynman trick and redefining properly the weight function (cf. Appendix A for details), one obtains from Eqs. (3.1.4) and (3.1.3):

$$\chi(k;P) = i\int_{-1}^{1} \mathrm{d}z \int_{-\infty}^{\infty} \mathrm{d}\gamma \frac{g(\gamma,z)}{\left[(k+z\frac{P}{2})^2 - (1-z^2)\kappa^2 - (z\overline{m}-\Delta)^2 - \gamma + i\varepsilon\right]^3}. \tag{3.1.5}$$

The support of g, which is left implicit in Eq. (3.1.5), depends on the particular kernel adopted for the BSE. For the scalar ladder BSE, Nakanishi has proved that $g(\gamma,z) = 0$ if $\gamma < 0$ [11], in which case Eq. (3.1.5) coincides with (3.1.1), but one must keep in mind that the rigorous proof of this result (as well as of the NIR itself) exists only for this particular case (in the general case, a lower bound for the γ variable can be determined by other physical arguments, see e.g. Subsec. 3.2.4).

3.2 Bound states on the light front

Light-Front (LF) quantum field theory offers a powerful and physically transparent framework for the description of relativistic bound states. In fact, in the LF framework, the bound state vector is represented by an infinite set of frame independent wave-functions, which describe the internal motion of the constituents in the bound state. This representation provides a clear-cut separation of the internal dynamics from the center-of-mass motion of the composite particle, in a way which closely resembles the non-relativistic description of bound states, but incorporates, at the same time, the necessity of dealing with an indefinite number of particles, as required by relativistic field theory.

In connection with the BSE, the LF framework provides a tool to explore the physical content of its solutions, e.g. through the LF distributions, in turn related with parton distributions. Moreover, the modern approach to the Minkowski space BSE is based on a LF projection of the BS amplitude, which finds its natural physical interpretation in the context of LF quantum field theory, as illustrated in what follows.

3.2.1 LF quantized field theories

In his seminal 1949 paper [36], Dirac made the insightful suggestion that a consistent form of relativistic dynamics could be obtained by choosing to evolve states in space-time starting from a general initial space-like hypersurface Σ, which intersects the worldlines of physical particles only once. In non-relativistic physics, where speeds are unbounded, there exists only one hypersurface with the above mentioned property, namely the "instant" $\Sigma_{\rm I} \equiv \{x|x^0 = \text{const.}\}$. On the other hand, in special relativity, one has much greater freedom in the choice of Σ, which results in the possibility to have manifold-different descriptions of relativistic dynamics.

In particular, Dirac brought attention to the partition of the generators of the Poincaré group induced by the choice of Σ: the "Hamiltonians", i.e. those generators that evolve the system from the initial hypersurface, and the remaining generators, which leave Σ invariant. Noteworthy the Hamiltonians contain the dynamics of the system, as the name chosen to indicate these generators suggests.

The LF form of Hamiltonian dynamics is probably the best realization of Dirac's idea. In the LF approach to quantum field theory [37, 38], one imposes canonical commutation rules (CCRs) to the fields on the null hyperplane[3] (see Fig. 3.2.1):

$$\Sigma_{\rm LF} \equiv \Big\{x|\, x^+ \equiv x^0 + x^3 = 0\Big\}. \tag{3.2.1}$$

Correspondingly, the initial state vector is defined on $\Sigma_{\rm LF}$, where the interacting (Heisenberg) fields coincide with the free fields. The coordinate x^+ plays, in the LF formalism, the role of an evolution parameter, exactly like the time x^0 in the usual instant form, and is thus called the *LF-time*.

This choice has several advantages. In the first place, the hypersurface (3.2.1) has the largest possible kinematical group, namely the subgroup of the Poincaré transformations which leaves Σ invariant[4]. In fact, it is seven-dimensional, in contrast to the kinematical group of the familiar "instant" $\Sigma_{\rm I}$, which is six-dimensional. As already recognized by Dirac, the generators of the kinematical group, or shortly kinematical generators, play a special role, since they are independent of interactions, i.e.

[3] It should be noted that $\Sigma_{\rm LF}$ is not genuinely space-like, since it has a light-like tangent direction $v = (1, 0, 0, -1)$. In the presence of massless particles, say photons, this means that $\Sigma_{\rm LF}$ contains entirely the worldline of a photon which moves along the $-\hat{z}$ direction. In $D+1$-dimensions, with $D \geq 2$, this is one out of ∞^{D-1} possible spatial orientations, and thus is not expected to lead to any serious complication. On the other hands, for $1+1$-dimensional QFT, the validity of the LF approach is still under debate [37].

[4] In order to have all points of Σ to be kinematically equivalent, one requires that any two points are connected by a kinematical transformation, i.e. the kinematical group acts transitively on Σ [39]. This implies that the kinematical group must be at least three-dimensional.

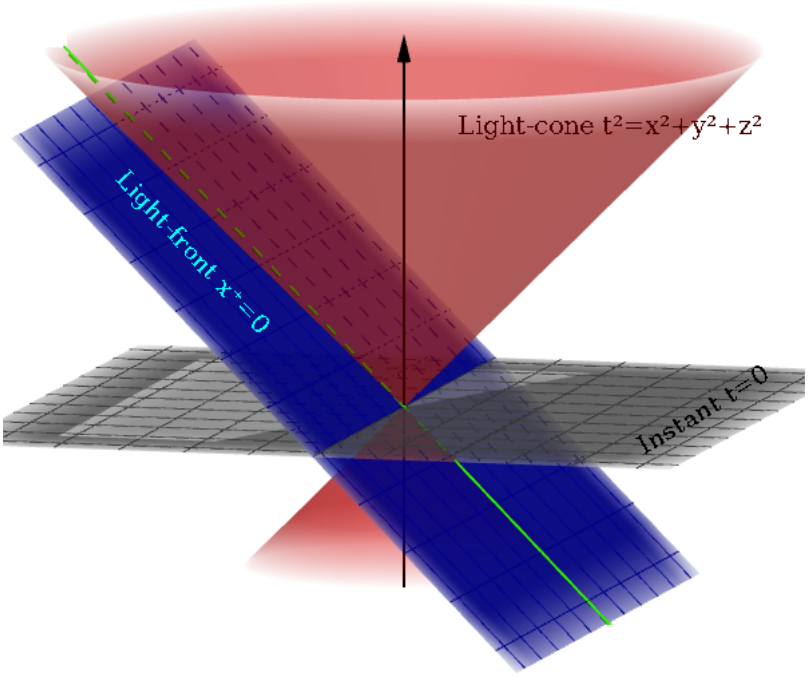

Figure 3.2.1. $2+1$-dimensional representation of the light-front $x^+ = 0$.

they are simply given by sums of free one-particle generators. In the context of Lagrangian QFT, this can be checked directly from the expressions[5]:

$$P^\mu \equiv \int_{\Sigma_{\rm LF}} \mathrm{d}\sigma_+ T^{+\mu}, \qquad M^{\mu\nu} \equiv \int_{\Sigma_{\rm LF}} \mathrm{d}\sigma_+ (x^\mu T^{+\nu} - x^\nu T^{+\mu}),$$

where $T^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathscr{L}$ is the energy-momentum tensor, P^μ and $M^{\mu\nu}$ are the four-momentum vector and angular momentum tensor, $\mathrm{d}\sigma_+$ denotes the infinitesimal element of $\Sigma_{\rm LF}$ (for more details on the LF notation, see Subsec. 3.2.2).

Particularly important elements of the kinematical group are the so-called LF boosts, which form themselves a subgroup, and act transitively on the positive mass-shell $\{P^2 = M^2,\ P^0 > 0\}$. As we shall see shortly, this particular fact implies that the LF wave-functions (LFWFs) of a bound state, which describe the internal motion of the constituents, are frame independent quantities.

The second important feature is that the expression of the exact vacuum in terms of Fock states is much simpler in the LF form than in the instant form. Indeed, for a theory without massless fields, it exactly coincides with the free vacuum [37, 38]. This feature allows one to establish a simple relation between the BS amplitude and the valence LFWF (see Subsec. 3.2.3).

Lastly, the Klein-Gordon equation is of the first order in the LF-time x^+, and thus the inverse propagator is of first order in the LF-energy. We anticipate that this latter feature plays a relevant role in the so-called LF projection of the BSE, which we shall introduce later on.

[5] With the usual caveat of non-derivative interactions [38], which, at any rate, also alter the CCRs. Moreover, the expression given for $M^{\mu\nu}$ is valid for scalar field theories.

3.2.2 Summary of LF notation

Tensorial algebra conventions. We define the LF coordinates by:

$$x^{\pm} \equiv x^0 \pm x^3, \qquad \mathbf{x}_\perp \equiv \left(x^1, x^2\right), \tag{3.2.2}$$

and denote more generally the $\pm$ components of a tensor by:

$$T^{\pm\mu\nu\rho\cdots} \equiv T^{0\mu\nu\rho\cdots} \pm T^{3\mu\nu\rho\cdots}. \tag{3.2.3}$$

The scalar product between two four vectors A and B takes the form:

$$A \cdot B = \frac{1}{2}A^+B^- + \frac{1}{2}A^-B^+ - \mathbf{A}_\perp \cdot \mathbf{B}_\perp \tag{3.2.4}$$

so that the metric tensor is given by:

$$g_{\rm LF} \equiv \begin{array}{c} \begin{matrix} \scriptstyle + & \scriptstyle 1 & \scriptstyle 2 & \scriptstyle - \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \end{array}. \tag{3.2.5}$$

The raising and lowering of indices is defined as usual by $A_\mu = g_{\mu\nu}A^\nu$, i.e.:

$$A_\pm = \frac{1}{2}A^\mp, \qquad A_{\perp i} = -A^i_\perp \tag{3.2.6}$$

The covariant components of the gradient are:

$$\partial_\pm \equiv \frac{\partial}{\partial x^\pm}, \quad \partial_{\perp i} \equiv \frac{\partial}{\partial x^i_\perp}. \tag{3.2.7}$$

The space-time volume and $\Sigma_{\rm LF}$ surface elements are given by[6]:

$$\mathrm{d}^4x = \frac{1}{2}\mathrm{d}x^+\mathrm{d}x^-\mathrm{d}^2\mathbf{x}_\perp, \tag{3.2.8}$$

$$\mathrm{d}\sigma_+ = \frac{1}{2}\mathrm{d}x^-\mathrm{d}^2\mathbf{x}_\perp. \tag{3.2.9}$$

We also indicate LF position and momentum three-vectors by the following notation:

$$\tilde{x} \equiv \left(x^-, \mathbf{x}_\perp\right), \qquad \tilde{p} \equiv \left(p^+, \mathbf{p}_\perp\right), \tag{3.2.10}$$

with scalar product:

$$\tilde{p}\tilde{x} \equiv \frac{1}{2}p^+x^- - \mathbf{p}_\perp \cdot \mathbf{x}_\perp. \tag{3.2.11}$$

The mass-shell Lorentz invariant measure satisfies:

$$\int \mathrm{d}^4p\, \delta(p^2 - m^2)\theta(p^0)f(p) = \int \frac{\mathrm{d}^3\tilde{p}}{2p^+}\theta(p^+)f(p), \tag{3.2.12}$$

where $f(p)$ is any function defined on the positive mass-shell $\{p^2 = m^2,\, p^0 > 0\}$, $\mathrm{d}^3\tilde{p} = \mathrm{d}p^+\mathrm{d}^2\mathbf{p}_\perp$, and the dispersion relation on the RHS of (3.2.12) is:

$$p^- = \frac{\mathbf{p}_\perp^2 + m^2}{p^+}. \tag{3.2.13}$$

[6]The surface element is obtained from the geometrical definition:

$$\mathrm{d}\sigma_\mu = \frac{1}{3!}\varepsilon_{\mu\nu\rho\sigma}\mathrm{d}x^\nu \otimes \mathrm{d}x^\rho \otimes \mathrm{d}x^\sigma,$$

which implies Eq. (3.2.9) since $\varepsilon_{+12-} = \frac{1}{2}$.

LF decomposition of the Poincaré algebra. Following Ref. [40], we employ the following conventions for the Poincaré algebra[7]:

$$[P^\mu, P^\nu] = 0, \tag{3.2.14}$$
$$[M^{\mu\nu}, P^\rho] = i(g^{\nu\rho}P^\mu - g^{\mu\rho}P^\nu), \tag{3.2.15}$$
$$[M^{\mu\nu}, M^{\rho\sigma}] = i(g^{\mu\sigma}M^{\nu\rho} + g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho}), \tag{3.2.16}$$

and define:

$$J^i = \frac{1}{2}\varepsilon_{ijk}M^{jk}, \qquad K^i = M^{0i}. \tag{3.2.17}$$

With these conventions, the operators $e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}}$ and $e^{-i\boldsymbol{\omega}\cdot\boldsymbol{K}}$ generate *active* rotations and boosts of the system respectively. For instance:

$$e^{i\theta J^3}\begin{pmatrix}P^1\\P^2\end{pmatrix}e^{-i\theta J^3} = \begin{pmatrix}\cos\theta P^1 - \sin\theta P^2\\ \sin\theta P^1 + \cos\theta P^2\end{pmatrix}, \tag{3.2.18}$$
$$e^{i\omega K^3}\begin{pmatrix}P^0\\P^3\end{pmatrix}e^{-i\omega K^3} = \begin{pmatrix}\cosh\omega P^0 + \sinh\omega P^3\\ \sinh\omega P^3 + \cosh\omega P^3\end{pmatrix}. \tag{3.2.19}$$

The LF components of P^μ and $M^{\mu\nu}$ are:

$$P^\mu = \left(P^+, P^1, P^2, P^-\right) \tag{3.2.20}$$

$$M^{\mu\nu} = \begin{array}{c}\overset{\quad\nu\quad}{\dashrightarrow}\\ \begin{pmatrix}0 & B^1 & B^2 & -2K^3\\ -B^1 & 0 & J^3 & -S^1\\ -B^2 & -J^3 & 0 & -S^2\\ 2K^3 & S^1 & S^2 & 0\end{pmatrix}\end{array}\begin{array}{l}\mu\\ \downarrow\end{array}, \tag{3.2.21}$$

where:

$$B^1 = K^1 + J^2, \qquad B^2 = K^2 - J^1, \tag{3.2.22}$$
$$S^1 = K^1 - J^2, \qquad S^2 = K^2 + J^1, \tag{3.2.23}$$

or, more compactly:

$$\mathbf{B}_\perp = \mathbf{K}_\perp - \mathbf{z}\times\mathbf{J}. \tag{3.2.24}$$
$$\mathbf{S}_\perp = \mathbf{K}_\perp + \hat{\mathbf{z}}\times\mathbf{J}, \tag{3.2.25}$$

The generators B^i and S^i are called transverse LF boosts and rotations, respectively. The three dynamical generators are P^- and $\mathbf{S}_\perp$. In particular, P^- generates translations in the + space-time direction (LF-time):

$$e^{\frac{i}{2}cP^-}\mathcal{O}(x)e^{-\frac{i}{2}cP^-} = \mathcal{O}(x^+ + c, x^-, \mathbf{x}_\perp),$$

and therefore it is called the *LF-energy*. The kinematical generators include the translations parallel to the null hyperplane ($\mathbf{P}_\perp$ and P^+), the rotations and boosts in the $\hat{z}$-direction (J^3 and K^3) and the transverse LF boosts $\mathbf{B}_\perp$. As mentioned above (see Appendix B for details), the LF boosts, i.e. the boosts generated by K^3 and $\mathbf{B}_\perp$, form a subgroup of the Lorentz group, which acts transitively on the positive mass shell $\{P^2 = M^2,\ P^0 > 0\}$. Together with their kinematical character, this fact implies that the LFWFs are frame independent.

For more details on the LF decomposition of the Poincaré algebra and, in particular, on the commutation rules of the LF generators, we address the interested reader to Refs. [40, 41].

[7]Notice that Kogut and Soper [40] define the ±-components as $x^\pm \equiv \frac{x^0 \pm x^3}{\sqrt{2}}$, differently from us (cf. Eq. (3.2.2)).

QFT conventions. We write the real scalar field at $x^+ = 0$ as:

$$\varphi(\tilde{x}) = \int \frac{\mathrm{d}^3\tilde{p}\,\theta(p^+)}{\sqrt{(2\pi)^3 2p^+}} \left\{ a(\tilde{p}) e^{-i\tilde{p}\tilde{x}} + a^\dagger(\tilde{p}) e^{i\tilde{p}\tilde{x}} \right\}, \tag{3.2.26}$$

$$a(\tilde{p}) = \sqrt{(2\pi)^3 2p^+} \int \frac{\mathrm{d}^3\tilde{x}}{2}\, e^{i\tilde{p}\tilde{x}} \varphi(\tilde{x}) \qquad (p^+ > 0). \tag{3.2.27}$$

The CCRs read [42]:

$$\left[\varphi(\tilde{x}), \partial^+\varphi(\tilde{y})\right] = i\delta^2(\mathbf{x}_\perp - \mathbf{y}_\perp)\delta(x^- - y^-), \tag{3.2.28}$$

$$\left[a(\tilde{p}), a^\dagger(\tilde{q})\right] = \delta^3(\tilde{p} - \tilde{q}). \tag{3.2.29}$$

Notice that, contrary to the annihilation operators in the instant form $t = 0$, the LF annihilation operators (3.2.27) are simply given by a Fourier transform of the field on the LF surface, as a consequence of the halved domain $[0, +\infty)$ of p^+.

The Fock vacuum is defined by:

$$a(\tilde{p})|0\rangle_{\mathrm{LF}} = 0 \qquad \forall \tilde{p} \tag{3.2.30}$$

and we will assume throughout that it coincides with the exact vacuum (see the discussion in Subsec. 3.2.1)[8]:

$$|0\rangle_{\mathrm{LF}} \equiv |0\rangle. \tag{3.2.31}$$

The interacting field is related to the field at $x^+ = 0$ by the Heisenberg equation:

$$\phi(x) \equiv e^{\frac{i}{2}P^- x^+} \varphi(\tilde{x}) e^{-\frac{i}{2}P^- x^+}. \tag{3.2.32}$$

The free field satisfies the Klein-Gordon equation:

$$(\partial^-\partial^+ - \boldsymbol{\partial}_\perp^2 + m^2)\phi = 0, \tag{3.2.33}$$

which is, as mentioned above, of first order in the time variable x^+, implying that the number of degrees of freedom is, in the present formulation, reduced by one-half with respect to the standard (instant) formulation (see Ref. [42] for further discussion).

3.2.3 The LFWF and its relation with the BS amplitude

The bound state $|b; P\rangle$ can be expanded in the complete Fock basis defined by the creation operators in (3.2.26). The expansion is usually written [22]:

$$|b; P\rangle = \int_0^1 \frac{\mathrm{d}\xi}{(2\pi)2\xi(1-\xi)} \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{(2\pi)^2} |2/P; \xi\mathbf{k}_\perp\rangle \Psi_2(\xi, \mathbf{k}_\perp) + \cdots, \tag{3.2.34}$$

where we have displayed explicitly only the so-called valence component of the state. The two particles states $|2/P; \xi\mathbf{k}_\perp\rangle$ are defined by:

$$|2/P; \xi\mathbf{k}_\perp\rangle \equiv \sqrt{(2\pi)^3 2p_1^+} a_1^\dagger(\tilde{p}_1) \sqrt{(2\pi)^3 2p_2^+} a_2^\dagger(\tilde{p}_2)|0\rangle, \tag{3.2.35}$$

[8] The simplicity of the vacuum stems from the fact that, according to the basic postulates of field theory, $|0\rangle$ must be a (proper) eigenvector of $\tilde{P}$ with eigenvalue $\tilde{p} = 0$. For a theory with only massive quanta, every eigenstate $|n\rangle$ of P^+ with $n \geq 1$ particles has an eigenvalue:

$$p^+ = \sum_{i=1}^n p_i^+ = \sum_{i=1}^n (p_i^0 + p_i^3) > 0,$$

so that its overlap with the vacuum must vanish, $\langle n|0\rangle = 0$.

where:

$$p_1^+ \equiv \xi P^+, \qquad \mathbf{p}_{1\perp} \equiv \xi \mathbf{P}_\perp + \mathbf{k}_\perp, \tag{3.2.36}$$

$$p_2^+ \equiv (1-\xi) P^+, \qquad \mathbf{p}_{2\perp} \equiv (1-\xi)\mathbf{P}_\perp - \mathbf{k}_\perp. \tag{3.2.37}$$

The scalar products between two such states is (assuming distinguishable particles):

$$\langle 2/P'; \xi' \mathbf{k}'_\perp | 2/P; \xi \mathbf{k}_\perp \rangle = (2\pi)^3 2P^+ \delta^3(\tilde{P} - \tilde{P}')(2\pi)^3 2\xi(1-\xi)\delta(\xi - \xi')\delta^2(\mathbf{k}_\perp - \mathbf{k}'_\perp), \tag{3.2.38}$$

so that:

$$\langle 2/P'; \xi' \mathbf{k}'_\perp | b; P \rangle = (2\pi)^3 2P^+ \delta^3(\tilde{P} - \tilde{P}') \Psi_2(\xi', \mathbf{k}'_\perp). \tag{3.2.39}$$

It should be remarked that Ψ_2, like any LFWF, is a frame independent quantity, as we prove in Appendix B.4. In fact, the quantities ξ and $\mathbf{k}_\perp$ are invariant with respect to the LF boosts generated by K^3 and $\mathbf{B}_\perp$ and describe therefore the intrinsic motion in the two-particles component of the state $|b\rangle$. As an inspection of Eq. (3.2.36) reveals, they are respectively the fraction of longitudinal momentum carried by particle 1, and its transverse momentum in a reference frame where $\mathbf{P}_\perp = \mathbf{0}$.

We can now establish a simple relation between the BS amplitude and the valence wave function Ψ_2, which provides a physical interpretation for the LF projection of the BS amplitude. We start by writing the time-ordered product appearing in the definition of the BS amplitude as a LF time-ordered product:

$$\chi(x;P) = \theta(x^+)\langle 0|\phi_1(\eta_2 x)\phi_2(-\eta_1 x)|b;P\rangle + \theta(-x^+)\langle 0|\phi_2(-\eta_1 x)\phi_1(\eta_2 x)|b;P\rangle. \tag{3.2.40}$$

The intuitive justification for this procedure is to regard the LF surface $\Sigma_{\rm LF}$ as the limit of a flat space-like surface. However, it must be recognized that the identity (3.2.40) is nontrivial. It is well known, for instance, that the T^+-ordered propagator of a free Dirac field differs from the T-ordered one by a non-covariant term, proportional to $\delta(x^+)\delta(\mathbf{x}_\perp)$ [40], [43]. Indeed, it follows from the assumption of microcausality that the T^+ and T product of two local fields can differ at most by a term concentrated on the light-like line $\{x | x^+ = 0, \mathbf{x}_\perp = \mathbf{0}\}$ contained in $\Sigma_{\rm LF}$ (the green line in Fig. 3.2.1).

We are not aware of any reference in the literature addressing this particular issue. In what follows, we shall simply ignore the issue and regard (3.2.40) as a valid relation. However, it is worth to recall that, in Ref. [43], the equivalence of T^+ and T products was proved perturbatively (i.e. order by order) for the Dyson series, viz.:

$$\mathrm{T}^+ \exp\left\{-i\int \mathrm{d}^4x\, \mathscr{H}_I^+\right\} = \mathrm{T}\exp\left\{-i\int \mathrm{d}^4x\, \mathscr{H}_I\right\},$$

where the subscript I denotes operators in the interaction representation. The LF interaction $\mathscr{H}_I^+$ coincides with $\mathscr{H}_I = -\mathscr{L}_I$ for a scalar theory with non-derivative interactions. In the presence of Dirac fields, $\mathscr{H}_I^+ - \mathscr{H}_I$ contains non-covariant terms, needed to cancel the corresponding non-covariant contributions arising from T^+-ordered fermion propagators.

We can directly check that when $x^+ = 0$ the two amplitudes on the RHS of (3.2.40) coincide, so that, at least on the formal level, we have:

$$\chi(x^+ = 0, \tilde{x}; P) = \lim_{\varepsilon \to 0^\pm} \chi(x^+ = \varepsilon, \tilde{x}; P) = \langle 0|\phi_1(\eta_2 \tilde{x})\phi_2(-\eta_1 \tilde{x})|b;P\rangle. \tag{3.2.41}$$

Notice that putting $x^+ = 0$ corresponds, in momentum space, to an integration of $\chi(k;P)$ over the minus component of the relative momentum. Indeed, using the plane wave expansion (3.2.26), the CCRs (3.2.29) and the bound state LF expansion (3.2.34), after some formal manipulations on Eq.

(3.2.41) (see Ref. [22] for details), we find[9]:

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}q^-}{2\pi} \chi(q;P) = \frac{1}{P^+\xi(1-\xi)} \Psi_2(\xi, \mathbf{k}_\perp), \tag{3.2.42}$$

where the variables in (3.2.42) are related by:

$$\xi = \eta_1 + \frac{q^+}{P^+}, \tag{3.2.43}$$

$$\mathbf{k}_\perp = \mathbf{q}_\perp - \frac{q^+}{P^+}\mathbf{P}_\perp. \tag{3.2.44}$$

It is thus seen that the LF projection of the BS amplitude is, apart from a kinematical factor, nothing but the valence LFWF of the bound state. In particular, the integration over q^- in Eq. (3.2.42), which in coordinate space amounts to set to zero the relative LF-time x^+, allows one to recover a probabilistic interpretation which, as discussed in Section 2.1, does not pertain to the BS amplitude.

Valence probability

A simple quantity which can be directly extracted from the LFWF is the so-called valence probability, which is defined as the probability to find the system in a state with two constituents. Roughly speaking, the valence probability can be regarded as a measure of the deviation from the non-relativistic picture of the bound state, as composed by two constituents in a bounded region of space.

To obtain the valence probability of the bound state, in order to deal with a physical (normalizable) state, let us consider a wave-packet:

$$|b;\Phi\rangle = \int \frac{\mathrm{d}^3\tilde{P}}{(2\pi)^3 2P^+} |b;P\rangle \Phi(P), \tag{3.2.45}$$

$$1 = \langle b;\Phi|b;\Phi\rangle = \int \frac{\mathrm{d}^3\tilde{P}}{(2\pi)^3 2P^+} \left|\Phi(P)\right|^2. \tag{3.2.46}$$

From the normalization (3.2.38) of 2-particle states, we see that the projector on the 2-particle sector of the Hilbert space may be written as:

$$\Pi_2 = \int \frac{\mathrm{d}^3\tilde{P}}{(2\pi)^3 2P^+} \frac{\mathrm{d}^2\mathbf{k}_\perp \mathrm{d}\xi}{(2\pi)^3 2\xi(1-\xi)} |2/P;\xi\mathbf{k}_\perp\rangle\langle 2/P;\xi\mathbf{k}_\perp|. \tag{3.2.47}$$

Therefore, the valence probability is:

$$P_{\mathrm{val}} = \langle b;\Phi|\Pi_2|b;\Phi\rangle = \int \frac{\mathrm{d}^2\mathbf{k}_\perp \mathrm{d}\xi}{(2\pi)^3 2\xi(1-\xi)} \left|\Psi_2(\xi,\mathbf{k}_\perp)\right|^2. \tag{3.2.48}$$

It is worth to stress that P_{val} is an intrinsic quantity, independent from the arbitrary wave-packet Φ, as a consequence of the frame independence of the LFWF.

[9] Notice that, contrary from us, Ref. [22] considers the constituents 1 and 2 as indistinguishable, and has therefore a factor $\sqrt{2}$ multiplying the RHS of (3.2.42).

3.2.4 NIR of the valence wave-function

The relation between the LFWF and the BS amplitude, Eq. (3.2.42), together with the NIR of the BS amplitude (3.1.5), allows to obtain immediately an integral representation for the LFWF.

The integral of (3.1.5) with respect to the relative LF-energy can be carried out explicitly, using the so-called LF integrals identities (see Appendix C.5.6), with the result:

$$\int \frac{\mathrm{d}q^-}{2\pi}\chi(q;P) = \frac{1}{P^+}\int_{-\infty}^{\infty}\mathrm{d}\gamma' \frac{g(\gamma', -\frac{2q^+}{P^+})}{\left[-\frac{q^+}{P^+}P^-q^+ - \mathbf{q}_\perp^2 - z'\mathbf{P}_\perp\cdot\mathbf{q}_\perp - \kappa^2 - \Delta^2 + 2z\overline{m}\Delta - \gamma' + i\varepsilon\right]^2}.$$

Comparing with (3.2.42), and using the definitions (3.2.43) and (3.2.44), we finally obtain[10]:

$$\Psi_2(\xi,\gamma) = \frac{1-z^2}{4}\int_{-\infty}^{\infty}\mathrm{d}\gamma' \frac{g(\gamma', z)}{\left[\gamma' + D_0(\gamma, z) - i\varepsilon\right]^2}, \tag{3.2.49}$$

where:

$$\gamma \equiv \mathbf{k}_\perp^2, \qquad z \equiv 1 - 2\xi \tag{3.2.50}$$

and

$$D_0(\gamma, z) \equiv \gamma + (1-z^2)\kappa^2 + (z\overline{m} - \Delta)^2. \tag{3.2.51}$$

Notice that $D_0(\gamma, z) \geq 0$ for a bound state, since $\kappa^2 \geq 0$. Using the distributional identity [13]

$$\frac{1}{[x - i\varepsilon]^n} = \mathrm{FP}\frac{1}{x^n} + i\pi\frac{(-1)^{n-1}}{(n-1)!}\delta^{(n-1)}(x),$$

where "FP" stands for the Hadamard finite part prescription [44], and the fact that g and Ψ_2 are both real[11], one may show that the NIR (3.2.49) implies that the weight function $g(\gamma', z)$ vanishes when $\gamma' + D_0(0, z) \leq 0$ (so that the $i\varepsilon$ term can be omitted from the denominator of Eq. (3.2.49)).

The inversion problem

As we have discussed, the LF projection of the BS amplitude, i.e. its integral over k^-, provides a map:

$$\chi \overset{L}{\mapsto} \Psi_2$$

which, given the full BS amplitude χ as input, returns the valence wave-function $\Psi_2 = L\chi$ as output. One could wonder if this map is one-to-one, i.e., if there exists a single valued inverse operator L^{-1}. Quite remarkably, the answer turns out to be positive and, even more interestingly, the bridge between the two functions is provided by the Nakanishi representation.

The formal existence of the inverse L^{-1} was already suggested in a series of papers (see Ref. [45] and references quoted therein) where the BSE was projected on the light-front, yielding an equation for the LFWF. By applying a formal operator L^{-1} to the solution Ψ_2 of the latter equation, it was shown that $\chi = L^{-1}\Psi_2$ provides a solution of the original BSE. However, no explicit expression of the operator L^{-1} was given, so that the possibility to invert the LF projection remained on a purely formal level. Notice that NIR was not used at all in those works.

Recently, a solution to the inversion issue under the assumption of a NIR for the BS amplitude has been proposed in Ref. [46], where it was shown that the relation (3.2.49) between Ψ_2 and the Nakanishi weight-function g can actually be inverted, given the knowledge of Ψ_2 in the full complex

[10] Incidentally, we observe that Eq. (3.2.49) explicitly displays the above mentioned frame independence of Ψ_2.

[11] The valence LFWF is real under the assumption of $\mathcal{PT}$ invariance, see Appendix B.4.

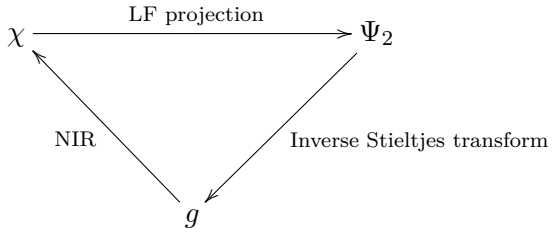

Figure 3.2.2. Relations between the BS amplitude, the valence LFWF and the Nakanishi weight function.

plane of its first argument $\gamma = \mathbf{k}_\perp^2$. Given the weight-function g, the BS amplitude is immediately recovered from its NIR.

The key observation of Ref. [46] is that the integral representation (3.2.49) is a particular case of a generalized Stieltjes transform:

$$F(x) = \int_0^\infty \frac{G(y)}{(y+x)^\rho} \mathrm{d}y. \tag{3.2.52}$$

One can show that, under very general assumptions on the function G, the LHS of Eq. (3.2.52) is an analytic function, a part from a cut along the negative real axis, and that (3.2.52) can be analytically inverted. In the case $\rho = 2$, the inversion formula is given by:

$$G(y) = \frac{y}{2\pi i} \int_C F(yw) \mathrm{d}w. \tag{3.2.53}$$

Here C is a contour in the w complex plane which starts and ends at $w = -1$, and has winding number 1 with respect to the origin $w = 0$. In particular, one has:

$$G(y) = y \int_{-\pi}^{\pi} \frac{\mathrm{d}\phi}{2\pi} e^{i\phi} F(y e^{i\phi}). \tag{3.2.54}$$

By applying this inversion formula to Eq. (3.2.49), one obtains (in the equal masses case):

$$g(\gamma, z) = \frac{4\gamma}{1-z^2} \int_{-\pi}^{\pi} \frac{\mathrm{d}\phi}{2\pi} e^{i\phi} \Psi_2(\gamma e^{i\phi} - m^2 z^2 - (1-z^2)\kappa^2, z). \tag{3.2.55}$$

Therefore the inverse operator L^{-1} of the LF projection is given by the composition of the Stieltjes inversion ($\Psi_2 \mapsto g$) and of the NIR ($g \mapsto \chi$). The relations between χ, Ψ_2 and g are summarized in Fig. 3.2.2.

It should be pointed out that, at least in principle, the knowledge of Ψ_2 for complex values of γ can be completely bypassed. Indeed, assuming only the Lebesgue integrability of G, one can construct from F a sequence of functions F_k such that:

$$\lim_{k\to\infty} F_k(y) = G(y)$$

at every Lebesgue point of G [47]. Therefore, the valence wave-function *in Minkowski space* is, in principle, sufficient to reproduce the BS amplitude. Unfortunately, the functions F_k are given by a complicated expression involving the $2k-1$-th derivative of F (see [47, eq. (5)]), so that this kind of "real" inversion appears practically unfeasible.

3.3 A NIR-LF based approach to the BSE

In this Section we introduce, in the simple case of scalar constituents, the method based on NIR and the LF projection for obtaining numerical solutions of the Minkowski space BSE. This approach has been so far applied: i) in the scalar constituents case, to the scalar exchange ladder [14, 18, 19] and ladder plus cross-ladder [15] kernels; ii) in the fermion constituents case, to the scalar, pseudoscalar and vector exchange ladder kernels [17, 20]. We shall illustrate the results for fermion-scalar constituents, interacting through the exchange of a massive scalar, in the next two Chapters, while the interaction through a massless vector will be discussed in [48].

It is worth mentioning that Refs. [14, 18, 19, 15, 17, 20] assume the indistinguishability of the constituent particles (in the vector-exchange case, the two fermions are particle and antiparticle). The full agreement of these results with those obtained from the Wick-rotated BSE (see, in particular, Ref. [19]) provides a first strong hint towards the general validity of the NIR-LF approach to the Minkowski space BSE.

3.3.1 Description of the method

The core of the method is the NIR of the BS amplitude discussed in Sec. 3.1, which we rewrite for convenience:

$$\chi(k;P) = i\int_{-1}^{1}\mathrm{d}z'\int_{0}^{\infty}\mathrm{d}\gamma'\frac{g(\gamma',z')}{\left[(k+z'\frac{P}{2})^2-(1-z'^2)\kappa^2-(z'\overline{m}-\Delta)^2-\gamma'+i\varepsilon\right]^3}. \tag{3.3.1}$$

(the support property $\gamma' > 0$ is assumed throughout). Once Eq. (3.3.1) is plugged into the BSE, the latter becomes a singular integral equation for the weight function g:

$$\chi(k;P) = i\int_{-1}^{1}\mathrm{d}z'\int_{0}^{+\infty}\mathrm{d}\gamma'\sum_{j=1,2}\widetilde{V}(k^2,Pk,\gamma',z')g(\gamma',z'), \tag{3.3.2}$$

where χ in the LHS is expressed in terms of g through Eq. (3.3.1). For the ladder kernel:

$$iK_{\mathrm{lad}}(k,k') = i\lambda\frac{i}{(k-k')^2-\mu^2+i\varepsilon}i\lambda \tag{3.3.3}$$

the RHS kernel $\widetilde{V}$ is given by:

$$\widetilde{V}(k^2,Pk,\gamma',z') = \frac{\lambda^2}{8\pi^2M^2}\int_0^1 v^2\mathrm{d}v\frac{1}{\left[(1-z)(k^- - k_d^-)+i\varepsilon\right]\left[(1+z)(k^- - k_u^-)-i\varepsilon\right]\left[k^-k_D^+ + l_D + i\varepsilon\right]^2}. \tag{3.3.4}$$

Here v is a Feynman parameter, reminiscent of the four-dimensional integration in k', and the quantities $k_{u,d}^-$, k_D^+ and l_D are defined in Appendix C.2 (with $m_{\psi/\phi} \to m_{1/2}$). The presence of poles in the various denominators, imply that the integrals in Eq. (3.3.2) have to be interpreted as Hadamard finite part integrals (see e.g. Ref. [44]; integral equations involving finite part integrals are called, in the mathematical literature, "hypersingular").

To obtain a regular equation for g, the whole equation is projected onto the light front, i.e., is integrated over k^-. Recall that, since the LF projection is invertible under the assumption of a NIR, the LF projected BSE is strictly equivalent to the original one if Eq. (3.3.1) is valid.

The final result is an integral equation of the form:

$$Bg = Vg, \tag{3.3.5}$$

where the kernels of the integral operators B and V are both regular. The LHS is, apart from a kinematical factor, the valence LFWF of the bound state (cf. Subsec. 3.2.49), and is given by:

$$(Bg)(z,\gamma) \equiv \int_0^{+\infty} \mathrm{d}\gamma' \frac{g(\gamma',z)}{\left[\gamma' + D_0(\gamma,z)\right]^2}, \tag{3.3.6}$$

with:

$$D_0(\gamma,z) \equiv \gamma + (1-z^2)\kappa^2 + (z\overline{m} - \Delta)^2 \tag{3.3.7}$$

The operator V in the RHS is determined by the irreducible kernel iK adopted for the original BSE. For the ladder kernel (3.3.3), in the equal masses case, it takes the form [18, 46]:

$$V = \alpha A, \tag{3.3.8}$$

where $\alpha \equiv \frac{g^2}{16\pi m^2}$ and:

$$(Ag)(\gamma,z) = \frac{m^2}{2\pi} \int_0^{+\infty} \mathrm{d}\gamma' \int_{-1}^{1} \mathrm{d}z' \left[\theta(z-z')F(\gamma,z,\gamma',z') + \theta(z'-z)F(\gamma,-z,\gamma',-z')\right] g(\gamma',z'), \tag{3.3.9}$$

where:

$$F(\gamma,z,\gamma',z') = \frac{(1-z)^2}{\gamma + z^2\frac{M^2}{4} + \kappa^2} \int_0^1 v^2 \mathrm{d}v \, \frac{1}{D(\gamma,z,\gamma' z')^2}, \tag{3.3.10}$$

$$\begin{aligned} D(\gamma,z,\gamma',z') = {} & v(1-v)(1-z')\gamma + v(1-z)\gamma' + v(1-z)(1-z')\left[1 + z(1-v) + vz'\right]\kappa^2 + \\ & + v\left[(1-v)(1-z')z^2 + vz'^2(1-z)\right] m^2 + (1-v)(1-z)\mu^2. \end{aligned} \tag{3.3.11}$$

Therefore, once the bound state mass M^2 is fixed, Eq. (3.3.5) becomes a generalized eigenvalue equation:

$$Ag = \lambda B g, \tag{3.3.12}$$

where $\lambda = \frac{1}{\alpha}$. The choice of solving Eq. (3.3.12) for $\lambda = \frac{1}{\alpha}$, instead that for α directly, is entirely due to numerical stability issues, since the routine employed for the numerical solution of the eigenvalue problem (DGGEV from LAPACK) treats asymmetrically the matrices A and B.

Notice that, if we invert the Stieltjes transform B in (3.3.12), we obtain an ordinary eigenvalue equation for the operator $C = B^{-1}A$, which has in general complex solutions (λ, g), since C is not necessarily symmetric. However, such an inversion was not performed in the literature, in the actual solutions of (3.3.12).

The bound states correspond to the real, positive eigenvalues. The actual search of the admissible eigenvalue for a given mass M of the ground state is driven by the request that the strength of the interaction (i.e. the value of the coupling constant α) is the smallest one giving rise to the bound state (this is the same pattern one meets in the study of the ground state for a non-relativistic potential well, where the shallowest well is requested). Therefore, the M vs. α dependence for the ground state is simply given by $\alpha(M) = \frac{1}{\lambda_{\max}(M)}$, where $\lambda_{\max}(M)$ is the greatest real eigenvalue in Eq. (3.3.12) for the fixed mass M. Excited states can be treated by a slight generalization of the method [19].

The eigenvalue problem (3.3.12) has been extensively investigated by exploiting standard methods. In Ref. [18], the integral operators A and B were discretized through the introduction of an orthonormal basis:

$$u_{j\ell}(\gamma,z) = \mathcal{L}_j(\gamma)\mathcal{G}_\ell(z) \qquad (\gamma,z) \in [0,+\infty) \times [-1,1]\,, \tag{3.3.13}$$

where $\mathcal{L}_j$ and $\mathcal{G}_\ell$ are given in terms of Laguerre L_j and Gegenbauer $C_\ell^{(\alpha)}$ polynomials respectively:

$$\mathcal{L}_j(\gamma) = \sqrt{a} L_j(a\gamma) e^{-\frac{a\gamma}{2}}, \tag{3.3.14}$$

$$\mathcal{G}_\ell(z) = 4(1-z^2)\Gamma(\frac{5}{2})\sqrt{\frac{(2\ell+\frac{5}{2})(2\ell!)}{\pi\Gamma(2\ell+5)}} C_{2\ell}^{(\frac{5}{2})}(z). \tag{3.3.15}$$

In the indistinguishable particles case, due to permutational symmetry, one has $g(\gamma, z) = g(\gamma, -z)$, hence only even degree polynomials were used in Ref. [18]. The Gegenbauer weight function $1 - z^2$ ensures the vanishing of the valence LFWF for $z \to \pm 1$, which corresponds physically to one of the constituents having an infinite momentum along the $\hat{z}$ direction. Also, the Laguerre weight function $e^{-\frac{a\gamma}{2}}$ ensures the vanishing of g for $\gamma \to \infty$, which is required in order to have a finite value at the origin for the density in transverse-coordinate space (i.e. Fourier-conjugated to the $\mathbf{k}_\perp$ space) [18].

On the other hand, Refs. [14, 17] employ a cubic spline basis to discretize the eigenvalue problem (3.3.12). The use of a local expansion has the advantage of overcoming the problem related to the wild oscillating behavior of high-order orthogonal polynomials in a compact interval, such as the Gegenbauer polynomials. However, the numerical analysis in a non-orthogonal basis is more involved. The results obtained for the scalar case with the two basis expansions agree with very high accuracy [18].

It is worth mentioning that the discretized matrices B turn out to be ill-conditioned, that is to say, to have a large condition number

$$\kappa(B) \equiv \|B\| \left\|B^{-1}\right\| = \frac{\beta_{\max}}{\beta_{\min}},$$

where $\beta_{\max/\min}$ denote the greatest and smallest eigenvalues of B, respectively. The numerical inversion of an ill-conditioned matrix is a delicate task, and both Refs. [14, 18] add a small value $\varepsilon \sim 10^{-4} \div 10^{-8}$ to the diagonal entries of B in order to achieve numerical stability.

3.3.2 A first quantitative view

We quote in this Subsection the main results obtained in Refs. [18, 19], using the method described above, for the scalar ladder BSE in the equal masses case (for a first investigation of the inhomogeneous BSE, see [22, 23]).

Table 3.3.1 and Figure 3.3.1 show the dependence of both the coupling constant α and the valence probability P_{val} from the binding energy $B \equiv 2m - M$. Notice that $\frac{\mu}{m} = 0.15$ was chosen in the literature just to call to mind the ratio $\frac{m_\pi}{m_N}$, where $m_{\pi,N}$ are the masses of the pion and the nucleon respectively.

As it can be seen, the coupling constant has, for $\mu > 0$, a non-vanishing limit for $\frac{B}{m} \to 0$, which means that there exists a threshold $\alpha_{\min} > 0$ below of which the ladder kernel can not support any bound state. This is in agreement with the non-relativistic result for the Yukawa potential [49]. On the other hand, for $\frac{B}{m} = 2$, it is shown a decreasing behavior with $\frac{\mu}{m}$ towards its asymptotic value $\alpha = 2\pi$, i.e. the one obtained in the Wick-Cutkosky model [25, pp. 489-490], where a massless exchange is considered.

The valence probability tends to its asymptotic value $P_{\text{val}} = 1$ when $\frac{B}{m} \to 0$, which can be interpreted as the extreme non-relativistic limit. Also, when $\frac{B}{m} = 2$, the valence probability decreases with $\frac{\mu}{m}$ towards $P_{\text{val}} = 0.64$, which is again the result obtained in the Wick-Cutkosky model [50].

$\frac{\mu}{m} = 0.05$

B/m	α	P_{val}
0.001	0.1685	0.94
0.01	0.3621	0.89
0.10	1.191	0.75
0.20	1.850	0.72
0.50	3.358	0.68
1.00	5.056	0.66
2.00	6.336	0.65

$\frac{\mu}{m} = 0.15$

B/m	α	P_{val}
0.001	0.3667	0.97
0.01	0.5716	0.94
0.10	1.437	0.80
0.20	2.099	0.75
0.50	3.611	0.70
1.00	5.314	0.67
2.00	6.598	0.66

$\frac{\mu}{m} = 0.5$

B/m	α	P_{val}
0.001	1.167	0.98
0.01	1.440	0.96
0.10	2.498	0.87
0.20	3.251	0.83
0.50	4.900	0.77
1.00	6.711	0.74
2.00	8.061	0.72

Table 3.3.1. Coupling constant α and valence probability P_{val} vs. binding energy $\frac{B}{m}$, for $\frac{\mu}{m} = 0.05, 0.15, 0.5$. After Ref. [18].

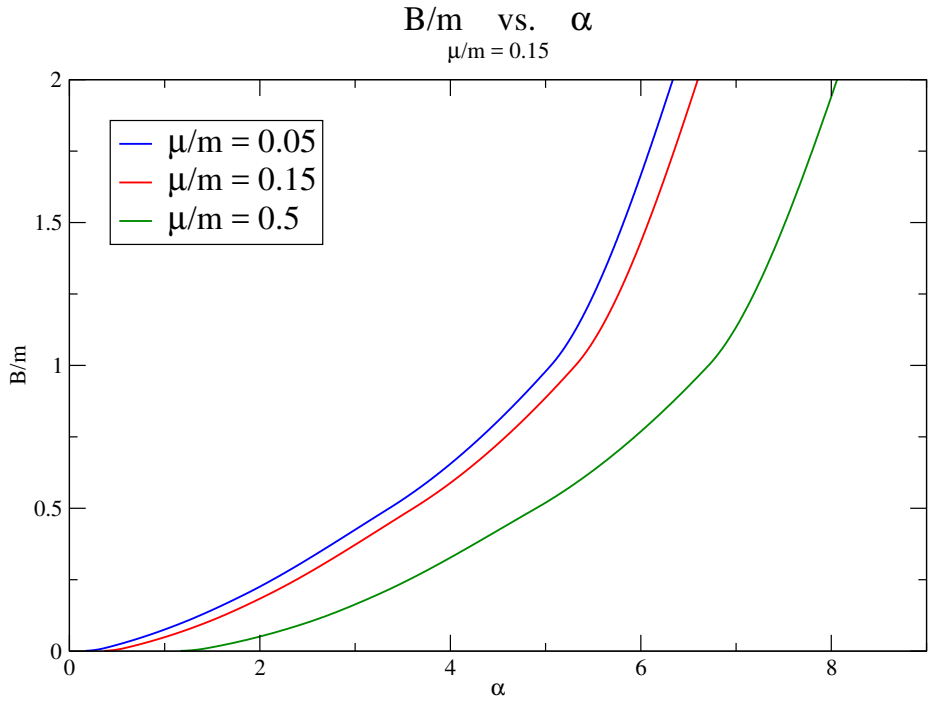

Figure 3.3.1. B/m vs. α for $\mu/m = 0.05, 0.15, 0.5$. The lines are Akima spline interpolations of the points in Table 3.3.1.

Cross-ladder effects and non-relativistic limit

It is well known that the cross-ladder contributions to the binding energy, i.e. those arising from irreducible diagrams of the form of Figure 3.3.2, are very large and attractive. The first investigation in Minkowski space was performed in Ref. [15], where the authors find:

$$\frac{\alpha_{\text{L}}(B) - \alpha_{\text{L+CL}}(B)}{\alpha_{\text{L}}(B)} \sim 0.1 \div 0.3$$

for $\frac{\mu}{m} = 0.5$ and $0 \leq \frac{B}{m} \leq 1$, where "L" and "L+CL" stand respectively for the "Ladder" and "Ladder+Cross-Ladder" BSE results. Noteworthy, the relative difference does not vanish in the non-relativistic limit, i.e. $B \to 0$, and increases with B, i.e. when the system becomes more relativistic.

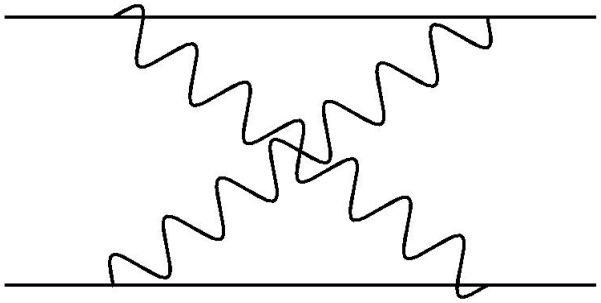

Figure 3.3.2. Cross-ladder diagram

It is also interesting to compare the results of the ladder and ladder+cross-ladder BSE with those obtained from the non-relativistic Schrödinger equation with a Yukawa potential. As shown in Ref. [15]:

$$\alpha_{\mathrm{L}}(B) > \alpha_{\mathrm{L+CL}}(B) > \alpha_{\mathrm{NR}}(B),$$

where "NR" stands for the non-relativistic result. The cross-ladder contribution are in the expected direction, but the difference $\alpha_{\mathrm{L+CL}} - \alpha_{\mathrm{NR}}$ is still about 10% for $\frac{B}{m} = 10^{-3}$ and $\frac{\mu}{m} = 0.5$.

To conclude, we address the reader to Ref. [34] for an in-depth analysis of cross-ladder effects on the valence LFWF and the electromagnetic form factor. All these results indicate that the terms ignored in the ladder approximation are actually important, both in the non-relativistic and in the high binding regimes.

Chapter 4

The fermion-scalar Bethe-Salpeter equation

We introduce in this Chapter the BSE for a bound state composed by a fermion and a scalar, with different masses. In Sec. 4.1 we extend the formalism based on the NIR of the BS amplitude and the LF projection, described for a two-scalar system, and in Sec. 4.2, we describe in detail our method for getting actual solutions of the fermion-scalar ladder BSE. For the sake of comparison, in Sec. 4.3 the Wick-rotated BSE for the system under scrutiny is introduced.

4.1 The BSE for a fermion-scalar bound state

The generalizations of the formalism developed in Chaps. 2 and 3 to the fermion-scalar case are, in principle, straightforward. However, as it is usual when one considers higher-spin structures, the calculations cope with some tricky points and become rather lengthy. For this reason, we shall mainly list the results needed for our numerical analysis to be presented in Chap. 5, referring to Appendices B and C for the formal details of the derivations.

4.1.1 General form of the BS amplitude

We consider a bound state of spin $j = \frac{1}{2}$, composed by a spin-$\frac{1}{2}$ and a spin-0 elementary particles described by the fields ψ and ϕ respectively. The corresponding BS amplitude is defined by:

$$\chi_s(x;P) \equiv \int \frac{\mathrm{d}^4k}{(2\pi)^4} e^{-ikx} \chi_s(k;P) \equiv \langle 0|\mathrm{T}\left\{\psi(\eta_2 x)\phi(-\eta_1 x)\right\}|b;P,s\rangle. \tag{4.1.1}$$

In the following, we shall choose to indicate with $s = \pm\frac{1}{2}$ the eigenvalue of the LF spin operator (see Appendix B.1):

$$j^3_{\mathrm{LF}} \equiv J^3 + \hat{\boldsymbol{z}} \cdot (\mathbf{B}_\perp \times \frac{\mathbf{P}_\perp}{P^+}). \tag{4.1.2}$$

Correspondingly, all spinors $u(P,s)$ appearing in what follows are understood to be LF spinors (see Appendix B.2).

In momentum space, the amplitude (4.1.1) satisfies the BSE (cf. Subsec. 2.1.1 and Table 2.1.1 for the definitions of the relevant Fourier transforms and of the global and relative variables):

$$\chi_s(k;P) = G_0(k;P) \int \frac{\mathrm{d}^4k'}{(2\pi)^4} iK(k,k';P)\chi_s(k';P), \tag{4.1.3}$$

where iK is the irreducible kernel and the two-particles propagator G_0 is given by:

$$G_0(k;P) \equiv iS'(\eta_1 P + k) i\Delta'(\eta_2 P - k), \tag{4.1.4}$$

$$iS'(q) \equiv \int \mathrm{d}^4x\, e^{iqx} \langle 0|\mathrm{T}\{\psi(x)\overline{\psi}(0)\}|0\rangle, \tag{4.1.5}$$

$$i\Delta'(q) \equiv \int \mathrm{d}^4x\, e^{iqx} \langle 0|\mathrm{T}\{\phi(x)\phi^\dagger(0)\}|0\rangle. \tag{4.1.6}$$

The proof of (4.1.3), which is valid separately for both spin components of the BS amplitude, goes through, with some minor modifications, as in Subsec. 2.2.3. Notice that the two-particles propagator G_0, as well as the kernel iK, are 4×4 matrices in Dirac space, while the BS amplitude χ is a Dirac spinor.

The spin components of the BS amplitude are not independent, since they must satisfy (see e.g. [51]):

$$\chi_s(k;P) = S^{-1}(\Lambda) \sum_{s'=\pm\frac{1}{2}} \chi_{s'}(\Lambda k; \Lambda P) \mathcal{D}^{(\frac{1}{2})}_{s's}(L_{\mathrm{LF}}(\Lambda, P)), \tag{4.1.7}$$

where Λ is any (proper, orthochronous) Lorentz transformation, $L_{\mathrm{LF}}(\Lambda, P)$ the LF Wigner rotation and $S(\Lambda)$ the Dirac representation of the homogeneous Lorentz group, which is introduced by the fermionic-field transformation law. Assuming parity invariance and exploiting the Dirac equation $\not{P} u(P,s) = M u(P,s)$, the BS amplitude can be simply expressed in terms of two Lorentz invariant amplitudes and independent Dirac structures (see Appendix C.1):

$$\chi_s(k;P) = \left[\phi_1(k^2, Pk) + \phi_2(k^2, Pk)\frac{\not{k}}{M}\right] u(P,s), \tag{4.1.8}$$

which automatically incorporates Eq. (4.1.7). The amplitudes $\phi_{1,2}$ are functions of the only Lorentz invariants which one can construct from the two four-vector k and P, i.e. k^2 and Pk ($P^2 = M^2$ is understood).

The parametrization (4.1.8) is valid under the assumption that the intrinsic parity of the bound state π_b equals the product $\pi_\psi \pi_\phi$, where:

$$\mathcal{P}^\dagger \psi(x) \mathcal{P} = \pi_\psi \gamma^0 \psi(x^0, -\mathbf{x}), \tag{4.1.9}$$

$$\mathcal{P}^\dagger \phi(x) \mathcal{P} = \pi_\phi \phi(x^0, -\mathbf{x}). \tag{4.1.10}$$

Indeed, from a non-relativistic point of view, one expects the two constituents to have relative orbital angular momentum $L = 0$ in the lowest lying bound state, which implies $\pi_b = \pi_\psi \pi_\phi$.

Assuming $\mathcal{PT}$ invariance, the conjugated amplitude, defined by:

$$\overline{\chi}_s(y;P) \equiv \int \frac{\mathrm{d}^4q}{(2\pi)^4} e^{iqy} \overline{\chi}_s(q;P) \equiv \langle b; P, s|\mathrm{T}\{\overline{\psi}(\eta_2 y)\phi^\dagger(-\eta_1 y)\}|0\rangle \tag{4.1.11}$$

satisfies (see Appendix C.1.2 for a derivation):

$$\overline{\chi}_s(y;P) = (-1)^{\frac{1}{2}-s} \gamma^1\gamma^3 \chi_{-s}(-y;P), \tag{4.1.12}$$

where we employ the following conventions[1]:

$$\mathcal{PT}|b;P,s\rangle = (-1)^{\frac{1}{2}-s}|b;P,-s\rangle, \tag{4.1.13}$$

$$\mathcal{PT}\psi(x_1)(\mathcal{PT})^\dagger = \gamma^1\gamma^3\gamma^0\psi(-x_1), \tag{4.1.14}$$

$$\mathcal{PT}\phi(x_2)(\mathcal{PT})^\dagger = \phi(-x_2). \tag{4.1.15}$$

[1]Notice that, since the operator $\mathcal{PT}$ is antiunitary, any constant phase factor in the RHSs of Eqs. (4.1.13)-(4.1.15) is unessential, and it can be put equal to one. For example, if $\mathcal{PT}\phi(x)(\mathcal{PT})^\dagger = e^{2i\theta}\phi(-x)$, we define the new field $\phi_1(x) = e^{i\theta}\phi(x)$, which satisfies $\mathcal{PT}\phi_1(x)(\mathcal{PT})^\dagger = \phi_1(-x)$ (if the field $\phi(x)$ is hermitian, its intrinsic phase can be absorbed as well in the definition of the Dirac field, without spoiling the validity of Eq. (4.1.12)).

In momentum space, Eq. (4.1.12) reads:

$$\overline{\chi}_s(q;P) = \overline{u}(P,s)\left[\phi_1(q^2,Pq) + \phi_2(q^2,Pq)\frac{\not{q}}{M}\right]. \tag{4.1.16}$$

4.1.2 NIR of the BS amplitude

As already mentioned in Subsec. 3.1.1, Nakanishi's analysis of Feynman diagrams can be generalized to theories involving higher-spin particles, since the denominator structure of a Feynman diagram is independent of the spin of the particles involved[2]. The additional numerator structure brings about a polynomial dependence in the Nakanishi weight function, which can be dealt with by decomposing the diagram in its independent tensorial structures allowed by the symmetries of the theory.

By exploiting the analysis of Subsec. 3.1.1 for the scalar case, one can assume, as a natural generalization of Nakanishi's result, a NIR for the invariant amplitudes $\phi_{1,2}$ in the decomposition (4.1.8), viz.:

$$\phi_i(k^2,Pk) = i\int_{-1}^{1}\mathrm{d}z'\int_{0}^{+\infty}\mathrm{d}\gamma'\frac{g_i(\gamma',z')}{\left[(k+z'\frac{P}{2})^2-(1-z'^2)\kappa^2-(z'\overline{m}-\Delta)^2-\gamma'+i\varepsilon\right]^3}. \tag{4.1.17}$$

As explained in detail in the following, plugging Eq. (4.1.17) into the BSE and performing the LF projection, we formally obtain a coupled system of integral equations for the Nakanishi weight functions g_i, which is the starting point of our work.

As seen in Subsec. 2.1.3, the actual calculation of physical quantities (in particular, the valence probabilities to be introduced in the next Subsection) from the BS amplitude requires a proper normalization. The normalization condition for χ takes the form (cf. Subsecs. 2.1.3 and 2.2.3):

$$\overline{\chi}_r N_\mu \chi_s = \delta_{rs} P_\mu, \tag{4.1.18}$$

where N_μ is the BS normalization kernel. In turn Eq. (4.1.18) is equivalent to a normalization condition for the Nakanishi vector $g = (g_1, g_2)$, which we write in the form:

$$\int_0^{+\infty}\mathrm{d}\gamma''\int_{-1}^{+1}\mathrm{d}z''\int_0^{+\infty}\mathrm{d}\gamma'\int_{-1}^{+1}\mathrm{d}z'\sum_{i,j=1,2} g_i(\gamma'',z'')\mathcal{F}_{ij}(\gamma'',z'',\gamma',z')g_j(\gamma',z') = 1, \tag{4.1.19}$$

or, more compactly:

$$(g|\mathcal{F}|g) = 1 \tag{4.1.20}$$

The kernel $\mathcal{F}$ is determined by the interaction kernel iK adopted for the BSE, and it is calculated explicitly in Appendix C.3 for a ladder interaction with a massive scalar exchange.

[2] One might, for example, employ identities like [52]:

$$\frac{p^\mu}{p^2-m^2+i\varepsilon} = \frac{1}{2}\int_{m^2}^{\infty}\mathrm{d}m'^2\left.\frac{\partial}{\partial t_\mu}\right|_{t=0}\frac{1}{(p+t)^2-m'^2+i\varepsilon},$$

which allow one to replace any propagator appearing in a Feynman diagram with a scalar one. The integration over the dummy masses m'^2 can be shown to be absorbed into the definition of the Nakanishi weight function, while the differentiations bring about the polynomial dependence mentioned in the text.

4.1.3 LF Fock decomposition of the bound state

The bound state $|b\rangle$ can be expanded in terms of the LF Fock states, which are in turn defined by the null hyperplane restriction of the elementary fields of the theory. We express the constituent fields at $x^+ = 0$ as[3]:

$$\phi(\tilde{x}) \equiv \int \frac{\mathrm{d}^3\tilde{p}\,\theta(p^+)}{\sqrt{(2\pi)^3 2p^+}} \left\{ a(\tilde{p}) e^{-i\tilde{p}\tilde{x}} + a^{c\dagger}(\tilde{p}) e^{i\tilde{p}\tilde{x}} \right\}, \tag{4.1.21}$$

$$\psi^{(+)}(\tilde{x}) \equiv \sum_{\lambda=\pm\frac{1}{2}} \int \frac{\mathrm{d}^3\tilde{p}\,\theta(p^+)}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{m_\psi}{p^+}} \left\{ b(p,\lambda) u^{(+)}(p,\lambda) e^{-i\tilde{p}\tilde{x}} + d^\dagger(p,\lambda) v^{(+)}(p,\lambda) e^{i\tilde{p}\tilde{x}} \right\}, \tag{4.1.22}$$

where the bosonic annihilation/creation operators satisfy the canonical commutation rules:

$$\left[a(\tilde{p}), a^\dagger(\tilde{q}) \right] = \left[a^c(\tilde{p}), a^{c\dagger}(\tilde{q}) \right] = \delta^3(\tilde{p} - \tilde{q}), \tag{4.1.23}$$

all others commutators being zero, while the fermionic operators satisfy canonical anticommutation rules:

$$\left\{ b(\tilde{p},\lambda), b^\dagger(\tilde{q},\mu) \right\} = \left\{ d(\tilde{p},\lambda), d^\dagger(\tilde{q},\mu) \right\} = \delta_{\lambda\mu}\delta^3(\tilde{p} - \tilde{q}), \tag{4.1.24}$$

all other anticommutators being zero. In Eq. (4.1.22), as explained below (cf. Subsec 4.1.4), the symbol $\psi^{(+)}$ indicates the "good" component $\psi^{(+)} = \Lambda^{(+)}\psi$ of the Dirac field, where the orthogonal projectors $\Lambda^{(\pm)}$ are defined by:

$$\Lambda^{(\pm)} \equiv \frac{\gamma^\mp\gamma^\pm}{4} = \frac{1 \pm \gamma^0\gamma^3}{2} \tag{4.1.25}$$

and satisfy:

$$\Lambda^{(\pm)\dagger} = \Lambda^{(\pm)}, \tag{4.1.26}$$

$$\Lambda^{(\pm)}\Lambda^{(\mp)} = 0, \tag{4.1.27}$$

$$\Lambda^{(+)} + \Lambda^{(-)} = 1. \tag{4.1.28}$$

The projected LF spinors $u^{(+)}$ and $v^{(+)}$ are given in Appendix B.2, and satisfy the following orthogonality condition:

$$u^{(+)\dagger}(p,\lambda)u^{(+)}(q,\mu) = v^{(+)\dagger}(p,\lambda)v^{(+)}(q,\mu) = \frac{\sqrt{p^+q^+}}{2m_\psi}\delta_{\lambda\mu}. \tag{4.1.29}$$

The Fock expansion of $|b\rangle$ starts from the two-particles (one fermion and one scalar) component:

$$|b;P,s\rangle = \sum_{\lambda=\pm\frac{1}{2}} \int_0^1 \frac{\mathrm{d}\xi}{(2\pi)2\xi(1-\xi)} \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{(2\pi)^2} |2/P;\xi\mathbf{k}_\perp\lambda\rangle \Psi_{2/s}(\xi,\mathbf{k}_\perp,\lambda) + \cdots, \tag{4.1.30}$$

where the two-particle state $|2/P\rangle$ is defined by:

$$|2/P;\xi\mathbf{k}_\perp\lambda\rangle \equiv \sqrt{(2\pi)^3\frac{p_1^+}{m_\psi}} b^\dagger(\tilde{p}_1,\lambda) \sqrt{(2\pi)^3 2p_2^+} a^\dagger(\tilde{p}_2)|0\rangle, \tag{4.1.31}$$

[3]For the sake of generality, we consider the scalar field ϕ as complex, so that $a^c \neq a$ in Eq. (4.1.21). Although this is totally irrelevant to the model considered in our numerical study (see next Section), it is nonetheless necessary to describe, e.g., a quark and a scalar (color anti-triplet) diquark interacting through the exchange of a gluon (a first investigation of the massless-vector-exchange ladder interaction will be presented in Ref. [48]).

with:

$$p_1^+ \equiv \xi P^+, \qquad \mathbf{p}_{1\perp} \equiv \xi \mathbf{P}_\perp + \mathbf{k}_\perp, \tag{4.1.32}$$

$$p_2^+ \equiv (1-\xi) P^+, \qquad \mathbf{p}_{2\perp} \equiv (1-\xi)\mathbf{P}_\perp - \mathbf{k}_\perp. \tag{4.1.33}$$

The valence wave-function Ψ_2 is independent of P, due to the invariance of the variables ξ, $\mathbf{k}_\perp$ and λ under LF boosts (see Appendix B.4, Eq. (B.4.3)), but of course depends on both the spins s and λ.

The four wave functions $\Psi_{2/s}(\cdot,\lambda)$ are clearly not independent. Let:

$$k^1 + ik^2 \equiv \sqrt{\gamma} e^{i\varphi} \qquad (0 \leq \varphi < 2\pi). \tag{4.1.34}$$

By applying the LF parity $\mathcal{P}_{\rm LF} \equiv e^{-i\pi J^1}\mathcal{P}$ and the LF time-reversal $\mathcal{T}_{\rm LF} = e^{-i\pi J^3}\mathcal{P}\mathcal{T}$ operators (which are both kinematical) to both sides of Eq. (4.1.30), we obtain the following constraints on the valence WFs (see Eq. (B.3.5))[4]:

$$\Psi_{2/-s}(\xi,\gamma,\pi-\varphi,-\lambda) = \Psi_{2/s}(\xi,\gamma,\varphi,\lambda) \qquad \text{(LF parity)}, \tag{4.1.35}$$

$$\Psi^*_{2/-s}(\xi,\gamma,\varphi+\pi,-\lambda) = \Psi_{2/s}(\xi,\gamma,\varphi,\lambda) \qquad \text{(LF time-reversal)}. \tag{4.1.36}$$

Notice that, in order to reproduce a state with spin s along the z-axis, the amplitude $\Psi_{2/s}(\cdot,\lambda)$ must describe a fermion-scalar pair with orbital angular momentum $l_z = s-\lambda$, i.e. its dependence from the azimuthal angle φ is given by $e^{i(s-\lambda)\varphi}$. Combining these information with Eqs. (4.1.35)-(4.1.36), we can write:

$$\Psi_{2/\pm\frac{1}{2}}(\xi,\gamma,\varphi,\pm\frac{1}{2}) \equiv \Psi_{\rm nf}(\xi,\gamma), \tag{4.1.37}$$

$$\Psi_{2/\pm\frac{1}{2}}(\xi,\gamma,\varphi,\mp\frac{1}{2}) \equiv \mp e^{\pm i\varphi}\Psi_{\rm f}(\xi,\gamma), \tag{4.1.38}$$

where the subscripts "nf" and "f" stand for "non-flip" and "flip" amplitudes. By Eqs. (4.1.35)-(4.1.36), these amplitudes are real:

$$\Psi^*_{\rm nf} = \Psi_{\rm nf}, \qquad \Psi^*_{\rm f} = \Psi_{\rm f}. \tag{4.1.39}$$

We can define non-flip and flip valence probabilities by:

$$P_{\rm nf/f} = \int_0^1 \frac{{\rm d}\xi}{4\pi\xi(1-\xi)} \int \frac{{\rm d}\gamma}{4\pi}\, \Psi^2_{\rm nf/f}(\xi,\gamma). \tag{4.1.40}$$

Hence, the total valence probability is $P_{\rm tot} = P_{\rm nf} + P_{\rm f}$.

The appearance of $|L_z| \neq 0$ amplitudes is entirely due to the non-trivial spin structure of the constituents. The total orbital momentum L^2 is *not* employed in the classification of the two-particles component of the bound state, since $\mathbf{L}_\perp = \mathbf{J}_\perp - \mathbf{S}_\perp$ is not defined as an operator on the two-particle sector of the Fock space ($\mathbf{J}_\perp$ is a dynamical operator). For the same reason, the conservation of the ordinary parity and time-reversal can not be checked component by component, as we did for the LF analogues of these operators.

As shown by Brodsky [53], the electromagnetic form factors of the bound state can be computed as overlaps of LFWFs. Using the formulas of Ref. [53] (see also Ref. [54] in this context), one can show that a nonzero spin-flip amplitude is necessary in order to have a valence contribution to the anomalous magnetic moment of the bound state.

[4]The conventions (4.1.14) and (4.1.15) correspond to an intrinsic $\mathcal{P}\mathcal{T}$-parity $\eta_{\mathcal{P}\mathcal{T}} = 1$ for both constituents.

4.1.4 LF projection of the BS amplitude

The valence wave-functions can be obtained from the BS amplitude by a LF projection technique. To this end, we define a projected BS amplitude:

$$\chi_s^{(+)}(x;P) \equiv \langle 0|\mathrm{T}\{\psi^{(+)}(\eta_2 x)\phi(-\eta_1 x)\}|b;P,s\rangle. \tag{4.1.41}$$

The reason for dealing with $\chi^{(+)}$, instead of χ directly, is the following: in the LF quantization formalism, the only independent degree of freedom associated to a Dirac field is the so-called "good" component $\psi^{(+)} = \Lambda^{(+)}\psi$, which has the plane-wave decomposition (4.1.22) at $x^+ = 0$. On the other hand, the bad component $\psi^{(-)} = \Lambda^{(-)}\psi$ is determined in terms of $\psi^{(+)}$, as well as of the other independent fields of the theory, and depends in particular on the interaction. In order to single out the valence component of $|b\rangle$ one has to use LF creation and annihilation operators; hence $\psi^{(+)}$, restricted to $x^+ = 0$, is the relevant fermionic field to be exploited. This is another instance of the halving of degrees of freedom in the LF formalism, which we have already encountered in Subsec. 3.2.2 (see Ref. [42] for a more detailed discussion).

We now make the assumption (cf. the discussion in Subsec. 3.2.3) that the T-product in Eq. (4.1.41) can be replaced by a T^+-product. Using Eqs. (4.1.21), (4.1.22) and (4.1.30), and performing analogous steps as in Subsec. 3.2.3 for the scalar case (see Ref. [22] for a full derivation), one obtains:

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}q^-}{2\pi}\chi_s^{(+)}(q;P) = \frac{\sqrt{2m_\psi}}{P^+\xi(1-\xi)} \sum_{\lambda=\pm\frac{1}{2}} \Psi_{2/s}(\xi,\mathbf{k}_\perp,\lambda)u^{(+)}(p_1,\lambda),$$

or:

$$\Psi_{2/s}(\xi,\mathbf{k}_\perp,\lambda) = \sqrt{2m_\psi}(1-\xi)\int_{-\infty}^{+\infty} \frac{\mathrm{d}q^-}{2\pi} u^{(+)}(p_1,\lambda)^\dagger \chi_s^{(+)}(q;P), \tag{4.1.42}$$

where:

$$\xi = \frac{p_1^+}{P^+} = \frac{1}{2} + \frac{q^+}{P^+}, \tag{4.1.43}$$

$$\mathbf{k}_\perp = \mathbf{p}_{1\perp} - \xi\mathbf{P}_\perp = \mathbf{q}_\perp - \frac{q^+}{P^+}\mathbf{P}_\perp. \tag{4.1.44}$$

Finally, introducing the representation (4.1.8) of the BS amplitude we obtain, after some algebra:

$$\Psi_{2/+\frac{1}{2}}(\xi,\gamma,\varphi,+\frac{1}{2}) = \Phi_1(\xi,\gamma) + (\xi-\frac{1}{2})\Phi_2(\xi,\gamma) \tag{4.1.45}$$

$$\Psi_{2/+\frac{1}{2}}(\xi,\gamma,\varphi,-\frac{1}{2}) = -\sqrt{\frac{\gamma}{M^2}}e^{i\varphi}\Phi_2(\xi,\gamma) \tag{4.1.46}$$

$$\Psi_{2/-\frac{1}{2}}(\xi,\gamma,\varphi,+\frac{1}{2}) = \sqrt{\frac{\gamma}{M^2}}e^{-i\varphi}\Phi_2(\xi,\gamma) \tag{4.1.47}$$

$$\Psi_{2/-\frac{1}{2}}(\xi,\gamma,\varphi,-\frac{1}{2}) = \Phi_1(\xi,\gamma) + (\xi-\frac{1}{2})\Phi_2(\xi,\gamma) \tag{4.1.48}$$

where:

$$\Phi_i(\xi,\gamma) = \frac{(1-\xi)\xi^{\frac{1}{2}}}{\sqrt{2M}}P^+\int\frac{\mathrm{d}q^-}{2\pi}\phi_i(q^2,Pq). \tag{4.1.49}$$

Comparing with Eqs. (4.1.37)-(4.1.38), we identify the non-flip and flip amplitudes as:

$$\Psi_{\mathrm{nf}}(\xi,\gamma) = \Phi_1(\xi,\gamma) + (\xi-\frac{1}{2})\Phi_2(\xi,\gamma), \tag{4.1.50}$$

$$\Psi_{\mathrm{f}}(\xi,\gamma) = \sqrt{\frac{\gamma}{M^2}}\Phi_2(\xi,\gamma). \tag{4.1.51}$$

If one assumes the NIR (4.1.17) then the amplitudes Φ_i are explicitly given by:

$$\Phi_i(\xi,\gamma) = \frac{\xi^{\frac{1}{2}}(1-\xi)}{\sqrt{2M}} \int_0^{+\infty} \mathrm{d}\gamma' \frac{g_i(\gamma',z)}{\left[\gamma' + D_0(\gamma,z)\right]^2}, \tag{4.1.52}$$

with $z = 1-2\xi$ and:

$$D_0(\gamma,z) = \gamma + (1-z^2)\kappa^2 + (z\overline{m} - \Delta)^2. \tag{4.1.53}$$

Notice that the flip (i.e. $|L_z| = 1$) amplitude (4.1.51) vanishes in the limit $\gamma \to 0$, which is correct, since a system with orbital angular momentum $L_z \neq 0$ can not have transverse momentum $\mathbf{k}_\perp = \mathbf{0}$. In the non-relativistic limit, one expects the LFWFs to be sharply peaked about $\xi \approx \frac{m_\psi}{m_\phi + m_\psi}$ and $\frac{\gamma}{M^2} \ll 1$, so that the $|L_z| = 1$ components disappear and one has:

$$\Psi_{\mathrm{nf}} \approx \Phi_1 + \frac{1}{2}\frac{m_\psi - m_\phi}{m_\psi + m_\phi}\Phi_2, \qquad \Psi_{\mathrm{f}} \approx 0.$$

4.2 Method of solution for the ladder BSE

We present in this Section our method of solution for the fermion-scalar ladder BSE. All detailed calculations can be found in Appendix C. Numerical results will be presented in the next Chapter.

4.2.1 Ladder BSE

We have solved the following ladder BSE:

$$\chi_s(k;P) = \frac{i}{\eta_1 \not{P} + \not{k} - m_\psi + i\varepsilon} \frac{i}{(\eta_2 P - k)^2 - m_\phi^2 + i\varepsilon} \int \frac{\mathrm{d}^4k'}{(2\pi)^4} iK_{\mathrm{lad}}(k,k';P)\chi_s(k';P), \tag{4.2.1}$$

where self-energy corrections in the propagators are neglected. The kernel is given by the one-scalar exchange diagram of Fig. 4.2.1, and has the expression:

$$iK_{\mathrm{lad}}(k,k';P) = i\lambda_\psi \frac{i}{(k-k')^2 - \mu^2 + i\varepsilon} i\lambda_\phi. \tag{4.2.2}$$

Notice that a point-like interaction vertex between the constituents and the exchanged boson is assumed. A parity preserving interaction leading to kernel (4.2.2) under our approximations is:

$$\mathscr{L} = \varphi\left\{\lambda_\psi \overline{\psi}\psi + \lambda_\phi \phi^\dagger \phi\right\}, \tag{4.2.3}$$

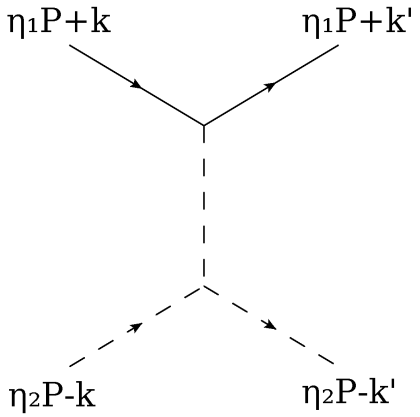

Figure 4.2.1. One-scalar exchange diagram (cf. Eq. (4.2.2)).

where φ is the field of the exchanged scalar, and the coupling constants λ_ψ and λ_ϕ have mass-dimension zero and one, respectively.

After introducing the parametrization (4.1.8) of the BS amplitude, the ladder BSE (4.2.1) turns into a coupled linear system for the invariant amplitudes $\phi_{1,2}$:

$$\phi_i(k^2,Pk)=\frac{i\lambda_\psi\lambda_\phi}{\Big[(\eta_1 P+k)^2-m_\psi^2+i\varepsilon\Big]\Big[(\eta_2 P-k)^2-m_\phi^2+i\varepsilon\Big]}\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\frac{\sum_{j=1,2}C_{ij}(k,k',P)\phi_j(k'^2,Pk')}{(k-k')^2-\mu^2+i\varepsilon},\tag{4.2.4}$$

where the coefficients C_{ij} are given in Appendix C.1.

There could be the objection that the Lagrangian (4.2.3) does not lead to a field theory with a well defined ground state, as shown by Baym for a ϕ^3-theory [55]. However, this does not need to concern us, since the boson-boson cubic interaction is being used only as a model for the irreducible kernel iK. In other words, we could always think of adding quartic terms such as $\mathscr{L}'=-g_{\phi\varphi}\phi^2\varphi^2-g_\phi\phi^4-g_\varphi\varphi^4$ to the Lagrangian, which would lead to a consistent renormalizable field theory, with the same symmetry $\phi\to-\phi$ of $\mathscr{L}$, but would not affect the form of the ladder BSE (with free propagators) considered here.

4.2.2 Solution of the Minkowski space BSE

To solve the coupled system (4.2.4), we assume that the invariant amplitudes are expressed through the following NIR:

$$\phi_i(k^2,Pk)=i\int_{-1}^{1}\mathrm{d}z'\int_0^{+\infty}\mathrm{d}\gamma'\widetilde{B}(k^2,Pk,\gamma',z')g_i(\gamma',z').\tag{4.2.5}$$

where:

$$\widetilde{B}(k^2,Pk,\gamma',z')=\frac{1}{\Big[(k+z'\frac{P}{2})^2-(1-z'^2)\kappa^2-(z'\overline{m}-\Delta)^2-\gamma'+i\varepsilon\Big]^3}\tag{4.2.6}$$

Once Eq. (4.2.5) is plugged into the coupled system (4.2.4), the k' dependence in the RHS integrand is made explicit and the integral can be carried out by means of the standard integrals (see Appendix C.2 for details):

$$\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2-s+i\varepsilon)^n}=i\frac{(-1)^n}{(4\pi)^2}\frac{1}{(n-1)(n-2)}\frac{1}{(s-i\varepsilon)^{n-2}}\qquad(n\geq3),\tag{4.2.7}$$

$$\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{q^\mu q^\nu}{(q^2-s+i\varepsilon)^n}=\frac{i}{2}\frac{(-1)^{n-1}}{(4\pi)^2}\frac{1}{(n-1)(n-2)(n-3)}\frac{g^{\mu\nu}}{(s-i\varepsilon)^{n-3}}\qquad(n\geq4),\tag{4.2.8}$$

The result is a coupled system of singular equations for the weight functions $g_{1,2}$, with a residual parametrical dependence on the invariants k^2 and Pk, which takes the form:

$$\int_{-1}^{1}\mathrm{d}z'\int_0^{+\infty}\mathrm{d}\gamma'\widetilde{B}(k^2,Pk,\gamma',z')g_i(\gamma',z')=\int_{-1}^{1}\mathrm{d}z'\int_0^{+\infty}\mathrm{d}\gamma'\sum_{j=1,2}\widetilde{V}_{ij}(k^2,Pk,\gamma',z')g_j(\gamma',z'),\tag{4.2.9}$$

where:

$$\widetilde{V}_{ij}(k^2,Pk,\gamma',z')=\frac{\lambda_\psi\lambda_\phi}{8\pi^2M^2}\int_0^1v^2\mathrm{d}v\frac{c_{ij}^{(0)}+c_{ij}^{(1)}k^-}{\Big[(1-z)(k^--k_d^-)+i\varepsilon\Big]\Big[(1+z)(k^--k_u^-)-i\varepsilon\Big]\Big[k^-k_D^++l_D+i\varepsilon\Big]^2}\tag{4.2.10}$$

(the quantities $k^-_{u,d}$, k^+_D, l_D and the coefficients $c^{(n)}_{ij}$ are defined in Appendix C.2, and the latters are independent of k^-).

To obtain a regular equation for the weight functions g_i, we perform a projection onto the null-hyperplane, i.e. we integrate the whole equation over k^-. The mathematical intuition behind such a procedure stems from the relation between the Nakanishi weight functions and the LFWFs (cf. Eqs. (4.1.50)-(4.1.52)), which is, we recall, essentially given by a generalized Stieltjes transform [46]. Indeed, the LF variables $\gamma = \mathbf{k}^2_\perp$ and $z = 1 - 2\xi$ share the same domain of the arguments γ' and z' of g_i, so that, after the LF projection, the kernels acting on g_i on the LHS and RHS of the equation can be formally considered as *operators* acting on a space of functions. Under mild regularity assumptions on g_i, the equation can thus be discretized and solved by expanding g_i on a suitable orthonormal basis.

As a technical remark, let us note that the LF projection of the kernel (4.2.10) could in principle yield a singular result for $k^+_D = 0$, in which case the k^- integral can not be evaluated by contour integration (if $c^{(1)}_{ij} \neq 0$ the integral has a potential logarithmic divergence). Indeed, the required result is a generalization of the so-called LF singular integral, which we have already employed in the NIR of the LFWF (cf. Subsec. 3.2.4), namely (see Appendix C.5.6 for a proof):

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}\beta}{2\pi} \frac{1}{[\beta x + z]^2} = -\frac{i}{z}\delta(x) \qquad (\mathrm{Im} z > 0). \tag{4.2.11}$$

The correct treatment of such singularities was first indicated in Ref. [20], which dealt with the ladder BSE for two fermions. In our case, where only the first power of k^- appears in the denominator of Eq. (4.2.10), there is no singularity for $k^+_D = 0$ and the result obtained by applying the residue theorem can be carried on in the limit $k^+_D \to 0$, as shown in detail in Appendix C.5.4.

After performing the LF projection, we obtain a system of regular integral equations for $g_{1,2}$, which takes the form:

$$\int_0^{+\infty} \mathrm{d}\gamma' B(\gamma, z; \gamma') g_i(\gamma', z) = \alpha \int_0^{\infty} \mathrm{d}\gamma' \int_{-1}^{+1} \mathrm{d}z' \sum_{i'=1,2} A_{ii'}(\gamma, z; \gamma', z') g_{i'}(\gamma', z'), \tag{4.2.12}$$

or, symbolically:

$$B g_i = \alpha \cdot \sum_{i'=1,2} A_{ii'} g_{i'} \tag{4.2.13}$$

where the integral operators B and $A_{ii'}$ are explicitly given in Appendix C.2 ($B(\gamma, z; \gamma')$ is a symmetric kernel for a generalized Stieltjes transform). The coupling constant α is defined by:

$$\alpha \equiv \frac{\lambda_\phi \lambda_\psi}{8\pi m_\phi}, \tag{4.2.14}$$

in order to ensure the correct non-relativistic limit, i.e. the effective Yukawa potential:

$$V(r) = -\frac{\alpha e^{-\mu r}}{r}$$

(see Appendix D.1). We recall from Subsec. 3.2.4 that B is an invertible operator, although its analytic inversion is not carried out in this Thesis, but left for future work.

To solve Eq. (4.2.13), as for the two-scalar or two-fermion case, we expand $g_{1,2}$ over an orthonormal basis:

$$g_i(\gamma, z) = \sum_{j=0}^{N_\gamma} \sum_{\ell=0}^{N_z} c^{j\ell}_i \mathcal{L}_j(\gamma) \mathcal{G}^{(i)}_\ell(z), \tag{4.2.15}$$

where $\mathcal{L}_j$ and $\mathcal{G}_\ell^{(i)}$ are given in terms of Laguerre L_j and Gegenbauer $C_\ell^{(\beta_i)}$ polynomials respectively:

$$\mathcal{L}_j(\gamma) = \sqrt{a} L_j(a\gamma) e^{-\frac{a\gamma}{2}}, \tag{4.2.16}$$

$$\mathcal{G}_\ell^{(i)}(z) = \sqrt{\frac{2^{2\beta_i-1}\Gamma(\beta_i)^2(\ell+\beta_i)\ell!}{\pi\Gamma(\ell+2\beta_i)}} C_\ell^{(\beta_i)}(z)(1-z^2)^{\frac{\beta_i}{2}-\frac{1}{4}}. \tag{4.2.17}$$

This choice of the basis is motivated by the same remarks of Sec. 3.3. The parameter a governs the extension of g in the γ-direction, while β governs the behavior of $F_\ell^{(\beta)}(z) \equiv C_\ell^{(\beta)}(z)(1-z^2)^{\frac{\beta}{2}-\frac{1}{4}}$ for $z^2 \to 1$, as well as the oscillatory behavior in the $-1 \leq z \leq 1$ range. Notice that, in our case, there is no exchange symmetry between the constituent particles, so that both even and odd degree Gegenbauer polynomials appear in the expansion (4.2.15).

Plugging the expansion (4.2.15) into (4.2.13) and projecting over a given basis element $\mathcal{L}_j\mathcal{G}_\ell^{(i)}$, we obtain the linear system:

$$\sum_{j'=0}^{N_\gamma}\sum_{\ell'=0}^{N_z} B_{i,jj',\ell\ell'} c_i^{j'\ell'} = \alpha \cdot \sum_{j'=0}^{N_\gamma}\sum_{\ell'=0}^{N_z}\sum_{i'=1,2} A_{ii',jj',\ell\ell'} c_{i'}^{j'\ell'}, \tag{4.2.18}$$

where:

$$B_{i,jj',\ell\ell'} \equiv \int_0^\infty \mathrm{d}\gamma \int_{-1}^{1} \mathrm{d}z \int_0^\infty \mathrm{d}\gamma'\, \mathcal{L}_j(\gamma)\mathcal{G}_\ell^{(i)}(z) B(\gamma, z; \gamma') \mathcal{L}_{j'}(\gamma')\mathcal{G}_{\ell'}^{(i)}(z), \tag{4.2.19}$$

$$A_{ii',jj',\ell\ell'} \equiv \int_0^\infty \mathrm{d}\gamma \int_{-1}^{1} \mathrm{d}z \int_0^\infty \mathrm{d}\gamma' \int_{-1}^{1} \mathrm{d}z'\, \mathcal{L}_j(\gamma)\mathcal{G}_\ell^{(i)}(z) A_{ii'}(\gamma, z; \gamma', z') \mathcal{L}_{j'}(\gamma')\mathcal{G}_{\ell'}^{(i')}(z'). \tag{4.2.20}$$

By introducing a collective index:

$$I = (i-1)(N_\gamma+1)(N_z+1) + j(N_z+1) + \ell + 1,$$

Eq. (4.2.18) becomes a generalized eigenvalue problem, viz.:

$$\sum_{I'=1}^{N} A_{II'} c_{I'} = \lambda \sum_{I'=1}^{N} B_{II'} c_{I'}, \tag{4.2.21}$$

where the dimension is $N = 2(N_\gamma+1)(N_z+1)$, and $\lambda = \frac{1}{\alpha}$ (this choice is suggested only by numerical purposes). The form (4.2.21) can finally be solved by standard methods. As explained in Sec. 3.3, for a given binding energy M, the ground state solution of the BSE is given by the greatest real eigenvalue $\lambda(M) = \frac{1}{\alpha(M)}$. In our work, Eq. (4.2.21) has been solved with the LAPACK routine DGGEV.

Let us conclude this Subsection by noting that, starting from (4.2.9), one may deduce a second integral equation for the weight function, once another relevant result by Nakanishi is introduced: the theorem stating the uniqueness of the weight function itself [29]. Such a theorem has been used in [13], [18] and [22] to investigate the alternative integral equation for the weight function, but for a two-scalar BSE. In our case, the direct application of the uniqueness theorem is complicated by the additional structure brought by the spin of the fermion constituent (the numerator of Eq. (4.2.10)) and it will be analysed elsewhere.

4.3 Wick-rotated ladder BSE

The mass/coupling constant relation obtained from the solution of (4.2.4) was cross-checked through the one obtained from the Wick-rotated BSE. The latter is obtained from Eq. (4.2.4) by the formal

substitutions $k^0 = ik^4$, $k'^0 = ik'^4$ in the rest frame $\mathbf{P} = \mathbf{0}$:

$$\hat{\phi}_i(\mathbf{k}^2, k^4) = \frac{\lambda_\psi \lambda_\phi}{\left[m_\psi^2 + \mathbf{k}^2 + (k^4 - i\eta_1 M)^2\right]\left[m_\phi^2 + \mathbf{k}^2 + (k^4 + i\eta_2 M)^2\right]} \times \\ \times \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{\sum_{j=1,2} \hat{C}_{ij}(\mathbf{k}, k^4, \mathbf{k}', k'^4)\hat{\phi}_j(\mathbf{k}'^2, k'^4)}{(k^4 - k'^4)^2 + (\mathbf{k} - \mathbf{k}')^2 + \mu^2}, \tag{4.3.1}$$

where:

$$\hat{\phi}_i(\mathbf{k}^2, k^4) \equiv \phi_i(-(k^4)^2 - \mathbf{k}^2, iMk^4), \tag{4.3.2}$$

$$\hat{C}_{ij}(\mathbf{k}, k^4, \mathbf{k}', k'^4) \equiv C_{ij}(\mathbf{k}, ik^4, \mathbf{k}', ik'^4, P)\Big|_{P=(M,\mathbf{0})}. \tag{4.3.3}$$

The Wick-rotated coefficients $\hat{C}_{ij}$ are tabulated in Appendix C.4.

The solution of the Wick-rotated BSE is much more direct. The four-dimensional equation (4.3.1) is reduced to a two-dimensional one by explicitly performing the angular integrations in the RHS, noticing that the amplitudes $\hat{\phi}_i$ are independent from the orientation of the three-vector $\mathbf{k}$. This yields the system:

$$\hat{\phi}_i(x, y) = \alpha \int_{-\infty}^{+\infty} \mathrm{d}x' \int_0^{+\infty} \mathrm{d}y' \sum_{j=1,2} \hat{V}_{ij}(x, y; x, y')\hat{\phi}_j(x', y'), \tag{4.3.4}$$

where $x = k^4$, $y = |\mathbf{k}|$ and the kernel $\hat{V}_{ij}$ is given in Appendix C.4. It should be pointed out that the amplitudes $\hat{\phi}_i$ are complex, so that the coupled system (4.3.4) corresponds effectively to four real coupled integral equations.

To solve the system (4.3.4), we first express the two unbounded variables x and y in terms of two bounded ones $\overline{x}, \overline{y}$:

$$x = \frac{\overline{x}}{1 - |\overline{x}|}, \qquad (-1 < \overline{x} < 1), \tag{4.3.5}$$

$$y = \frac{\overline{y}}{1 - \overline{y}}, \qquad (0 \le \overline{y} < 1), \tag{4.3.6}$$

and define the new amplitudes:

$$f_i(\overline{x}, \overline{y}) \equiv y^2 \hat{\phi}_i(x, y) \tag{4.3.7}$$

which satisfy the integral equation:

$$f_i(\overline{x}, \overline{y}) = \alpha \int_{-1}^{1} \mathrm{d}\overline{x}' \int_0^{1} \mathrm{d}\overline{y}' \sum_{j=1,2} W_{ij}(\overline{x}, \overline{y}; \overline{x}', \overline{y}')f_j(\overline{x}', \overline{y}'), \tag{4.3.8}$$

$$W_{ij}(\overline{x}, \overline{y}; \overline{x}', \overline{y}') \equiv \frac{1}{(1 - |\overline{x}'|)^2(1 - \overline{y}')^2} \frac{y^2}{y'^2} \hat{V}_{ij}(x, y; x', y'). \tag{4.3.9}$$

The system (4.3.8) is finally discretized and solved by introducing a local cubic spline expansion for the amplitude f_i. We will not give details of the numerical solution, since the interested reader can find more information in Ref. [56, Appendix A].

Chapter 5

Numerical results for the fermion-scalar ladder BSE

In this Chapter, our numerical results are illustrated in details. On one side, what follows shows *per se* the reliability of the NIR-LF approach to the Minkowski space BSE, extending the analyses performed for two-scalar and two-fermion systems. On the other side, the adopted ladder approximation, which is essential in determining the dynamics at small distances, though quite simple, puts in evidence unexpected features when a scalar-exchange is acting the spin degrees of freedom are turned on.

Let us remind that within the NIR-LF approach we obtained a coupled system of integral equations for the Nakanishi weight functions, formally equivalent to the initial BSE for the fermion-boson system. Furthermore, the coupled system can be transformed into a generalized eigenvalue problem, once the BS kernel in ladder approximation is adopted (the cross-ladder contribution will be analysed elsewhere). This allows one to easily get eigenvalues (the coupling constants) and eigenvectors (the Nakanishi weight functions) for the coupled system, making NIR a viable tool for actually solving the ladder BSE.

In Sec. 5.1 some generalities and notations are given. In Sec. 5.2, the coupling constants and the Nakanishi weight functions are presented and what we call saturation of the binding energies is introduced. In Sec. 5.3, our quantitative results for LF wave functions and corresponding probabilities are discussed.

5.1 Generalities

For the sake of clarity, we have collected in this Section some general definitions to be used for presenting our numerical investigation of the ladder BSE. In particular, the following notations are adopted:

$$m_\psi \equiv \text{fermion mass}, \tag{5.1.1}$$

$$m_\phi \equiv \text{scalar mass}, \tag{5.1.2}$$

$$M \equiv \text{bound state mass}, \tag{5.1.3}$$

$$\mu \equiv \text{exchanged mass}, \tag{5.1.4}$$

$$\overline{m} \equiv \frac{m_\phi + m_\psi}{2}, \tag{5.1.5}$$

$$r \equiv \frac{m_\psi}{m_\phi}. \tag{5.1.6}$$

The binding energy is defined by:

$$B \equiv m_\phi + m_\psi - M = 2\overline{m} - M, \tag{5.1.7}$$

and ranges in the interval $0 \leq B \leq 2\overline{m}$, since both B and M are positive definite. Of course the positivity of B, namely $M < m_\phi + m_\psi$, is required for the stability of the bound state, since we are considering physical (i.e. not confined) constituents.

We denote by a hat quantities scaled by $\overline{m}$, e.g.:

$$\hat{B} \equiv B/\overline{m} \equiv 2 - \hat{M}.$$

Apart from a global mass-scale $\overline{m}$, the ground state solution is entirely characterized by the adimensional quantities $\hat{B} \in [0, 2]$, $r \in [0, \infty]$ and $\hat{\mu} \in [0, \infty]$, and we denote by $\alpha(\hat{B}, r, \hat{\mu})$ the corresponding value of the coupling constant.

We shall present our results for the coupling constants in terms of the quantity:

$$\overline{\alpha}(\hat{B}, r, \hat{\mu}) \equiv \lambda(r)\alpha(\hat{B}, r, \lambda(r)\hat{\mu}), \tag{5.1.8}$$

where:

$$\lambda(r) \equiv \frac{2\sqrt{r}}{1+r}. \tag{5.1.9}$$

This choice is motivated by the fact that, for the solution of the non-relativistic Schrödinger equation with a Yukawa potential, $\overline{\alpha}(\hat{B}, r, \hat{\mu})$ does not depend on r at all, viz.:

$$\overline{\alpha}(\hat{B}, r, \hat{\mu}) = \alpha(\hat{B}, 1, \hat{\mu}) \qquad \text{[Schrödinger equation]} \tag{5.1.10}$$

as a result of the transformation property of the Yukawa Hamiltonian under the canonical transformation (see Appendix D.2 for a derivation):

$$\mathbf{r} \to \theta\mathbf{r}, \qquad \mathbf{p} \to \frac{1}{\theta}\mathbf{p}. \tag{5.1.11}$$

The non-flip and flip valence LFWFs (see Subsec. 4.1.3) are denoted by $\Psi_{\rm nf/f}$ respectively, and the corresponding valence probabilities by $P_{\rm nf/f}$. The LFWFs, defined by Eqs. (4.1.50), (4.1.51) and (4.1.52), are obtained from the properly normalized Nakanishi weight functions (see Subsec. 4.1.2).

The numerical solution of the generalized eigenvalue problem in Eq. (4.2.12) was obtained with (cf. Eqs. (4.2.16)-(4.2.17)) an $N_\gamma \times N_z = 32 \times 32$ polynomial basis, with parameters:

$$a = 6.00/\overline{m}^2, \qquad \beta_1 = 1, \qquad \beta_2 = 3.$$

To solve the discretized eigenvalue problem, we have employed the LAPACK routine DGGEV.

The results for the coupling constants have been cross-checked, for several values of $\hat{B}$, r and $\hat{\mu}$, with those obtained from the solution of the Wick-rotated BSE, described in Sec. 4.3. The outcomes agree with high accuracy.

It should be pointed out that the accuracy achieved in the present calculations requires a quite modest amount of computing resources. In perspective (e.g., the extension to cross-ladder exchanges), this low-demanding need could be a valuable feature.

5.2 Coupling constants and Nakanishi weight functions

A first sample of our numerical results for the coupling constant is shown in Table 5.2.1, where the coupling constants are presented for equal-mass constituents, i.e. $r = 1$, three values of the masses for the exchanged quanta and binding energies up to $\hat{B} = 1$. This cut, together with Fig. 5.2.1(a) where

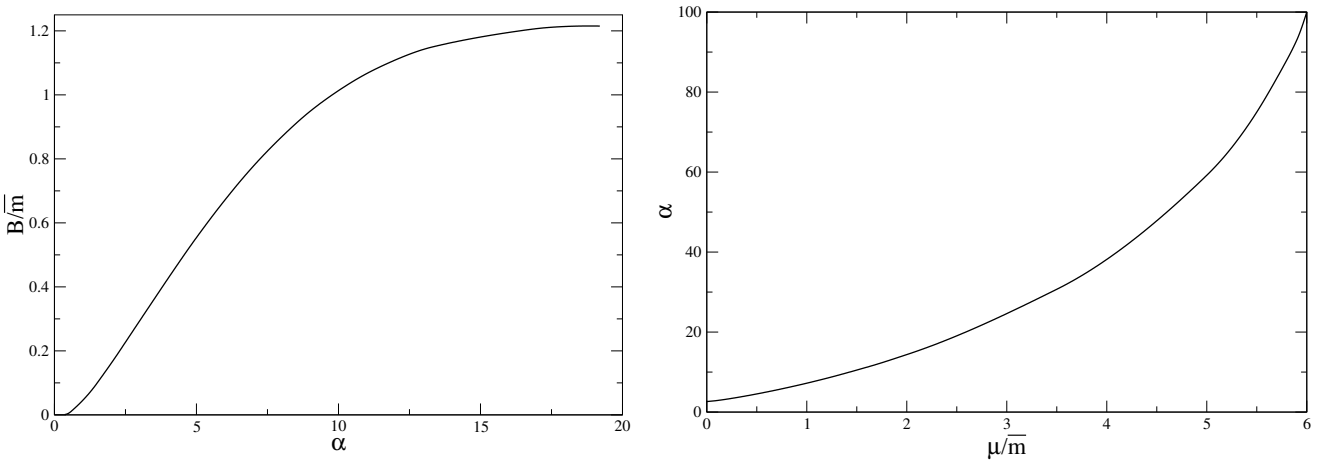

Figure 5.2.1. Coupling constants for equal-mass constituents, i.e. $r = 1$. (a) $\hat{B}$ vs. α at fixed $\hat{\mu} = 0.15$. (b) α vs. $\hat{\mu}$ at fixed $\hat{B} = 0.3$.

it is shown the binding energy as a function of the coupling constant α, and fixed value $\hat{\mu} = 0.15$, introduces the interesting feature of the saturation of the binding in the case of a scalar interaction. Indeed, the graph in Fig. 5.2.1(a) has an horizontal asymptote $\hat{B}_{\rm crit} \approx 1.2$, and for $\hat{B} \geq \hat{B}_{\rm crit}$ no bound state solution is found.

$\hat{B}$	$\hat{\mu} = 0$	$\hat{\mu} = 0.15$	$\hat{\mu} = 0.5$
0.001	0.072	0.3645	1.1705
0.01	0.2655	0.5760	1.4628
0.1	1.1679	1.5057	2.656
0.2	1.9408	2.297	3.6244
0.5	4.133	4.568	6.404
1	8.97	9.777	13.74

Table 5.2.1. Coupling constants for $r = 1$, $\hat{\mu} = 0$, 0.15, 0.5 and $\hat{B} \in [0, 1]$.

From the numerical point of view, the critical value manifests itself in a rather simple way: for $\hat{B} < \hat{B}_{\rm crit}$ we observe full convergence of the solution with respect to the $N_\gamma \times N_z$ basis employed in the expansion of $g_{1,2}$ (cf. Subsec. 4.2.2); for $\hat{B} > \hat{B}_{\rm crit}$ there is no longer convergence. We stress that the same value for $\hat{B}_{\rm crit}$ is found in the Wick-rotated solution of the BSE. This important test confirms that the observed behavior is not due to the NIR, nor to the numerical method employed in the solution of the Minkowski space BSE.

The critical value $\hat{B}_{\rm crit}$ is found to be decreasing with increasing the exchanged mass $\hat{\mu}$. In Table 5.2.2, some approximate values of $\hat{B}_{\rm crit}$ for $r = 1$ and different values of $\hat{\mu}$ are given.

In Fig. 5.2.1(b) the $\hat{\mu}$-dependence of α is shown for a large interval of exchanged masses, for $\hat{B} = 0.3$ and $r = 1$. Notice the large values of α in the high-$\hat{\mu}$ region. Indeed, if we fix $\hat{B}$ as in Fig. 5.2.1(b), since $\hat{B}_{\rm crit} = \hat{B}_{\rm crit}(\hat{\mu}; r = 1)$ decreases with $\hat{\mu}$, it will eventually approach $\hat{B}$ from above. Since the coupling constant tends to infinity for $\hat{B} \to \hat{B}_{\rm crit}$, we see that the curve is developing a vertical asymptote for $\hat{\mu} \gtrsim 6$.

The existence of a critical binding energy is already suggested by the structure of the coupled system for the Nakanishi weight functions, Eq. (4.2.12), whose RHS kernel is explicitly given in

$\hat{\mu}$	0	0.15	0.5	1.5	5
$\hat{B}_{\rm crit}$	1.24	1.21	1.14	1.00	0.68

Table 5.2.2. Critical binding energies for $r = 1$ and different values of $\hat{\mu}$.

Appendix C.2, and has the form:

$$A_{ij}(\gamma, z; \gamma', z') \equiv \frac{m_\phi}{4\pi D_0} \int_0^1 v^2 \mathrm{d}v \left(\theta(z' - z) \frac{N_{ij}^{(u)}}{D_u^2} + \theta(z - z') \frac{N_{ij}^{(d)}}{D_d^2} \right). \tag{5.2.1}$$

The denominators D_0 and $D_{u,d}$ are regular in the limit $M \to 0$ (i.e. $\hat{B} \to 2$), but the two numerators $N_{12}^{(u,d)}$ diverge:

$$N_{12}^{(u)} \sim -\frac{1-v}{1+z} \left[(1-z) m_\phi^2 + 2\gamma \right] \frac{1}{M}, \tag{5.2.2}$$

$$N_{12}^{(d)} \sim (1-v) m_\psi^2 \frac{1}{M}. \tag{5.2.3}$$

It is worth to observe that the kernel A for a two-scalar system (cf. Sec. 3.3) has the same structure of (5.2.1), with numerators $N^{(u)} = N^{(d)} = 1$. In our case, the negative divergent term in $N_{12}^{(u)}$ appears to be causing the saturation. Indeed, dropping this term from the kernel, we obtain a convergent and increasing value for the coupling constant in the whole interval $0 \leq \hat{B} \leq 2$, as in the case of the two-scalar system.

Figure 5.2.2 better illustrates the overall behavior of the saturation, when we change the mass ratio r. It is clear[1] that the value of $\hat{B}_{\rm crit}$ increases rapidly with r, as also suggested by the divergent structure of $N_{12}^{(u,d)}$ (cf. Eqs. (5.2.2)-(5.2.3) and notice that the negative contribution $N_{12}^{(u)}$ becomes less and less important). Indeed, for $r \gtrsim 5$ the convergence is obtained up to $\hat{B} \simeq 2$ (see Fig. 5.2.2, right panel), i.e. there is no critical binding energy. Interestingly, all $\hat{B}(\alpha)$ curves have a flex inside the interval $0 \leq \hat{B} \leq 2$ (for instance, for the $r = \frac{1}{3}, 1, 3$ lines in Fig. 5.2.2, the flex-points are approximately given by $\hat{B}_{\rm flex} = 0.15, 0.3, 0.5$, respectively).

We would like to put some emphasis on the appearance of flexes in the mass/coupling constant functional dependence. This peculiarity is not observed in the corresponding scalar-scalar system, for which the binding energy plot is convex (see Fig. 3.3.1). Indeed, as illustrated below, in the region $\hat{B} \leq \hat{B}_{\rm flex}$ our results are in very good agreement (when appropriately interpreted) with those found in Ref. [18] for the scalar case. However, it should be reminded that a sort of saturation effect for scalar exchange was found for a system composed by two equal-mass fermions, suggesting the basic role played by the spin degrees of freedom in determining the above mentioned effect. In Ref. [46], a plot analogous to Fig. 5.2.1 shows a flex, while this feature disappears when a pseudoscalar exchange or a vector one are adopted.

It would be, of course, highly desirable to understand the *physical* motivation of the saturation. Since the involved binding energies are of the order of the constituent masses, this clearly calls for an analysis of relativistic effects in the one-boson exchange, in particular of the features which depend on the spins of the particles involved. As a first step, we may consider all terms up to $O(\beta^2)$ in a potential reduction of the one-boson (scalar, pseudoscalar or vector) exchange interactions. Notice, for instance, that the scalar coupling of the Dirac field, i.e. $\overline{\psi}\psi$, yields a spin-orbit interaction of opposite

[1] Notice that the rescaled coupling constant $\overline{\alpha}$ for a given value of r is obtained with an exchanged mass $\hat{\mu} = \lambda(r) \times 0.15$ (cf. Eqs. (5.1.8) and (5.1.9)). However, as seen from Table 5.2.2, the critical value $\hat{B}_{\rm crit}$ depends weakly on the exchanged mass in the interval $0 \leq \hat{\mu} \leq 0.15$, so that the comparison of the critical regions in Figs. 5.2.2(a)-(b) is meaningful.

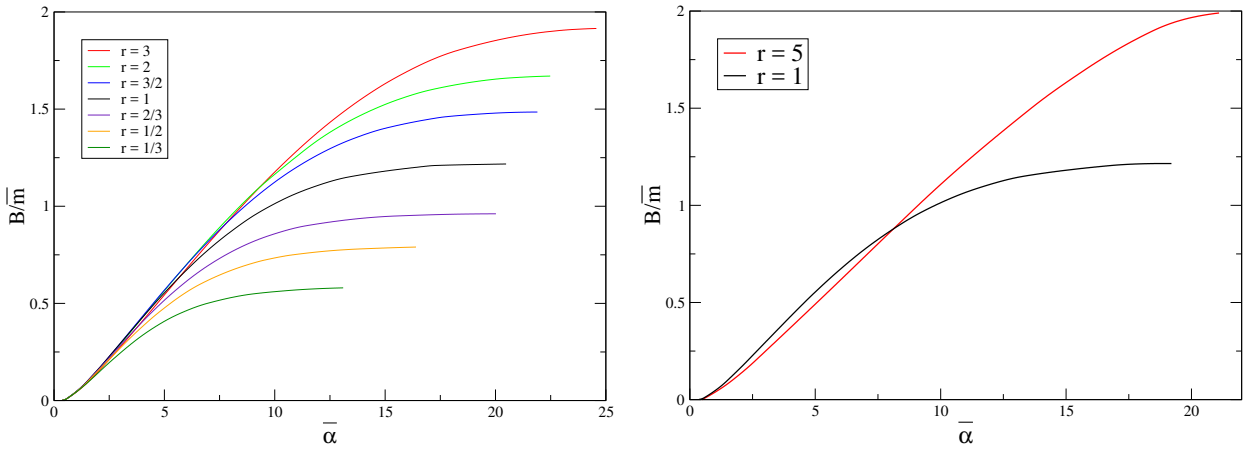

Figure 5.2.2. Rescaled coupling constant $\overline{\alpha}$ (cf. Eq. (5.1.8)) for various mass ratios r and fixed $\hat{\mu} = 0.15$.

sign to the case of a vector coupling, i.e. $\overline{\psi}\gamma^\mu\psi$ (see Appendix D.1, Eq. (D.1.13); in the vector case, the spin-orbit term is produced by the charge-density $\overline{\psi}\gamma^0\psi = \psi^\dagger\psi$, where the low components of ψ are coupled with a plus sign). Furthermore, the potential produced by a pseudoscalar coupling[2], the so-called OPEP (one pion exchange potential), has no spin-orbit term at all. Along these lines, it is also worth noticing that the spin-orbit interaction is the *only* $O(\beta^2)$ spin-dependent term in the one-scalar exchange potential (both for a two-fermion and a fermion-scalar system).

In view of the above considerations, which suggest a potential role of the spin-orbit correlation in the determination of the binding saturation[3], the comparison of the scalar exchange kernel considered here with a vector exchange one seems to be of primary importance, and will be presented in a future paper [48]. In the following, we shall give a detailed discussion of the solution in the low-binding region, which we may approximately identify as $\hat{B} \lesssim \hat{B}_{\rm flex}$, postponing a more complete discussion to Ref. [48].

Focusing on the low-binding region, the solution shows a definitely physical behavior. In the first place we observe that, for very low bindings, our coupling constants for $r = 1$ are very close to those obtained from the scalar-scalar ladder BSE in Ref. [18] (see Subsec. 3.3.2). Of course, this is expected from the non-relativistic point of view, since the spin of the fermion decouples in the extreme non-relativistic limit.

As a second remark, notice that the non-relativistic relation (5.1.10) is generally quite well satisfied from the ladder BSE solutions for low binding energies, as it can be seen from Figs. 5.2.2. In particular, the eigenvalues $\overline{\alpha}(\hat{B}, r, \hat{\mu})$ for $1 \leq r \leq 3$, are in very nice agreement with Eq. (5.1.10) up to $\hat{B} \lesssim 0.5$, while the agreement is restricted to a smaller region when $r < 1$. This can be understood by observing that the spin-orbit correction to the Yukawa effective potential (see Eq. (D.1.13)) depends on the inverse squared mass of the fermion, and is therefore enhanced when $r < 1$. However for $r \gtrsim 5$, the agreement is again restricted to a very small region, showing that the deviations from Eq. (5.1.10) are important not only for large binding energies, but also when the constituent masses are largely different. Furthermore, it is clear that the dependence of $\overline{\alpha}(\hat{B}, r, \hat{\mu})$ from r is not monotonic, and indeed the curves in Figs. 5.2.2 exhibit several crossings in the low-binding region.

We conclude this Section with a brief discussion of the Nakanishi weight functions $g_{1,2}(\gamma, z)$. In Fig. 5.2.3, $g_{1,2}$ are shown for fixed z and running γ, and viceversa, for the case $r = 1$, and two values of $\hat{B}$, corresponding to weak binding $\hat{B} = 0.2$, and strong binding $\hat{B} = 1.2$ (more relativistic

[2]We remark that a simple fermion-scalar interaction through a pseudoscalar π in the t-channel is not allowed, since the lagrangian density $\mathscr{L} = \pi\left\{\lambda_\psi\overline{\psi}i\gamma^5\psi + \lambda_\phi\phi^\dagger\phi\right\}$ is not parity invariant, whatever the intrinsic parity of ϕ be.

[3]As a side note, we observe that the pseudoscalar or vector potentials for a two-fermion system include the so-called tensor forces, i.e. terms of the form $S_{12}(\hat{\mathbf{n}}) \sim (\boldsymbol{\sigma}_1\cdot\hat{\mathbf{n}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{n}}) - \frac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$, where $\hat{\mathbf{n}} = \frac{\mathbf{r}}{|\mathbf{r}|}$ or $\frac{\mathbf{p}}{|\mathbf{p}|}$. In particular, this potentials do not commute with L^2, the total relative orbital momentum.

regime). Of particular interest are the lines corresponding to $\gamma = 0$ and running z, which are nearly symmetric for $\hat{B} = 0.2$, while the asymmetry becomes more pronounced for larger $\hat{B}$. Again, this is clearly a consequence of the decoupling of the fermion spin for $\hat{B} \to 0$: in this limit (in the $r = 1$ case) the constituents can be regarded as indistinguishable, hence the Nakanishi weight functions become symmetric, as a consequence of the corresponding $\xi \to 1 - \xi$ (i.e. $z \to -z$) symmetry of the valence LFWFs (see the next Subsection).

As a technical remark, it is worth to mention that the convergence of the eigenvectors $g = (g_1, g_2)$ requires much more numerical effort than that of the eigenvalues α. Indeed, while a fairly good convergence of α is generally already obtained with a basis of $N_\gamma \times N_z = 20 \times 20$ polynomials, the convergence of g is much slower. For instance, to obtain the plots of Fig. 5.2.3, we had to extend the basis up to $N_\gamma \times N_z = 44 \times 44$ polynomials. In any case, the valence probabilities and the LFWFs, which are given by integrals of $g_{1,2}$, converge essentially with the same speed of the coupling constant.

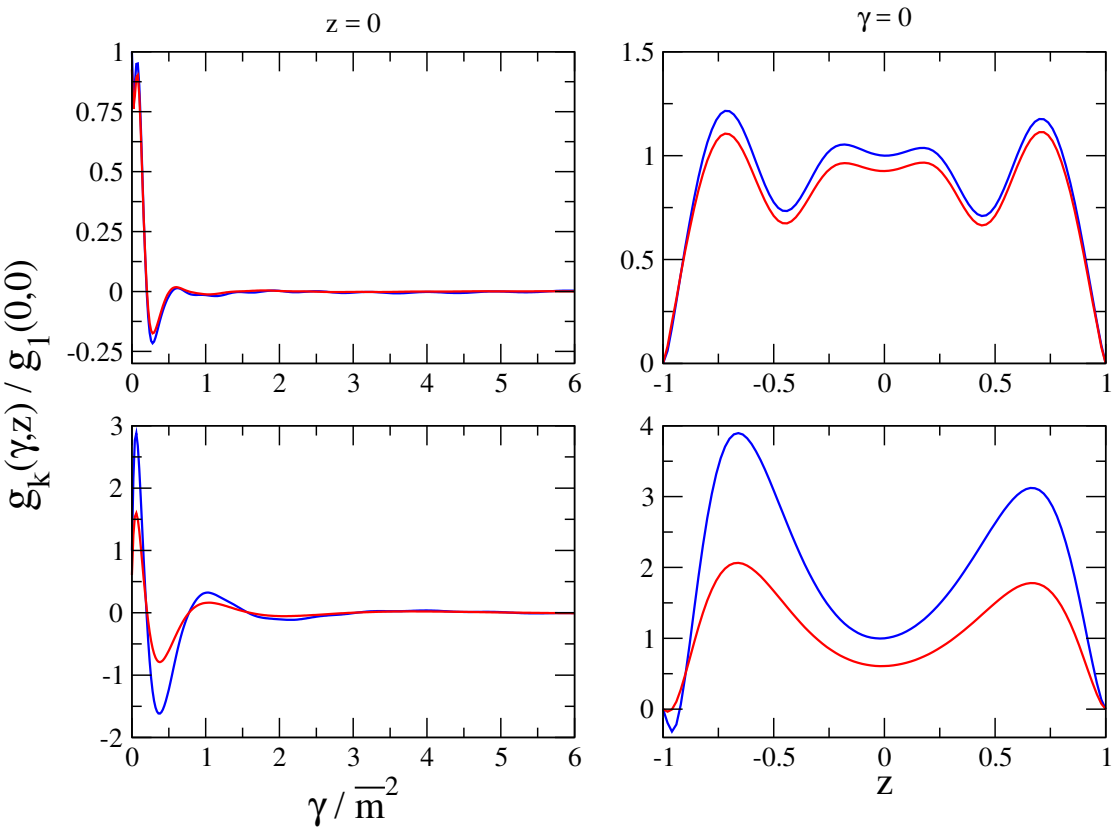

Figure 5.2.3. Nakanishi weight functions for $r = 1$, $\hat{\mu} = 0.15$, and $\hat{B} = 0.2$ (upper panels), $\hat{B} = 1.2$ (lower panels). Blue lines: $g_1(\gamma, z)/g_1(0, 0)$. Red lines: $g_2(\gamma, z)/g_1(0, 0)$.

5.3 Valence LFWFs and probabilities

We shall present our results for the LFWFs and the corresponding probabilities in the binding energy interval $0 \leq \hat{B} \leq \hat{B}_{\rm flex}$. The results for the full interval $0 \leq \hat{B} \leq \hat{B}_{\rm crit}$, which require a more careful analysis and comparisons with the vector-exchange case, will be presented in Ref. [48]. To help the

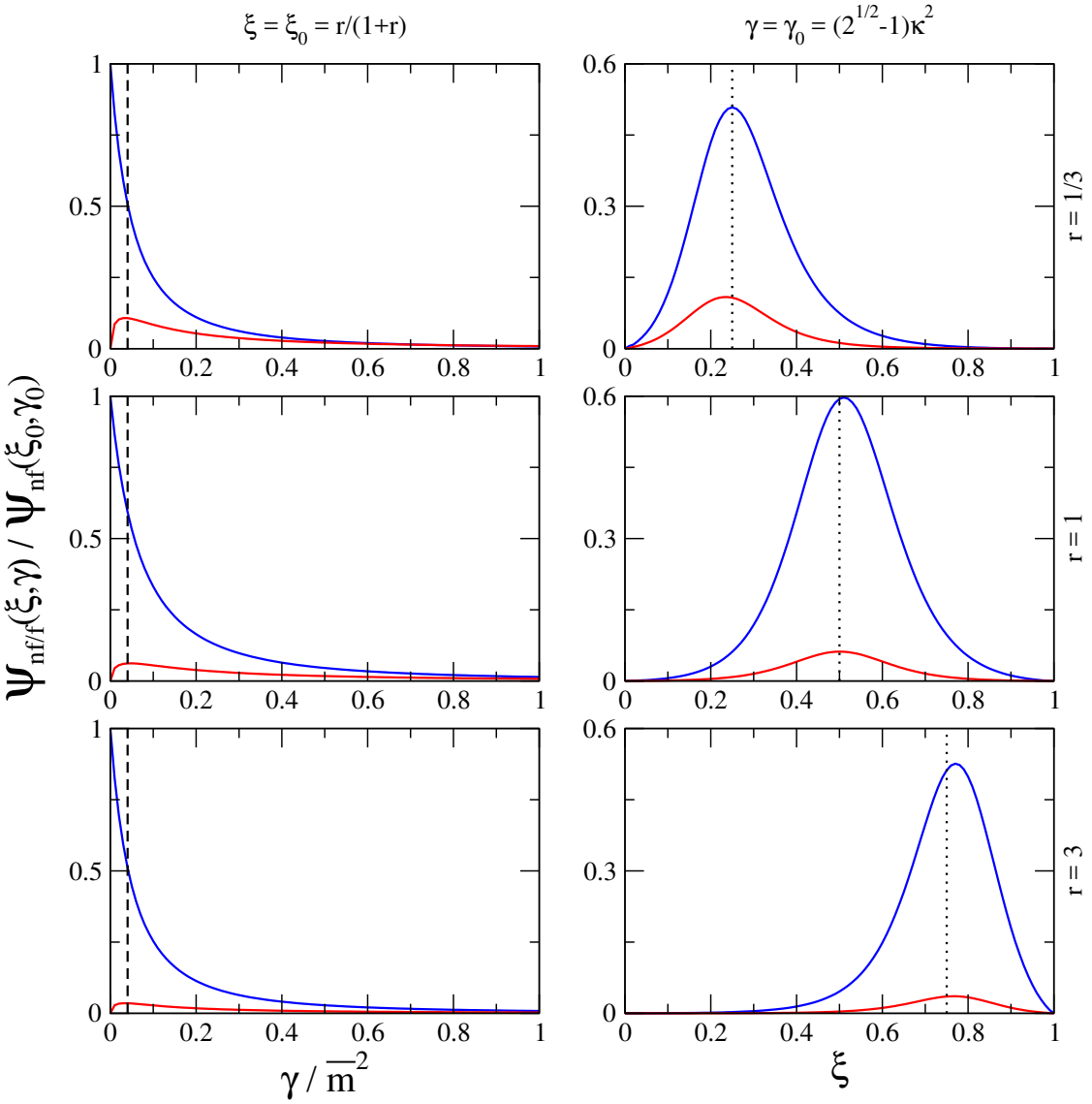

Figure 5.3.1. Valence wave-functions $\Psi_{\rm nf/f}(\xi,\gamma)/\Psi_{\rm nf}(\frac{r}{1+r},0)$, for $\hat{B}=0.1$, $\hat{\mu}=\lambda(r)\times 0.15$ and $r=\frac{1}{3}$, 1, 3 (from the top panel). Blue lines: non-flip amplitude. Red lines: flip amplitude. The dashed lines on the left are $\gamma=(\sqrt{2}-1)\kappa^2$, while the dotted lines on the right are $\xi=\frac{r}{1+r}$.

reader, we report the definitions in Eqs. (4.1.50) and (4.1.51) of the non-flip and flip amplitudes:

$$\Psi_{\rm nf}(\xi,\gamma)=\Phi_1(\xi,\gamma)+(\xi-\frac{1}{2})\Phi_2(\xi,\gamma), \tag{5.3.1}$$

$$\Psi_{\rm f}(\xi,\gamma)=\sqrt{\frac{\gamma}{M^2}}\Phi_2(\xi,\gamma), \tag{5.3.2}$$

where $\Phi_{1,2}$ are given by integrals of the Nakanishi weight functions (see Eq. (4.1.52)).

Several interesting insights into the inner dynamics can be gained from Fig. 5.3.1, where the valence LFWFs for $\hat{B}=0.1$ and $r=\frac{1}{3}$, 1, 3 are presented. In the right panels, the curves at fixed $\gamma=\gamma_0$ and running $\xi=\frac{1-z}{2}$ are shown, while in the left panels $\xi=\xi_0$ and γ varies.

From Fig. 5.3.1 we see that the amplitudes are peaked about $\xi_0=\frac{r}{1+r}$ (dotted lines in the right panels of Fig. 5.3.1), which is what one expects from the non-relativistic point of view, since for small momenta and bindings one has $\xi=\frac{p^0_\psi+p^3_\psi}{P^0+P^3}\approx\frac{m_\psi}{m_\phi+m_\psi}$. Furthermore, the lines for $r=1$ are nearly

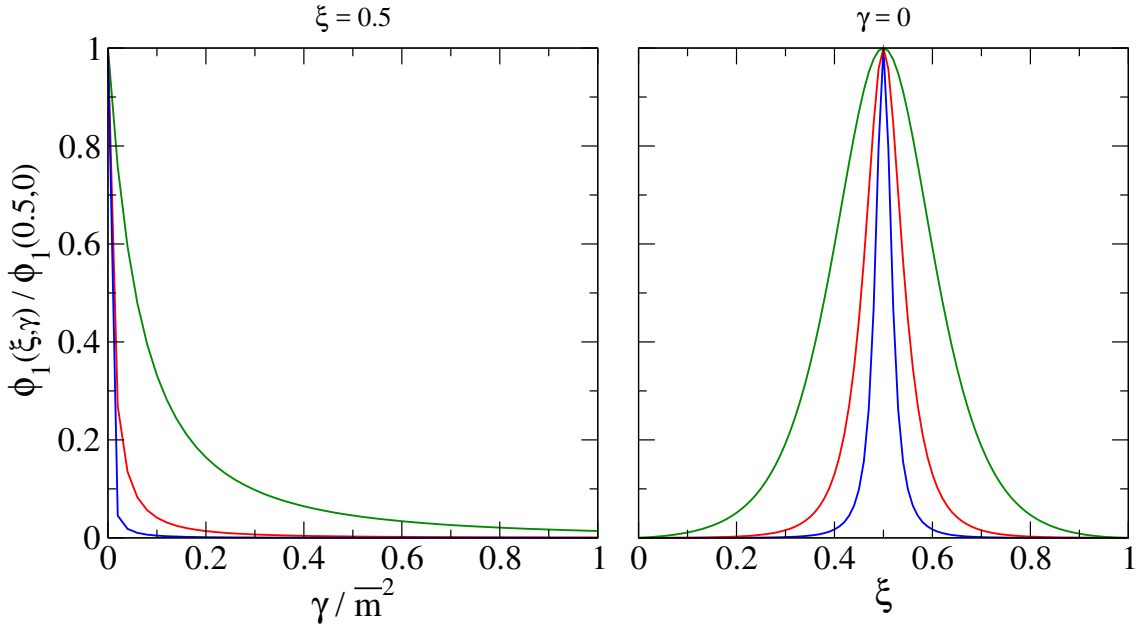

Figure 5.3.2. behavior of the amplitude $\Phi_1(\gamma,\xi)$ for $\hat{B}\to 0$. Green lines: $\hat{B}=0.1$. Red lines: $\hat{B}=0.01$. Blue lines: $\hat{B}=0.001$.

symmetric about $\xi_0=\frac{1}{2}$. As discussed above for the Nakanishi weight functions, this is a consequence of the decoupling of the fermion spin in the non-relativistic limit.

The $\xi=\xi_0$ lines show that the γ-width at half maximum of the non-flip amplitude is approximately given by:

$$\gamma_0=(\sqrt{2}-1)\kappa^2, \tag{5.3.3}$$

where:

$$\kappa^2=\overline{m}^2-\frac{M^2}{4}=\overline{m}B-\frac{B^2}{4}. \tag{5.3.4}$$

This value is suggested from the analytic expression of the valence wave-function in the Wick-Cutkosky model (see below, Eq. (5.3.7)), where the exchanged mass $\mu=0$. As seen from the figures, the position of the peak in the flip amplitude is given by $\gamma\approx\gamma_0$ as well. Thus, one can argue that the size of the bound system is approximately $R\sim\frac{1}{\kappa}$, which may be used as a phenomenological guidance to have a rough estimate of the average mass of the constituents, if both R and M are known. Notice that, for $B\ll\overline{m}$, this reduces to $R^2\sim\frac{1}{\overline{m}B}$, which is merely a statement of the uncertainty principle.

The asymptotic behaviors in the γ-tails should be those implied by the NIR of $\Phi_{1,2}$ in Eqs. (5.3.1) and (5.3.2) (see Eq. (4.1.52)), viz.[4]:

$$\Psi_{\rm nf}\sim\gamma^{-2},\quad \Psi_{\rm f}\sim\gamma^{-\frac{3}{2}}\qquad(\gamma\to\infty). \tag{5.3.5}$$

We did not, however, perform a systematic study of the asymptotics of $\Psi_{\rm nf/f}$, which appear to require some care from the numerical point of view, and are left for future work.

It is interesting to observe that, in the limit $\hat{B}\to 0$, the amplitudes Φ_1 and Φ_2 in Eqs. (5.3.1) and (5.3.2) become sharply peaked about $\xi=\frac{r}{1+r}$ and $\gamma=0$ as one expects (see Fig. 5.3.2; the behavior of Φ_2 is identical). Thus:

$$\Psi_{\rm nf}\approx\Phi_1+\frac{1}{2}\frac{r-1}{r+1}\Phi_2,\qquad \Psi_{\rm f}\approx 0\qquad(\hat{B}\to 0). \tag{5.3.6}$$

[4]These estimates strictly follow if the Nakanishi weight functions g_i are absolutely integrable.

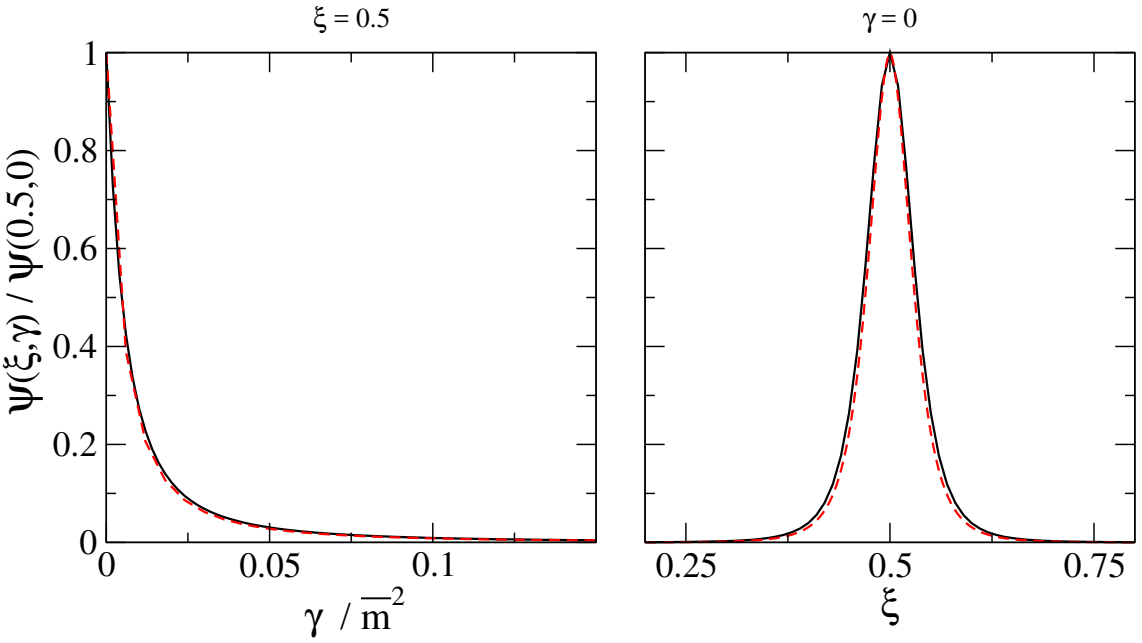

Figure 5.3.3. Comparison between the non-flip amplitude $\Psi_{\rm nf}$ for $\hat{B} = 0.01$, $r = 1$ and $\hat{\mu} = 0$ and the analytic expression obtained in the Wick-Cutkosky model (cf. Eq. (5.3.7)).

Finally, we can compare the non-flip wave-function for $r = 1$, $\hat{\mu} = 0$ and $\hat{B} \approx 0$, with the analytic expression obtained in the Wick-Cutkosky model, which reads [34]:

$$\Psi^{(\rm WC)}(\xi, \gamma) = \frac{C(\xi)}{\left[\gamma + m^2 - \xi(1-\xi)M^2\right]^2}, \tag{5.3.7}$$

where m is the common mass of the constituents, and the function $C(\xi)$ is given, for $\hat{B} \ll 1$, by:

$$C(\xi) = \frac{\kappa^4}{8}\xi(1-\xi)\left(\frac{1}{2} - \left|\frac{1}{2} - \xi\right|\right) \qquad (\hat{B} \ll 1). \tag{5.3.8}$$

The normalization is chosen in such a way that $\Psi^{(\rm WC)}(0.5, 0) = 1$. As seen from Figure 5.3.3 for $\hat{B} = 0.01$, the agreement between our calculations and the analytic expression is quite remarkable.

Our results for the corresponding valence probabilities are shown in Fig. 5.3.4, where the probabilities for $r = \frac{1}{3}$, 1, 3 and $\hat{\mu} = \lambda(r) \times 0.15$ are shown for $0 \leq \hat{B} \leq \hat{B}_{\rm flex}$.

In the low-binding energy region, the behavior of the probability has the features one should expect on the non-relativistic ground: the non-flip probability approaches unity as $\hat{B} \to 0$ and decreases for growing binding energy, while the flip probability tends to zero in the same limit and increases with $\hat{B}$ (cf. Eqs. (5.3.1) and (5.3.6)). Moreover, as a further consistency check, we remark that our results for the total probability at low $\hat{B}$ are very close to those obtained, in the scalar-scalar case, in Ref. [18] (cf. Subsec. 3.3.2), as we have already seen for the coupling constants.

From 5.3.4, we see that the flip probability appears to decrease when we increase the ratio r. A monotonic behavior of this kind is not completely unexpected since, as already observed, the spin-orbit term (the only $O(\beta^2)$ spin-dependent term in the non-relativistic reduction of the Yukawa interaction) is proportional to $\frac{1}{m_\psi^2}$. However, within our LF description, it is not *a priori* immediately clear if the interaction of the fermion spin favors, from the energetic point of view, the flip or non-flip component. This is so because, as already observed in Subsec. 4.1.3, only $L_z = -i\frac{\partial}{\partial\varphi}$ does not change the number of LF constituents (since it is a kinematical operator), so that the spin-orbit interaction $V_{\rm s.o.} \sim \mathbf{L} \cdot \mathbf{S}$ does not have a direct interpretation in terms of LF two-particle states.

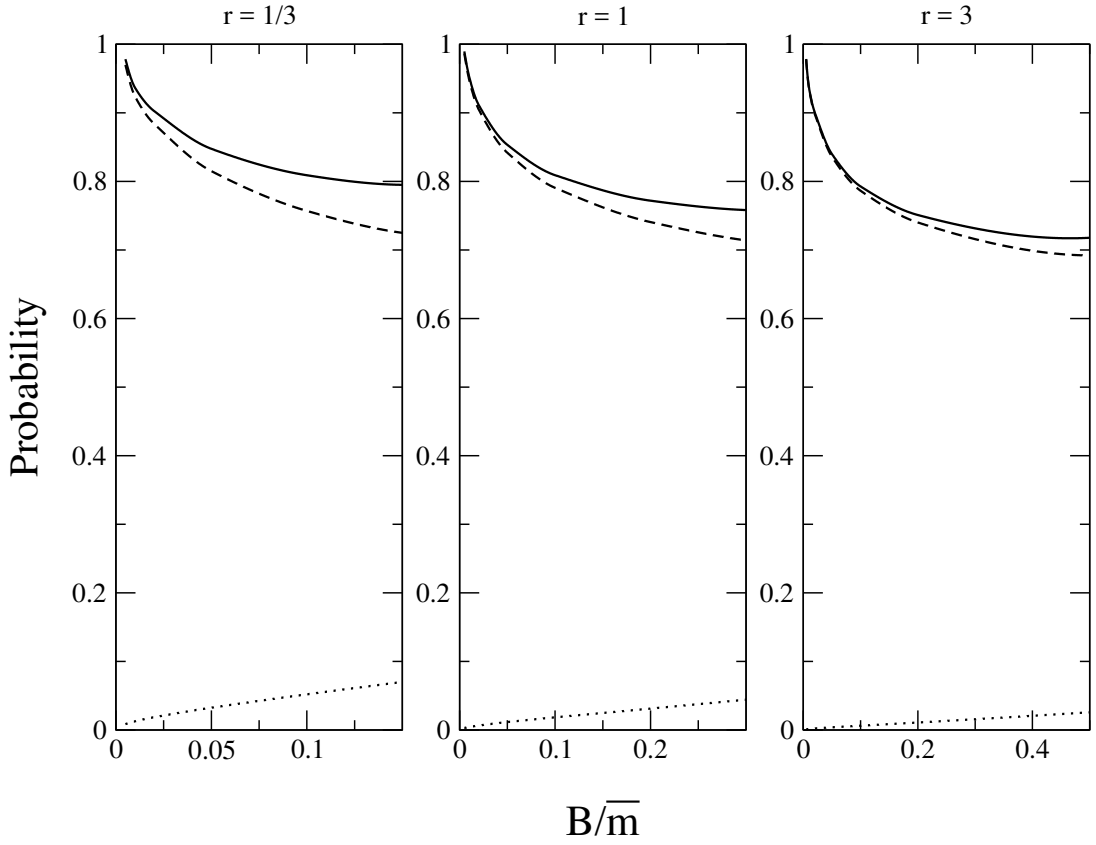

Figure 5.3.4. Valence probabilities for $r = \frac{1}{3}$, 1, 3 and $\hat{\mu} = \lambda(r) \times 0.15$. Dashed line: non-flip probability. Dotted line: flip probability. Solid line: total probability.

Also not easy to anticipate is the $\hat{\mu}$-dependence of the valence probabilities for fixed $\hat{B}$ and r. Indeed, this dependence may be pictorially regarded as the effect of two competing contributions: i) for large $\hat{\mu}$, it is kinematically less favorable for the constituents to emit a virtual boson, i.e. it is difficult to populate higher Fock components, suggesting an higher valence probability; however ii) for a fixed $\hat{B}$, the coupling constant increases with $\hat{\mu}$, so that the emission of a boson becomes favored dynamically. As seen in Fig. 5.3.5, the first physical mechanism appears to be the dominant one for $\hat{B} = 0.1$ and $r = 1$, but the non-flip probability alone is not monotonic in $\hat{\mu}$.

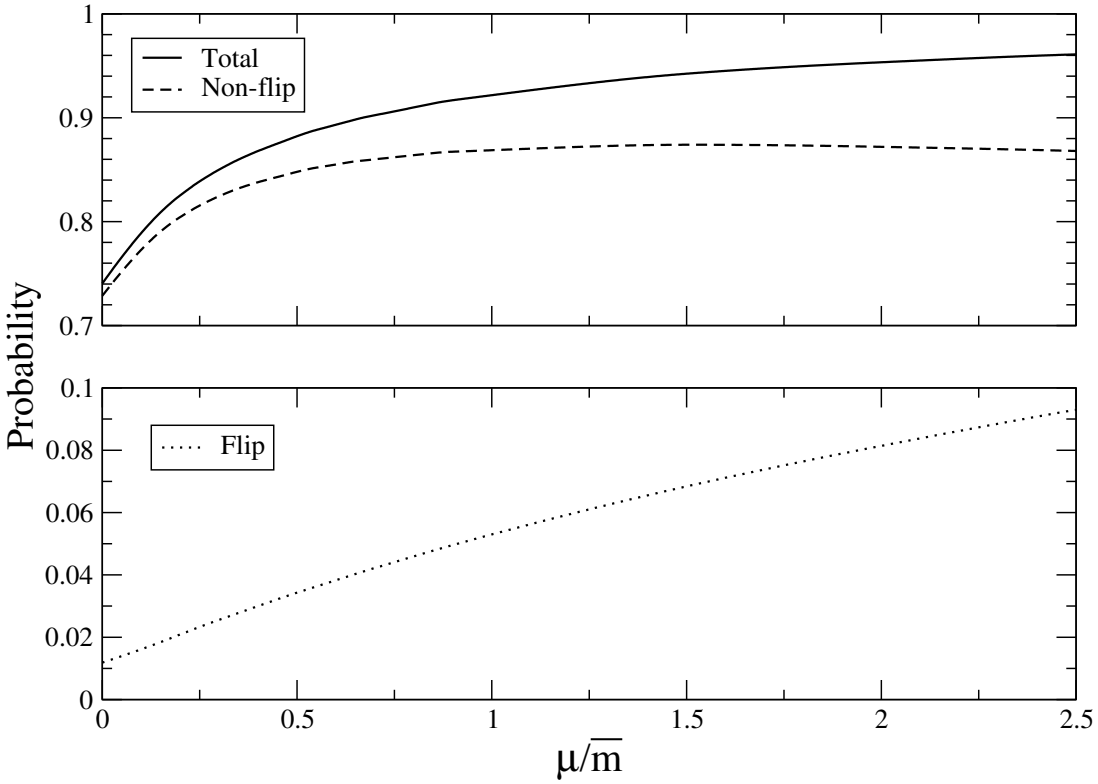

Figure 5.3.5. Valence probabilities for $r = 1$ and $\hat{B} = 0.1$.

Chapter 6

Conclusions and Perspectives

In this Thesis, we have addressed the homogeneous Bethe-Salpeter equation in Minkowski space, for a bound state composed by a fermion and a scalar, freely propagating inside the system, but interacting through the exchange of a massive scalar. A workable equation, in ladder approximation for the interaction kernel, has been obtained within a framework based on i) the Nakanishi integral representation of the BS amplitude and ii) the so-called light-front projection technique. It has to be emphasized that the NIR allows one to perform analytically the integration on the minus-component of the relative four-momentum, namely the LF-projection onto the null hyperplane of the BSE, and to single out singular integrals generated by the presence of spin degrees of freedom. After rigorously treating such singular contributions, that in our case are shown to be vanishing (cf. the discussion following Eq. (4.2.10)), one obtains a formally exact reduction of the original equation into a system of integral equations for the Nakanishi weight-functions. Notably, one eventually deals with a generalized eigenvalue problem in the ladder approximation. It should be pointed out that the existence of solutions for such an eigenvalue problem validates, a posteriori, the NIR-LF approach for solving the ladder BSE. Furthermore, the obtained eigenvalues provide the usual binding energy/coupling constant functional dependence, and the eigenvectors allow one to completely reconstruct the Minkowski space BS amplitude.

Our numerical results for the coupling constants were cross-checked through the solution of the Wick-rotated BSE. The peculiar and unique feature of solving the BSE directly in Minkowski space is represented by the possibility to evaluate also dynamical quantities, like the LF distributions. As a first step of a wider investigation, we have presented an analysis of the so-called LF valence wave-functions and the corresponding probabilities.

For moderate binding energies, the solution of the fermion-scalar ladder BSE has most of the features one expects both from intuition and from the non-relativistic limit. The coupling constants and the valence probabilities for low binding energies are very close to those obtained by solving the ladder BSE for a two-scalar system [18], showing that spin degrees of freedom decouple in the non-relativistic limit, as one expects. Moreover, a non-relativistic relation for the coupling constant dependence upon the bound system parameters (see Eq. (5.1.10)) is found to be fairly well satisfied from our solutions. The Nakanishi weight functions and the valence LFWFs for equal-mass constituents have an approximate symmetry $z \to -z$ (i.e. $\xi \to 1-\xi$), which becomes exact when $B \to 0$. In the same limit, the spin-flip amplitude vanishes and the non-flip amplitude becomes sharply peaked about $\gamma = 0$ and $\xi = \frac{m_\psi}{m_\phi + m_\psi}$, while the total valence probability tends to one. Also, noteworthy, in the massless-exchange case, the non-flip amplitudes for very low bindings coincide with the wave-function obtained analytically in the Wick-Cutkosky model (cf. Eq. 5.3.3). It is very rewarding to see these several non-relativistic features automatically emerging from a field-theoretical and fully covariant approach. Summarizing the low-energy studies, we can state that the numerical solutions are under control and well understood.

On the other hand, for higher binding energies, the solution shows an unexpected behavior, i.e. what we call binding saturation. Indeed, for not too high values of the ratio $r = \frac{m_\psi}{m_\phi}$, the interval of binding energies $0 \leq B \leq m_\phi + m_\psi$ is not entirely covered by the BSE solution but, on the contrary, there exists a critical binding energy $B_{\rm crit} < m_\phi + m_\psi$ above of which no bound-state solution is found. In general, the $B(\alpha)$ plots show a flex, which is not observed in the corresponding two-scalar system [18], but is interestingly present for a 0^+ two-fermion system with scalar-exchange [17]. Therefore a working hypothesis is yielded by the particular role played by the spin degrees of freedom in the fermion-scalar case, where the interaction could be differently tuned by the parallel and antiparallel configurations of the constituent LF spin and the composite system one.

In order to gain crucial insights into the issue, the comparison with a vector-exchange ladder kernel, or with a ladder+cross-ladder kernel (to assess in more detail the ladder contribution), seems to be of primary importance. In view of this, both binding/coupling correlations and valence LFWFs were presented only for low binding energies, where the solution appears to be coherent with previous results, postponing the desired more complete analysis to a forthcoming work [48]. In spite of these not yet completely understood features, which appear to be intrinsic of the scalar exchange considered here, it is worth recalling that our initial results validate the NIR for the fermion-scalar ladder BSE, and that the introduction of unequal masses (for the first time in a NIR-LF study of the ladder BSE) opens a new window on the ongoing dynamical game inside a bound system.

The natural development of our analysis will be the study of the fermion-vector system, more demanding from the formal side, given the presence of higher-spin components that request a careful study of light-cone singularities one has to face with while performing the LF projection.

It should be pointed out that the method presented here is by no means restricted to the ladder BSE, which is in any case essential for describing the asymptotic behavior of the LF distributions. It would be extremely interesting to try to apply the same approach with more realistic kernels and propagators. From a fundamental point of view, efforts for implementing self-energies in the constituent propagators have to be undertaken, with the final goal of making the present approach a viable one, e.g., in the description of a nucleon in terms of quark and diquark degrees of freedom, given the need to take into account the confined nature of its colored constituents.

As a final note, we would like to emphasize that all numerical calculations presented here were performed with a home computer. Indeed, thanks to NIR, the integrations in the Minkowski space BSE are performed analytically as far as possible, thereby reducing it to a two-dimensional, regular integral equation of a canonical form, which is a relatively simple numerical problem.

To close this Thesis, we would like to make some comments on the present state of art of the BSE, and on the relevance of these studies to our understanding of the bound-state problem in the relativistic domain.

As we hope to have clarified in the Introduction, the development of the Minkowski space approach to the BSE is still in an early stage, and it has to be improved in many respects before becoming a broad-spectrum tool for phenomenological applications. In spite of this, we believe that NIR-LF studies of the BSE could provide some helpful insights into the non-perturbative regime of QFT, basically investigated through the well-recognized main tool, namely the the path-integral method. This hope is based on the ability of the NIR-LF approach to elaborate the description of the bound system directly in the physical 4D space. In order to appreciate the relevance of this last issue, one should remind that within lattice QCD, though valuable results have been achieved, e.g., in the calculations of both hadron masses and decay constants, only a few Mellin moments of the parton distribution functions are at present available. In this context, the great ambition is to investigate the internal dynamics as unveiled by the PDFs, or more generally the transverse-momentum distributions and generalized parton distributions. In particular, in view of recent caveats [57] on the feasibility of performing lattice calculations of PDFs in full, one could conjecture that studies of the BSE in Minkowski space be able to offer useful hints. Also in condensed matter one could foresee applications,

when relativistic bound and unbound systems have to be carefully described.

In conclusion, there is no risk to overemphasize that the bound-state problem is still a relatively unexplored field, and the above considerations, together with the recent progress in solving the BSE, suggest that these studies could in perspective play a relevant and complementary role in understanding the bound-state structure in the physical Minkowski space.

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Appendix A

NIR: from the vertex to the BS amplitude

In this Appendix we shall give a formal derivation of the NIR of the BS amplitude for a scalar bound state of scalar constituents, starting from the one of the corresponding vertex given in Subsec 3.1.1. The same representation is adopted for the invariant amplitudes in the decomposition of the fermion-scalar BS amplitude (cf. Subsec. 4.1.1).

We recall from 3.1.1 that the bound state vertex $i\Gamma$ is defined by:

$$\chi(k;P) \equiv i\Delta_1'(\frac{P}{2}+k)i\Delta_2'(\frac{P}{2}-k)i\Gamma(k;P), \tag{A.1}$$

and we assume that it admits the following NIR:

$$\Gamma(k;P) = \int_0^1 \mathrm{d}\zeta \int_{-\infty}^{\infty} \mathrm{d}\nu \frac{\varphi(\zeta,\nu)}{\zeta s_1 + (1-\zeta)s_2 - \nu + i\varepsilon}, \tag{A.2}$$

where:

$$s_1 = (\frac{P}{2}+k)^2, \qquad s_2 = (\frac{P}{2}-k)^2. \tag{A.3}$$

The support of φ in the unbounded variable ν is left implicit.

Neglecting self-energy corrections in the propagators we obtain, after a Feynman parametrization:

$$\chi(k;P) = -2i \int_{[0,1]^2} \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \theta(1-\alpha_1-\alpha_2) \int_0^1 \mathrm{d}\zeta \int_{-\infty}^{+\infty} \mathrm{d}\nu \frac{\varphi(\zeta,\nu)}{[D(\zeta,\nu,\alpha,s_1,s_2)+i\varepsilon]^3}, \tag{A.4}$$

where:

$$\begin{aligned} D(\zeta,\nu,\alpha,s_1,s_2) = {} & s_1(\alpha_1+\zeta(1-\alpha_1-\alpha_2)) + s_2(\alpha_2+(1-\zeta)(1-\alpha_1-\alpha_2)) + \\ & - \alpha_1 m_1^2 - \alpha_2 m_2^2 - (1-\alpha_1-\alpha_2)\nu. \end{aligned} \tag{A.5}$$

Define z and γ through:

$$\frac{1+z}{2} = \alpha_1 + \zeta(1-\alpha_1-\alpha_2) \tag{A.6}$$

$$\alpha_1 m_1^2 + \alpha_2 m_2^2 + (1-\alpha_1-\alpha_2)\nu = \frac{1+z}{2}m_1^2 + \frac{1-z}{2}m_2^2 + \gamma. \tag{A.7}$$

Making this change of variables into Eq. (A.4), we obtain:

$$\chi(k;P) = -i \int\limits_{[0,1]^2} \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \int\limits_{-1}^{+1} \mathrm{d}z \int\limits_{-\infty}^{+\infty} \mathrm{d}\gamma \times$$
$$\times \frac{\theta(\frac{1+z}{2}-\alpha_1)\theta(\frac{1-z}{2}-\alpha_2)}{\left[1-\alpha_1-\alpha_2\right]^2} \frac{\varphi(\frac{\gamma+(\frac{1+z}{2}-\alpha_1)m_1^2+(\frac{1-z}{2}-\alpha_2)m_2^2}{1-\alpha_1-\alpha_2}, \frac{\frac{1+z}{2}-\alpha_1}{1-\alpha_1-\alpha_2})}{\left[D(\gamma,z,s_1,s_2)+i\varepsilon\right]^3}, \tag{A.8}$$

$$D(\gamma,z,s_1,s_2) = \frac{1+z}{2}(s_1-m_1^2) + \frac{1-z}{2}(s_2-m_2^2) - \gamma. \tag{A.9}$$

Finally, defining the Nakanishi weight function of the BSA as:

$$g(\gamma,z) = -\int\limits_{[0,1]^2} \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \frac{\theta(\frac{1+z}{2}-\alpha_1)\theta(\frac{1-z}{2}-\alpha_2)}{\left[1-\alpha_1-\alpha_2\right]^2} \times$$
$$\times \varphi(\frac{\gamma+(\frac{1+z}{2}-\alpha_1)m_1^2+(\frac{1-z}{2}-\alpha_2)m_2^2}{1-\alpha_1-\alpha_2}, \frac{\frac{1+z}{2}-\alpha_1}{1-\alpha_1-\alpha_2}) \tag{A.10}$$

we obtain from (A.8) the final form:

$$\chi(k;P) = i \int\limits_{-1}^{+1} \mathrm{d}z \int\limits_{-\infty}^{+\infty} \mathrm{d}\gamma \frac{g(\gamma,z)}{\left[\frac{1+z}{2}(s_1-m_1^2) + \frac{1-z}{2}(s_2-m_2^2) - \gamma + i\varepsilon\right]^3}. \tag{A.11}$$

As explained in Subsec. 3.1.1, we shall assume that $g(\gamma,z)=0$ for $\gamma<0$. Expressing $s_{1,2}$ in terms of P and k through Eq. (A.3) and defining:

$$\overline{m} \equiv \frac{m_1+m_2}{2}, \quad \Delta \equiv \frac{m_2-m_1}{2}, \quad \kappa^2 \equiv \overline{m}^2 - \frac{M^2}{4}. \tag{A.12}$$

we obtain the form of the NIR used throughout this Thesis:

$$\chi(k;P) = i \int\limits_{-1}^{1} \mathrm{d}z \int\limits_{0}^{\infty} \mathrm{d}\gamma \frac{g(\gamma,z)}{\left[(k+z\frac{P}{2})^2 - (1-z^2)\kappa^2 - (z\overline{m}-\Delta)^2 - \gamma + i\varepsilon\right]^3}. \tag{A.13}$$

Appendix B

Special topics in LF dynamics

In this Appendix we discuss some technicalities related to the Poincaré group in the LF formalism, which are necessary in order to deal with a system with spin. In Sec. B.1 we review the standard construction of a spin operator from the Poincaré generators, and specialize to the light front spin. In Sec. B.2 we derive the expressions of the LF spinors. Finally, in Sec. B.3, we discuss the issue of discrete symmetries in the LF formalism, and introduce LF parity and LF time reversal operators.

In what follows, we denote by $U(\Lambda, a)$ the unitary operator, acting on the Hilbert space of the quantum theory, corresponding to a (proper, orthochronous) Poincaré transformation $x \mapsto \Lambda x + a$ (we adopt the shorthand notation $U(\Lambda)$ if $a = 0$). It is assumed that U forms a representation of the Poincaré group:

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2),$$
$$U(1, 0) = 1.$$

As a consequence:

$$U(\Lambda, a)^\dagger = U(\Lambda^{-1}, -\Lambda^{-1}a).$$

The irreducible subrepresentations of U are the vacuum state:

$$U(\Lambda, a)|0\rangle = |0\rangle,$$

and the one-particle states, which are (normalizable) eigenstates of the mass operator $M = \sqrt{P^\mu P_\mu}$. We assume that the theory contains no massless particles, i.e. that $p^\mu = 0$ is an isolated point of the joint spectrum of the four-momentum P^μ. This implies, in particular, that the operator $\frac{1}{M} \equiv M^{-1}$ is well defined on the subspace orthogonal to the vacuum.

B.1 LF boosts and LF spin

As is well known, in relativistic quantum mechanics there are several ways to construct a spin operator, that is to say, a triplet $\boldsymbol{j} = (j_1, j_2, j_3)$ of self-adjoint operators which satisfies the algebra of the rotation group:

$$[j_l, j_k] = i\varepsilon_{lkm} j_m, \tag{B.1.1}$$

and whose action on states at rest is that of $\mathbf{J}$, the angular momentum generator.

The construction starts from the Pauli-Lubanski four vector, which is defined by:

$$W^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma. \tag{B.1.2}$$

From the commutation rules of the Poincaré generators $M_{\nu\rho}$ and P_σ, one can easily prove that the operator:

$$W^2 = W^\mu W_\mu, \tag{B.1.3}$$

commutes with all the generators of the Poincaré group, i.e., it is a Casimir operator for the Poincaré group representation. Indeed, it is more common to consider, rather than (B.1.2), the operator defined by:

$$j^2 = -\frac{W^2}{M^2}, \tag{B.1.4}$$

because, as the notation suggests, Eq. (B.1.4) reduces to $\sum_{i=1}^{3} J_i^2$, when applied on a state at rest.

Writing down explicitly the components of (B.1.2), one finds that:

$$W^0 = \mathbf{P}\cdot\mathbf{J}, \tag{B.1.5}$$

$$\mathbf{W} = H\mathbf{J} - \mathbf{P}\times\mathbf{K}, \tag{B.1.6}$$

which imply:

$$W^\mu|\mathbf{P}=\mathbf{0}\rangle = (0, M\mathbf{J})\,|\mathbf{P}=\mathbf{0}\rangle, \tag{B.1.7}$$

where $|\mathbf{P}=\mathbf{0}\rangle$ is an arbitrary state with zero three-momentum.

The second ingredient of the construction is the definition of a boost for the mass-shell $P^2 = M^2$, that is to say, of a particular parametrization $B(P)$ of Lorentz transformations such that the (classical) equation[1]:

$$B(P)_\mu{}^\nu P^\mu = (M, \mathbf{0}) \tag{B.1.8}$$

holds for any P such that $P^2 = M^2$. If $|P\rangle$ is an arbitrary state of momentum P, then $U(B(P))^\dagger|P\rangle$ has three-momentum $\mathbf{P'} = \mathbf{0}$ zero. Therefore, (B.1.7) implies:

$$W^\mu U(B(P))^\dagger|P\rangle = (0, M\mathbf{J})\,U(B(P))^\dagger|P\rangle,$$

or:

$$\frac{1}{M}B(P)_\mu{}^\nu W^\mu|P\rangle = \left(0, U(B(P))\mathbf{J}U(B(P))^\dagger\right)|P\rangle. \tag{B.1.9}$$

Note that P and M in the LHS of (B.1.9) can be interpreted as *operators,* so that Eq. (B.1.9) says that the operator $\frac{1}{M}B(P)_\mu{}^\nu W^\mu$ has only the three space components, which act like $\mathbf{J}$ on a state of zero momentum. Furthermore, the triplet defined by:

$$(0, \boldsymbol{j}) = \frac{1}{M}B(P)_\mu{}^\nu W^\mu \tag{B.1.10}$$

satisfies the commutation rules (B.1.1)[2] and, therefore, perfectly qualifies as a spin operator. From (B.1.10), we also see that:

$$\boldsymbol{j}\cdot\boldsymbol{j} = -\frac{W^2}{M^2} \tag{B.1.11}$$

---

[1] The components of the inverse Lorentz transformations Λ^{-1} are denoted by $\Lambda_\mu{}^\nu$.

[2] To see this, one can use the commutation relations:

$$[W^\mu, P^\nu] = 0, \qquad [W^\mu, W^\nu] = i\varepsilon^{\mu\nu\rho\sigma}W_\rho P_\sigma.$$

Writing:

$$S^\nu = \frac{1}{M}B_\mu{}^\nu(P)W^\mu, \qquad W^\mu = MB^\mu{}_\nu(P)S^\nu, \qquad P^\mu = MB^\mu{}_0(P),$$

which are valid as operator relations, one has:

$$\begin{aligned}[S^\nu, S^\rho] &= \frac{1}{M^2}B(P)_\mu{}^\nu B(P)_\sigma{}^\rho i\varepsilon^{\mu\sigma\tau\kappa}W_\tau P_\kappa \\ &= B(P)_\mu{}^\nu B(P)_\sigma{}^\rho i\varepsilon^{\mu\sigma\tau 0}B_\tau{}^\gamma(P)S_\gamma B_\kappa{}^0(P) \\ &= i\varepsilon^{\nu\rho\gamma 0}S_\gamma,\end{aligned}$$

where we have used the fact that $\det B(P) = 1$. This is clearly equivalent to (B.1.1).

is the Casimir operator j^2 defined by (B.1.4).

Therefore, the definition of a spin operator $\boldsymbol{j}$ amounts essentially to the choice of a boost $B(P)$, which maps the standard vector $(M, \mathbf{0})$ to P [3]. A particularly convenient choice in the LF formulation is the LF boost $B_{\rm LF}(P)$ defined by ($x' = B_{\rm LF}(P)x$):

$$x'^{+} = e^{\omega} x^{+}, \tag{B.1.12}$$

$$\mathbf{x}'_{\perp} = \mathbf{x}_{\perp} + x^{+}\boldsymbol{\alpha}_{\perp}, \tag{B.1.13}$$

$$x'^{-} = e^{-\omega} x^{-} + e^{-\omega}\boldsymbol{\alpha}_{\perp}^{2} x^{+} + 2e^{-\omega}\boldsymbol{\alpha}_{\perp}\cdot\mathbf{x}_{\perp}, \tag{B.1.14}$$

where $x^{\pm} = x^0 \pm x^3$ and:

$$\boldsymbol{\alpha}_{\perp} = \frac{\mathbf{P}_{\perp}}{M}, \qquad e^{\omega} = \frac{P^{+}}{M}. \tag{B.1.15}$$

The LF boost (B.1.12)-(B.1.14) leaves the light front $x^+ = 0$ invariant, so that the corresponding unitary transformation $U(B_{\rm LF}(P))$ is kinematical. Indeed, in terms of the LF generators of the Poincaré group, one has:

$$U(B_{\rm LF}(P)) = e^{-i\omega K^3} e^{-i\boldsymbol{\alpha}_{\perp}\cdot\mathbf{B}_{\perp}}. \tag{B.1.16}$$

Also, the LF boosts form a subgroup of the Lorentz group. This is particularly clear in the $\mathrm{SL}(2,\mathbb{C})$ notation of Lorentz transformations [51], in which $B_{\rm LF}(P)$ is represented by the 2×2 complex matrix:

$$B_{\rm LF}(P) \doteq e^{-\frac{\omega}{2}} \begin{pmatrix} e^{\omega} & 0 \\ \alpha_1 + i\alpha_2 & 1 \end{pmatrix}. \tag{B.1.17}$$

The spin operator corresponding to the boost $B_{\rm LF}$ is:

$$j^3_{\rm LF} = \frac{1}{P^+}\left[P^+ J^3 + \hat{\mathbf{z}}\cdot(\mathbf{B}_{\perp}\times\mathbf{P}_{\perp})\right], \tag{B.1.18}$$

$$\mathbf{j}_{{\rm LF}\perp} = \frac{1}{M}\left\{\hat{\mathbf{z}}\times\left[\frac{1}{2}(P^-\mathbf{B}_{\perp} - P^+\mathbf{S}_{\perp}) + \mathbf{P}_{\perp}K^3\right] - \frac{\mathbf{P}_{\perp}}{P^+}\left[P^+J^3 + \hat{\mathbf{z}}\cdot(\mathbf{B}_{\perp}\times\mathbf{P}_{\perp})\right]\right\}. \tag{B.1.19}$$

We observe that the third component of the LF spin (B.1.18) is purely kinematical, and it reduces to J^3 in a frame of reference such that $\mathbf{P}_{\perp} = \mathbf{0}$. To understand the physical meaning of $j^3_{\rm LF}$, observe that [41]:

$$\begin{aligned} e^{i\omega K^3}\mathbf{P}\cdot\mathbf{J}e^{-i\omega K^3} &= P^1(\cosh\omega M^{23} + \sinh\omega M^{20}) + P^2(\cosh\omega M^{31} + \sinh\omega M^{01}) \\ &\quad + (\cosh\omega P^3 + \sinh\omega P^0)J^3 \\ &= P^1\left(\frac{e^{\omega}B_2 + e^{-\omega}S_2}{2}\right) - P^2\left(\frac{e^{\omega}B_1 - e^{-\omega}S_1}{2}\right) + \left(\frac{e^{\omega}P^+ - e^{-\omega}P^-}{2}\right)J^3. \\ e^{i\omega K^3}\,|\mathbf{P}|\,e^{-i\omega K^3} &= \sqrt{(P^1)^2 + (P^2)^2 + \left(\frac{e^{\omega}P^+ - e^{-\omega}P^-}{2}\right)^2}. \end{aligned}$$

Hence, in the limit $\omega\to\infty$, we have:

$$\lim_{\omega\to\infty} e^{i\omega K^3}\frac{\mathbf{P}\cdot\mathbf{J}}{|\mathbf{P}|}e^{-i\omega K^3} = j^3_{\rm LF}. \tag{B.1.20}$$

An eigenstate of $j^3_{\rm LF}$ becomes, for a Lorentz observer which moves along the $-\hat{z}$ direction with the speed of light, i.e. in the so-called "infinite momentum frame", an eigenstate of the helicity $h = \frac{\mathbf{P}\cdot\mathbf{J}}{|\mathbf{P}|}$. This explains why the LF spin $j^3_{\rm LF}$ is sometimes called the LF helicity.

To conclude, in an irreducible representation of the Poincaré group, one labels the states through the eigenvalues of a complete set of commuting operators, which can be chosen as the two Casimir operators M^2 and j^2, the LF trimomentum $\tilde{P} = (P^+, \mathbf{P}_{\perp})$ and the third component of $\boldsymbol{j}_{\rm LF}$.

[3] Note that any two such boosts B_1 and B_2 are connected by a rotation (the so-called "Melosh rotation"), because $R_{21}(P) = B_2^{-1}(P)B_1(P)$ maps the standard vector $(M, \mathbf{0})$ into itself.

B.2 LF spinors

In this section we construct explicitly the spinors associated to the eigenstates of the LF spin operator $j^3_{\rm LF}$ for a free spin-$\frac{1}{2}$ field.

The momentum four-vector and the angular momentum four-tensor of a free spin-$\frac{1}{2}$ field are given by:

$$P^\mu = \int_\Sigma \mathrm{d}\sigma_\rho\, \overline{\psi}\gamma^\rho i\partial^\mu\psi \tag{B.2.1}$$

$$M^{\mu\nu} = \int_\Sigma \mathrm{d}\sigma_\rho \left[x^\mu\overline{\psi}\gamma^\rho i\partial^\nu\psi - x^\nu\overline{\psi}\gamma^\rho i\partial^\mu\psi + \overline{\psi}\gamma^\rho\frac{1}{2}\sigma^{\mu\nu}\psi\right], \tag{B.2.2}$$

where Σ is a space-like hypersurface. If Σ is chosen to be the LF surface $x^+ = 0$, we have:

$$P^\mu = \int \mathrm{d}^3\tilde{x}\, \psi^{(+)\dagger} i\partial^\mu\psi^{(+)} \tag{B.2.3}$$

$$M^{\mu\nu} = \int \mathrm{d}^3\tilde{x} \left[\psi^{(+)\dagger}(x^\mu i\partial^\nu - x^\nu i\partial^\mu)\psi^{(+)} + \psi^{(+)\dagger}\frac{1}{2}\sigma^{\mu\nu}\psi\right], \tag{B.2.4}$$

where $\mathrm{d}^3\tilde{x} = \mathrm{d}^2\mathbf{x}_\perp \mathrm{d}x^-$ and:

$$\psi^{(\pm)} = \Lambda^{(\pm)}\psi, \tag{B.2.5}$$

with:

$$\Lambda^{(\pm)} = \frac{1}{2}\gamma^0\gamma^\pm = \frac{1 \pm \alpha^3}{2}, \tag{B.2.6}$$

$$\Lambda^{(\pm)}\Lambda^{(\mp)} = 0, \tag{B.2.7}$$

$$\Lambda^{(+)} + \Lambda^{(-)} = 1. \tag{B.2.8}$$

In particular, we have (cf. Subsec. 3.2.2):

$$J^3 = M^{12} = \int \mathrm{d}^3\tilde{x} \left[\psi^{(+)\dagger}(x^1 i\partial^2 - x^2 i\partial^1)\psi^{(+)} + \psi^{(+)\dagger}\frac{1}{2}\sigma^{12}\psi^{(+)}\right], \tag{B.2.9}$$

$$B^k = M^{+k} = \int \mathrm{d}^3\tilde{x}\psi^{(+)\dagger}(-x^k i\partial^+)\psi^{(+)}. \tag{B.2.10}$$

Notice that P^μ, J^3 and $\mathbf{B}_\perp$ are given directly as functionals of the projected field $\psi^{(+)}$, which is the only independent component of the field in the LF framework (see e.g. [42]).

We may expand the free Dirac field in terms of annihilation and creation operators:

$$\psi(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{\mathrm{d}^3\tilde{p}\,\theta(p^+)}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{m}{p_+}} \left[b(p,s)u(p,s)e^{-ipx} + d^\dagger(p,s)v(p,s)e^{ipx}\right], \tag{B.2.11}$$

$$b(p,s) = (2\pi)^{-\frac{3}{2}}\sqrt{\frac{p^+}{m}} \int \frac{\mathrm{d}^3\tilde{x}}{2}\, e^{i\tilde{p}\tilde{x}}\overline{u}(p,s)\psi(x^+ = 0, \tilde{x}), \tag{B.2.12}$$

$$d(p,s) = -(2\pi)^{-\frac{3}{2}}\sqrt{\frac{p^+}{m}} \int \frac{\mathrm{d}^3\tilde{x}}{2}\, e^{i\tilde{p}\tilde{x}}\overline{\psi}(x^+ = 0, \tilde{x})v(p,s), \tag{B.2.13}$$

where $p^- = \frac{m^2+\mathbf{p}_\perp^2}{p^+}$ is understood and:

$$\left\{b(p,s), b^\dagger(q,s)\right\} = \left\{d(p,s), d^\dagger(q,s)\right\} = \delta^3(\tilde{p} - \tilde{q})\delta_{rs}, \tag{B.2.14}$$

all other anticommutators being zero. The spinors u and v satisfy:

$$(\not{p} - m)u(p,s) = 0, \qquad (\not{p} + m)v(p,s) = 0 \tag{B.2.15}$$

and are normalized according to:

$$\overline{u}(p,r)u(p,s) = -\overline{v}(p,r)v(p,s) = \delta_{rs}. \tag{B.2.16}$$

The corresponding expansion for $\psi^{(+)}$ is:

$$\psi^{(+)}(x) = \sum_{s=\pm\frac{1}{2}} \int_{p^+>0} \frac{\mathrm{d}^3\tilde{p}}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{m}{p^+}} \left[b(p,s)u^{(+)}(p,s)e^{-ipx} + d^\dagger(p,s)v^{(+)}(p,s)e^{ipx} \right], \tag{B.2.17}$$

with:

$$u^{(+)}(p,s) = \Lambda^{(+)}u(p,s), \qquad v^{(+)}(p,s) = \Lambda^{(+)}v(p,s). \tag{B.2.18}$$

We want to determine a basis of spinors $u(p,s)$ and $v(p,s)$ in such a way that the states $|b(p,s)\rangle \equiv \sqrt{(2\pi)^3 2p^+} b^\dagger(p,s)|0\rangle$ are eigenstates of the LF spin (B.1.18).

Since $|b(p,s)\rangle$ are eigenstates of the four momentum P^μ, we have:

$$j^3_{\mathrm{LF}}|b(p,s)\rangle = J^3|b(p,s)\rangle + \hat{\boldsymbol{z}} \cdot (\mathbf{B}_\perp \times \frac{\mathbf{p}_\perp}{p^+})|b(p,s)\rangle. \tag{B.2.19}$$

Let us decompose the total angular momentum J^3 in an orbital and a spin part:

$$J^3 = L^3 + S^3, \tag{B.2.20}$$

$$L^3 = -i \int \mathrm{d}^3\tilde{x} \left[\psi^{(+)\dagger} \hat{\boldsymbol{z}} \cdot (\mathbf{x}_\perp \times \boldsymbol{\partial}_\perp) \psi^{(+)} \right], \tag{B.2.21}$$

$$S^3 = \int \mathrm{d}^3\tilde{x} \psi^{(+)\dagger} \frac{1}{2} \sigma^{12} \psi^{(+)}. \tag{B.2.22}$$

A tedious but straightforward calculation shows that:

$$L^3|b(p,s)\rangle = -\hat{\boldsymbol{z}} \cdot (\mathbf{B}_\perp \times \frac{\mathbf{p}_\perp}{p^+})|b(p,s)\rangle = \hat{\boldsymbol{z}} \cdot (\frac{i\partial}{\partial \mathbf{p}_\perp} \times \mathbf{p}_\perp)|b(p,s)\rangle. \tag{B.2.23}$$

$$j^3_{\mathrm{LF}}|b(p,s)\rangle = S^3|b(p,s)\rangle = \sum_{r=\pm\frac{1}{2}} |b(p,r)\rangle \frac{2m}{p^+} u^{(+)}(p,r)^\dagger \frac{\sigma^{12}}{2} u^{(+)}(p,s). \tag{B.2.24}$$

Since the normalization (B.2.16) and the definition (B.2.18) imply:

$$u^{(+)}(p,r)^\dagger u^{(+)}(p,s) = \delta_{rs} \frac{p^+}{2m}, \tag{B.2.25}$$

we see that $|b(p,s)\rangle$ are eigenstates of j^3_{LF} provided that:

$$\frac{\sigma^{12}}{2} u^{(+)}(p,s) = \lambda_s u^{(+)}(p,s). \tag{B.2.26}$$

The spinors $u^{(+)}$ are also subject to the constraint:

$$\Lambda^{(+)} u^{(+)}(p,s) = u^{(+)}(p,s), \tag{B.2.27}$$

given the definition (B.2.28) and the nature of the projector $\Lambda^{(+)}$ (see Eqs. (B.2.7)-(B.2.8)). There are two linearly independent solutions:

$$u^{(+)}(p, \pm\frac{1}{2}) = \sqrt{\frac{p^+}{2m}} X_{\pm\frac{1}{2}}, \tag{B.2.28}$$

where:

$$X_{+\frac{1}{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad X_{-\frac{1}{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \tag{B.2.29}$$

corresponding to the eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$ respectively. Observe that, apart from the trivial factor $\frac{1}{\sqrt{2m}}$, which comes from our choice of normalization (B.2.16), the projected LF spinors (B.2.28) are independent from the mass of the field. Indeed, a glance at the expansion (B.2.17) reveals that $\psi^{(+)}$, restricted on the LF surface $x^+ = 0$, is independent from the mass of the field.

To obtain the full eigenspinors, consider the Dirac equation:

$$(i\frac{\gamma^+}{2}\partial^- + i\frac{\gamma^-}{2}\partial^+ - i\boldsymbol{\gamma}_\perp \cdot \boldsymbol{\partial}_\perp - m)\psi = 0. \tag{B.2.30}$$

By projecting with $\Lambda^{(\pm)}$, we obtain the two equations:

$$i\partial^-\psi^{(+)} = (-i\boldsymbol{\alpha}_\perp \cdot \boldsymbol{\nabla}_\perp + \beta m)\psi^{(-)}, \tag{B.2.31}$$

$$i\partial^+\psi^{(-)} = (-i\boldsymbol{\alpha}_\perp \cdot \boldsymbol{\nabla}_\perp + \beta m)\psi^{(+)}. \tag{B.2.32}$$

For a spinor solution $\psi(x) = u(p,s)e^{-ipx}$, equation (B.2.32) implies:

$$p^+u^{(-)}(p,s) = (\boldsymbol{\alpha}_\perp \cdot \mathbf{p}_\perp + \beta m)u^{(+)}(p,s).$$

Therefore, the full spinors are given by:

$$u(p,s) = \frac{1}{\sqrt{2mp^+}} \left[p^+ + \boldsymbol{\alpha}_\perp \cdot \mathbf{p}_\perp + \beta m\right] X_s, \tag{B.2.33}$$

The corresponding antiparticle spinors $v(p,s)$ can be obtained from the $u(p,s)$ through charge conjugation:

$$v(p,s) = \mathscr{C}\overline{u}^T(p,s),$$

with $\mathscr{C} = i\gamma^0\gamma^2$. This results in:

$$v(p,s) = \frac{1}{\sqrt{2mp^+}} \left[p^+ + \boldsymbol{\alpha}_\perp \cdot \mathbf{p}_\perp - \beta m\right] X_{-s}. \tag{B.2.34}$$

B.3 Discrete symmetries in the LF formalism

One of the major drawbacks in the LF formalism is that the underlying rotational symmetry is made much less apparent by the choice of a particular space direction $\hat{z}$. This is reflected by the fact that, in the LF framework, only the third components J^3 and j^3_{LF}, of $\mathbf{J}$ and $\boldsymbol{j}_{\mathrm{LF}}$ respectively, are kinematical operators, while $\boldsymbol{j}_{\mathrm{LF}\perp}$ and $\mathbf{J}_\perp$ (as well as $\mathbf{J}^2$ and $\boldsymbol{j}^2_{\mathrm{LF}} = -\frac{W^2}{M^2}$) are dynamical.

The same considerations apply to the discrete symmetries of parity and time reversal, which have both the effect of exchanging the two fronts $x^+ = 0$ and $x^- = 0$, and act on the $\pm$ components of the four momentum P^μ as:

$$\mathcal{P}P^\pm\mathcal{P}^\dagger = P^\mp, \qquad \mathcal{T}P^\pm\mathcal{T}^\dagger = P^\mp$$

Thus, parity and time reversal are, in the LF formalism, both dynamical operators, and their action on the exact eigenstates of the interacting four momentum P^μ differs from their action on free Fock states, eigenstates of the free four momentum P^μ_0.

Operator O	$\mathcal{P}^\dagger O \mathcal{P}$	$\mathcal{P}^\dagger_{\rm LF} O \mathcal{P}_{\rm LF}$	$\mathcal{T}^\dagger_{\rm LF} O \mathcal{T}_{\rm LF}$
P^-	P^+	P^-	P^-
P^+	P^-	P^+	P^+
$\mathbf{P}_\perp$	$-\mathbf{P}_\perp$	$(-P^1, P^2)$	$-\mathbf{P}_\perp$
J^3	J^3	$-J^3$	$-J^3$
K^3	$-K^3$	K^3	$-K^3$
$\mathbf{B}_\perp$	$-\mathbf{S}_\perp$	$(-B^1, B^2)$	$-\mathbf{B}_\perp$
$\mathbf{S}_\perp$	$-\mathbf{B}_\perp$	$(-S^1, S^2)$	$\mathbf{S}_\perp$
$j^3_{\rm LF}$	$J^3 + \varepsilon_{ij} S^i P^j$	$-j^3_{\rm LF}$	$-j^3_{\rm LF}$

Table B.3.1. behavior of Poincaré generators under parity, LF parity, and LF time reversal [41, Chapter II].

In the LF formulation, it is much more fruitful to consider light front parity and time-reversal operators, defined by [41, Chapter II]:

$$\mathcal{P}_{\rm LF} = e^{-i\pi J^1}\mathcal{P}, \tag{B.3.1}$$

$$\mathcal{T}_{\rm LF} = e^{-i\pi J^3}\mathcal{P}\mathcal{T}. \tag{B.3.2}$$

These are both kinematical operators, which act on coordinates as:

$$\left(x^-, x^+, x_1, x_2\right) \overset{\mathcal{P}_{\rm LF}}{\mapsto} \left(x^-, x^+, -x_1, x_2\right), \tag{B.3.3}$$

$$\left(x^-, x^+, x_1, x_2\right) \overset{\mathcal{T}_{\rm LF}}{\mapsto} \left(-x^-, -x^+, x_1, -x_2\right). \tag{B.3.4}$$

The action of $\mathcal{P}_{\rm LF}$ and $\mathcal{T}_{\rm LF}$ on the Poincaré generators is summarized in Table B.3.1, where we show for comparison also the action of the ordinary parity operator.

The action of the LF parity and time-reversal on one-particle states (here λ is the eigenvalue with respect to the LF-spin $j^3_{\rm LF}$) is given by [41, Chapter II]:

$$\mathcal{P}_{\rm LF}|p^+, p^1, p^2, \lambda\rangle = \eta_{\mathcal{P}}(-i)^{2j}|p^+, -p^1, p^2, -\lambda\rangle, \tag{B.3.5}$$

$$\mathcal{T}_{\rm LF}|p^+, p^1, p^2, \lambda\rangle = \eta_{\mathcal{P}}\eta_{\mathcal{T}}(+i)^{2j}|p^+, -p^1, -p^2, -\lambda\rangle, \tag{B.3.6}$$

where $\eta_{\mathcal{P}}$ and $\eta_{\mathcal{T}}$ are the (ordinary) intrinsic parity and T-parity respectively, and j is the spin of the particle in question.

Finally, we note that the space-time inversion $\mathcal{I} \equiv \mathcal{P}\mathcal{T}$ can be expressed in terms of $\mathcal{T}_{\rm LF}$ and J^3 as:

$$\mathcal{I} = e^{i\pi J^3}\mathcal{T}_{\rm LF} \tag{B.3.7}$$

and is therefore a kinematical operator, a result which was used extensively throughout this Thesis.

B.4 Frame independence and symmetries of the LFWFs

The state $|b; P, s\rangle$ transforms under LF boosts as [38]:

$$e^{-i\omega K^3} e^{-i\boldsymbol{\alpha}_\perp \cdot \mathbf{B}_\perp}|b; P, s\rangle = |b; P', s\rangle. \tag{B.4.1}$$

where

$$P'^+ = e^\omega P^+, \qquad \mathbf{P}'_\perp = \mathbf{P}_\perp + \boldsymbol{\alpha}_\perp P^+. \tag{B.4.2}$$

Eqs. (B.4.1)-(B.4.2) follow from the commutation rules of the Poincaré algebra (see, e.g., Ref. [40]). In particular, the LF helicity s is invariant under LF boosts, as a consequence of:

$$\left[K^3, j^3_{\rm LF}\right] = \left[\mathbf{B}_\perp, j^3_{\rm LF}\right] = 0.$$

Notice that the scaling property of the $\pm$ components under longitudinal boosts is nothing but a rephrasing of the standard Lorentz boost of a four-vector:

$$\begin{cases} A'^0 &= \cosh\omega A^0 + \sinh\omega A^3 \\ A'^3 &= \sinh\omega A^0 + \cosh\omega A^3 \end{cases} \iff A'^{\pm} = e^{\pm\omega} A^{\pm},$$

which, incidentally, shows explicitly that longitudinal boosts fix the light-front $x^+ = 0$.

On the other hand, since the generators K^3 and $\mathbf{B}_\perp$ are interaction independent, from the definitions (3.2.35), (3.2.36) and (3.2.37) we immediately obtain:

$$e^{-i\omega K^3} e^{-i\boldsymbol{\alpha}_\perp\cdot\mathbf{B}_\perp} |2/P; \xi\mathbf{k}_\perp\lambda_1\lambda_2\rangle = |2/P'; \xi\mathbf{k}_\perp\lambda_1\lambda_2\rangle, \tag{B.4.3}$$

where the LF helicity of the constituents has also been taken into account. Equations (B.4.1), (B.4.2) and (B.4.3) clearly imply that the valence LFWFs appearing in Eqs. (3.2.39) and (4.1.30) are independent of P, the bound state four-momentum. It is worth to emphasize that such a feature is unique to LF quantum field theory: the two fundamental pieces of information are that the LF boosts generated by K^3 and $\mathbf{B}_\perp$ are kinematical, and that they act transitively on the positive mass-shell $\{P^2 = M^2,\ P^0 > 0\}$, as is clear from (B.4.2). The same argument applies to higher Fock components wave functions.

Let us now, for simplicity, limit ourselves to a scalar bound state of scalar constituents. We choose our reference frame as the rest frame of the bound state b. In this frame, its Fock expansion begins with:

$$|b; P_{\rm CM}\rangle = \int_0^1 \frac{{\rm d}\xi}{(2\pi)2\xi(1-\xi)} \int \frac{{\rm d}^2\mathbf{k}_\perp}{(2\pi)^2} |2/P_{\rm CM}; \xi\mathbf{k}_\perp\rangle \Psi_2(\xi, \mathbf{k}_\perp) + \cdots,$$

where $(P_{\rm CM})^\pm = M$, $(\mathbf{P}_{\rm CM})_\perp = \mathbf{0}$. Since the bound state $|b\rangle$ is spinless, by applying a rotation $e^{-i\phi J_z}$ about the z-axis, which has the effect:

$$e^{-i\phi J_z} |2/P_{\rm CM}; \xi, k_x, k_y\rangle = |2/P_{\rm CM}; \xi, \cos\phi k_x - \sin\phi k_y, \sin\phi k_x + \cos\phi k_y\rangle,$$

we see that the LFWF depends only upon $\gamma \equiv \mathbf{k}^2_\perp$.

To conclude, we consider the implications of parity and time reversal invariance. We assume that the operator $\mathcal{PT}$ satisfies:

$$\mathcal{PT}\, a^\dagger_{1,2}(\tilde{p}) (\mathcal{PT})^\dagger = a^\dagger_{1,2}(\tilde{p}), \tag{B.4.4}$$

where $a^\dagger_{1,2}$ are the creation operators for the constituents. We have omitted a possible phase factor in the right hand side of (B.4.4), which can always be absorbed in the definition of the a's by using the antiunitary character of $\mathcal{PT}$.

Applying $\mathcal{PT}$ to both sides of Eq. (3.2.34), we see that:

$$|b; P\rangle = \int_0^1 \frac{{\rm d}\xi}{(2\pi)2\xi(1-\xi)} \int \frac{{\rm d}^2\mathbf{k}_\perp}{(2\pi)^2} |2/P; \xi\mathbf{k}_\perp\rangle \Psi^*_2(\xi, \mathbf{k}_\perp) + \cdots \tag{B.4.5}$$

which implies:

$$\Psi_2(\xi, \mathbf{k}_\perp) = \Psi^*_2(\xi, \mathbf{k}_\perp), \tag{B.4.6}$$

i.e., the valence wave-function is *real.*

Appendix C

Analytic derivations for the fermion-scalar BSE

In this appendix we present our calculations for the fermion-scalar BSE in full detail.

In Sec. C.1 we decompose the BS amplitude and the conjugated amplitude in its independent Dirac structures allowed by parity invariance. Assuming time-reversal invariance as well, we show that the same two invariant amplitudes enter both the decompositions. An integral equation for these amplitudes is derived under the approximations employed in our numerical work.

In Sec. C.2 we obtain the integral equations satisfied by the Nakanishi weight functions of the invariant amplitudes, through the LF projection of the BSE.

In Sec. C.3 we explicitly write the normalization condition of the BS amplitude in terms of the Nakanishi weight functions.

In Sec. C.4 we discuss the Wick-rotated BSE, and reduce it to a two-dimensional integral equation on a bounded domain.

Finally, Sec. C.5 contains some mathematical tools employed in the derivations of the previous Sections.

In what follows, Dirac matrices are in the usual Pauli representation:

$$\gamma^0 = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} O & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & O \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} O & I \\ I & O \end{pmatrix},$$

and spinors are normalized according to:

$$\sum_{s=\pm\frac{1}{2}} u(p,s)\overline{u}(p,s) = \frac{\not{p} + m}{2m}.$$

C.1 The BS amplitude for a fermion-scalar system

In this Section, we derive some formal properties of the fermion-scalar BS amplitude. The results of Subsecs. C.1.1 and C.1.2 are valid under the assumption of parity and parity plus time-reversal invariance respectively, but are in any other respect completely general. In Subsec. C.1.3, on the other hand, we specialize to the case studied in our numerical work.

C.1.1 General form of the amplitude

The BS amplitude for a bound state of spin $j = \frac{1}{2}$ composed by a fermion and a scalar, has the following general structure (see Table 2.1.1 for the definitions of coordinate and momentum space

variables):

$$\begin{aligned}\chi_s(k;P) \equiv& \int \mathrm{d}^4x\, e^{ikx}\langle 0|\mathrm{T}\left\{\psi(\eta_2 x)\phi(-\eta_1 x)\right\}|b;P,s\rangle \\ =&\Lambda(k,P)u(P,s)\end{aligned} \tag{C.1.1}$$

where ψ and ϕ are the fermion and scalar fields respectively, $u(P,s)$ is the Dirac spinor corresponding to the state $|b;P,s\rangle$, and $\Lambda(k,P)$ is a 4×4 matrix, which can be expressed through the Dirac basis and the four-momenta at disposal as follows:

$$\Lambda(k,P) = \phi_S + \phi_P\gamma^5 + (\phi_{V,1}\frac{k_\mu}{M} + \phi_{V,2}\frac{P_\mu}{M})\gamma^\mu + (\phi_{A,1}\frac{k_\mu}{M} + \phi_{A,2}\frac{P_\mu}{M})\gamma^\mu\gamma^5 + \phi_T\,\frac{k_\mu P_\nu}{M^2}\sigma^{\mu\nu}. \tag{C.1.2}$$

For the BS amplitude to transform properly under Lorentz transformations, the functions $\phi_{(\cdot)}$ must depend only upon the invariants which one can construct from the two vectors k^μ and P^μ, i.e. Pk and k^2 ($P^2 = M^2$ is understood).

We can reduce the number of independent amplitudes to two, by taking into account parity invariance. As discussed in Subsec. 4.1.1, we assume that the intrinsic parity of the ground state $|b\rangle$ coincides with the product of the intrinsic parities of ψ and ϕ, defined by:

$$\mathcal{P}^\dagger\psi(x)\mathcal{P} = \pi_\psi\gamma^0\psi(x_P), \tag{C.1.3}$$

$$\mathcal{P}^\dagger\phi(x)\mathcal{P} = \pi_\phi\phi(x_P), \tag{C.1.4}$$

where $x_\mathcal{P} = (x^0, -\mathbf{x})$, $P_\mathcal{P} = (P^0, -\mathbf{P})$. For the purpose of this and the following derivations (Subsec. C.1.2), we may assume that $|b;P,s\rangle$ represents an eigenstate of the helicity $h = \frac{\mathbf{P}\cdot\mathbf{J}}{|\mathbf{P}|}$, so that:

$$\mathcal{P}|b;P,s\rangle = \pi_\psi\pi_\phi|b;P_\mathcal{P},-s\rangle. \tag{C.1.5}$$

The corresponding spinors $u(P,s)$ are given by:

$$u(P,s) = \sqrt{\frac{P^0+M}{2M}}\begin{pmatrix}\varphi_s(\hat{\mathbf{P}})\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{P}}{P^0+M}\varphi_s(\hat{\mathbf{P}})\end{pmatrix}, \qquad \boldsymbol{\sigma}\cdot\hat{\mathbf{P}}\varphi_{\pm\frac{1}{2}}(\hat{\mathbf{P}}) = \pm\varphi_{\pm\frac{1}{2}}(\hat{\mathbf{P}}), \tag{C.1.6}$$

and satisfy:

$$u(P_\mathcal{P},-s) = \gamma^0 u(P,s), \tag{C.1.7}$$

provided $\varphi_{\pm\frac{1}{2}}(\hat{\mathbf{P}}) = \varphi_{\mp\frac{1}{2}}(-\hat{\mathbf{P}})$. Using Eqs. (C.1.3), (C.1.4), (C.1.5) and (C.1.7), one can see that the matrix $\Lambda(k,P)$ must satisfy:

$$\Lambda(k,P)u(P,s) = \gamma^0\Lambda(k_\mathcal{P},P_\mathcal{P})\gamma^0 u(P,s). \tag{C.1.8}$$

which implies[1]:

$$\Lambda(k,P) = \gamma^0\Lambda(k_\mathcal{P},P_\mathcal{P})\gamma^0. \tag{C.1.9}$$

In turn, this requires:

$$\phi_P = \phi_{A,1} = \phi_{A,2} \equiv 0. \tag{C.1.10}$$

Observe now that the spinor $u(P,s)$ satisfies:

$$\begin{aligned}\not{P}u(P,s) &= Mu(P,s),\\ k_\mu P_\nu\sigma^{\mu\nu}u(P,s) &= i(M\not{k} - Pk)u(P,s).\end{aligned}$$

[1]This argument actually requires some care, since the matrix $\Lambda(k,P)$ is defined by Eq. (C.1.1) only up to a term of the form $\Lambda_1(k,P)\frac{\not{P}-M}{2M}$. A more careful procedure would be to define uniquely $\Lambda(k,P)$ by requiring, e.g. $\Lambda(k,P)\frac{\not{P}+M}{2M} = \Lambda(k,P)$, which leads to some additional relations between the amplitudes $\phi_{(\cdot)}$. For such a matrix, which satisfies automatically $\Lambda(k,P)v(P,s) = \gamma^0\Lambda(k_\mathcal{P},P_\mathcal{P})\gamma^0 v(P,s) = 0$, Eq. (C.1.8) implies (C.1.9).

Therefore, we can write the BS amplitude as:

$$\chi_s(k;P) = \left[\phi_1(k^2,Pk) + \phi_2(k^2,Pk)\frac{\not{k}}{M}\right] u(P,s), \tag{C.1.11}$$

where the dependence of the various quantities upon the arguments k, P and s has been made explicit, and:

$$\phi_1 = \phi_S + \phi_{V,2} - i\frac{Pk}{M^2}\phi_T, \tag{C.1.12}$$

$$\phi_2 = \phi_{V,1} + i\phi_T. \tag{C.1.13}$$

C.1.2 The conjugate amplitude

The conjugate BS amplitude is defined by:

$$\overline{\chi}_s(q;P) \equiv \int \mathrm{d}^4y\, e^{-iqy}\langle b;P,s|\mathrm{T}\{\overline{\psi}(\eta_2 y)\phi^\dagger(-\eta_1 y)\}|0\rangle \tag{C.1.14}$$

(notice the different signs of the exponentials in Eqs. (C.1.1) and (C.1.14)). When $\mathcal{PT}$ is a good symmetry of the theory, there exists a simple relation between χ and $\overline{\chi}$, which may be derived as follows.

We employ the following conventions[2] for the $\mathcal{PT}$ operator (see, e.g., [**?**]):

$$\mathcal{PT}|b;P,s\rangle = (-1)^{\frac{1}{2}-s}|b;P,-s\rangle, \tag{C.1.15}$$

$$\mathcal{PT}\psi(x)(\mathcal{PT})^\dagger = \gamma^1\gamma^3\gamma^0\psi(-x), \tag{C.1.16}$$

$$\mathcal{PT}\phi(x)(\mathcal{PT})^\dagger = \phi(-x). \tag{C.1.17}$$

Recalling that $\mathcal{PT}$ is antiunitary, from these we easily obtain:

$$\langle 0|\mathrm{T}\left\{\psi(\eta_2 y)\phi(-\eta_1 y)\right\}|b;P,s\rangle = (-1)^{\frac{1}{2}-s}\gamma^1\gamma^3\langle b;P,-s|\mathrm{T}\{\overline{\psi}(-\eta_2 y)\phi^\dagger(\eta_1 y)\}|0\rangle^T. \tag{C.1.18}$$

From the definition (C.1.14) we find, in momentum space:

$$\chi_s(q;P) = \gamma^1\gamma^3(-1)^{\frac{1}{2}-s}\overline{\chi}_{-s}(q;P)^T \tag{C.1.19}$$

When $\mathcal{P}$ and $\mathcal{T}$ are separately conserved, we may write the conjugate amplitude in the form:

$$\overline{\chi}_s(q;P) = \overline{u}(P,s)\left[\overline{\phi}_1(q^2,Pq) + \overline{\phi}_2(q^2,Pq)\frac{\not{q}}{M}\right], \tag{C.1.20}$$

where the amplitudes $\overline{\phi}_i$ are, *a priori*, different from ϕ_i. Since $\not{q}^T = \gamma^1\gamma^3\not{q}\gamma^3\gamma^1$ we have:

$$\gamma^1\gamma^3\overline{\chi}_{-s}(q;P)^T = \left[\overline{\phi}_1(q^2,Pq) + \overline{\phi}_2(q^2,Pq)\frac{\not{q}}{M}\right](-1)^{\frac{1}{2}-s}\gamma^1\gamma^3\overline{u}(P,-s)^T. \tag{C.1.21}$$

By appropriately choosing a common global phase for $u(P,\pm\frac{1}{2})$, we may assume that:

$$\gamma^1\gamma^3\overline{u}(P,-s)^T = (-1)^{\frac{1}{2}-s}u(P,s). \tag{C.1.22}$$

[2]This can always be done, given the antiunitary character of $\mathcal{PT}$; e.g. if $\mathcal{PT}\phi(x)(\mathcal{PT})^\dagger = e^{2i\theta}\phi(-x)$, we define the new field $\phi_1(x) = e^{i\theta}\phi(x)$, which satisfies $\mathcal{PT}\phi_1(x)(\mathcal{PT})^\dagger = \phi_1(-x)$.

The factor $(-1)^{\frac{1}{2}-s}$ is consistent with our choice (C.1.15) for the $\mathcal{PT}$ transformation of the bound state[3]. With these conventions, Eqs. (C.1.19) and (C.1.21) imply:

$$\left[\overline{\phi}_1(q^2,Pq)+\overline{\phi}_2(q^2,Pq)\frac{\not{q}}{M}\right]u(P,s)=\left[\phi_1(q^2,Pq)+\phi_2(q^2,Pq)\frac{\not{q}}{M}\right]u(P,s)\qquad(s=1,2),\quad\text{(C.1.23)}$$

which in turn yields $\overline{\phi}_i\equiv\phi_i$, since ϕ_1 and ϕ_2 are uniquely defined by Eq. (C.1.11) (see the next Subsection, Eqs. (C.1.30) and (C.1.31)). In conclusion:

$$\overline{\chi}_s(q;P)=\overline{u}(P,s)\left[\phi_1(q^2,Pq)+\phi_2(q^2,Pq)\frac{\not{q}}{M}\right].\tag{C.1.24}$$

C.1.3 Ladder coupled system for the invariant amplitudes

The relation between the BS amplitude and the invariant amplitudes $\phi_{1,2}$, Eq. (C.1.11), can be inverted by using the projection identity:

$$\sum_{s=\pm\frac{1}{2}}u(P,s)\overline{u}(P,s)=\frac{\not{P}+M}{2M}.\tag{C.1.25}$$

This yields:

$$T_1\equiv\mathrm{Tr}\left\{\sum_{s=1,2}\chi_s(k;P)\overline{u}(P,s)\right\}\tag{C.1.26}$$

$$=2\phi_1+2\frac{Pk}{M^2}\phi_2,\tag{C.1.27}$$

$$T_2\equiv\mathrm{Tr}\left\{\frac{\not{k}}{M}\sum_{s=1,2}\chi_s(k;P)\overline{u}(P,s)\right\}\tag{C.1.28}$$

$$=2\frac{Pk}{M^2}\phi_1+2\frac{k^2}{M^2}\phi_2.\tag{C.1.29}$$

Solving for ϕ_1 and ϕ_2 we obtain:

$$\phi_1=\frac{1}{2}\frac{M^2k^2T_1-M^2P\cdot k\,T_2}{M^2k^2-(P\cdot k)^2},\tag{C.1.30}$$

$$\phi_2=\frac{1}{2}\frac{M^4T_2-M^2P\cdot k\,T_1}{M^2k^2-(P\cdot k)^2}.\tag{C.1.31}$$

Observe that $k^2M^2-(P\cdot k)^2$ is proportional to the determinant of the above linear system, and it vanishes if $k\propto P$. As we will show below, the determinant actually drops out from the equations for the Nakanishi weight functions of $\phi_{1,2}$.

Eqs. (C.1.30) and (C.1.31) can be used to derive a coupled system of integral equations satisfied by the invariant amplitudes $\phi_{1,2}$. We write the BSE for the fermion-scalar system under study as:

$$\chi_s(k;P)=\frac{i}{\eta_1\not{P}+\not{k}-m_\psi+i\varepsilon}\frac{i}{(\eta_2P-k)^2-m_\phi^2+i\varepsilon}\int\frac{\mathrm{d}^4k'}{(2\pi)^4}iK(k,k';P)\chi_s(k';P),\tag{C.1.32}$$

[3]For a shorthand derivation of Eq. (C.1.22), one can consider the asymptotic fields $b_{\mathrm{in/out}}(x)$ of the bound state, which transform like:

$$\mathcal{PT}b_{\mathrm{in}}(x)(\mathcal{PT})^\dagger=\eta\gamma^1\gamma^3b_{\mathrm{out}}(-x),$$

under $\mathcal{PT}$ transformations. The phase choice $\eta=1$, together with our convention (C.1.15), yields Eq. (C.1.22).

where self-energy corrections are neglected. To obtain the corresponding equations for the invariant amplitudes amplitudes $\phi_{1,2}$, we first need equations for the traces T_1 and T_2 defined by (C.1.26) and (C.1.28). Plugging the BSE (C.1.32) into the definitions (C.1.26) and (C.1.28) we obtain:

$$T_j = \frac{i}{(\eta_1 P + k)^2 - m_\psi^2 + i\varepsilon} \frac{i}{(\eta_2 P - k)^2 - m_\phi^2 + i\varepsilon} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} S_j(k, k', P), \tag{C.1.33}$$

with:

$$S_1(k, k', P) \equiv \mathrm{Tr}\left\{ (\eta_1 \not{P} + \not{k} + m_\psi) iK(k, k'; P)(\phi_1 + \phi_2 \frac{\not{k}'}{M}) \frac{\not{P} + M}{2M} \right\}, \tag{C.1.34}$$

$$S_2(k, k', P) \equiv \mathrm{Tr}\left\{ \frac{\not{k}}{M} (\eta_1 \not{P} + \not{k} + m_\psi) iK(k, k'; P)(\phi_1 + \phi_2 \frac{\not{k}'}{M}) \frac{\not{P} + M}{2M} \right\}. \tag{C.1.35}$$

Then, from (C.1.30) and (C.1.31), we obtain:

$$\phi_1 = \frac{i}{(\eta_1 P + k)^2 - m_\psi^2 + i\varepsilon} \frac{i}{(\eta_2 P - k)^2 - m_\phi^2 + i\varepsilon} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{1}{2} \frac{M^2 k^2 S_1 - M^2 P \cdot k\, S_2}{M^2 k^2 - (P \cdot k)^2}, \tag{C.1.36}$$

$$\phi_2 = \frac{i}{(\eta_1 P + k)^2 - m_\psi^2 + i\varepsilon} \frac{i}{(\eta_1 P - k)^2 - m_\phi^2 + i\varepsilon} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{1}{2} \frac{M^4 S_2 - M^2 P \cdot k\, S_1}{M^2 k^2 - (P \cdot k)^2}. \tag{C.1.37}$$

We now take iK as the one-scalar exchange ladder kernel:

$$iK(k, k'; P) = i\lambda_\psi \frac{i}{(k - k')^2 - \mu^2 + i\varepsilon} i\lambda_\phi. \tag{C.1.38}$$

The traces (C.1.34) and (C.1.35) read

$$S_j(k, k', P) = i\lambda_\psi \frac{i}{(k - k')^2 - \mu^2 + i\varepsilon} i\lambda_\phi \mathscr{T}_j(k, k', P), \tag{C.1.39}$$

where the $\mathscr{T}_j$'s are given in Subsec. C.5.1. Substituting eqs. (C.1.39) in (C.1.36) and (C.1.37), and using the formulas (C.5.3) and (C.5.4), we obtain:

$$\phi_i(k^2, Pk) = \frac{i\lambda_\psi \lambda_\phi}{\Big[(\eta_1 P + k)^2 - m_\psi^2 + i\varepsilon\Big]\Big[(\eta_2 P - k)^2 - m_\phi^2 + i\varepsilon\Big]} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{\sum_{j=1,2} C_{ij}(k, k', P)\phi_j(k'^2, Pk')}{(k - k')^2 - \mu^2 + i\varepsilon}, \tag{C.1.40}$$

The coefficients C_{ij} are the ones tabulated in Subsec. C.5.1 and rewritten here, for convenience, with $\eta_1 = \eta_2 = \frac{1}{2}$:

$$C_{11} = \frac{M}{2} + m_\psi, \tag{C.1.41}$$

$$C_{12} = \frac{(\frac{M}{2} + m_\psi)k^2(Pk') + (\frac{M}{2} - m_\psi)(Pk)(kk') + Mk^2(kk') - \frac{1}{M}(Pk)^2(Pk') - \frac{k^2}{M}(Pk)(Pk')}{M^2 k^2 - (Pk)^2}, \tag{C.1.42}$$

$$C_{21} = M, \tag{C.1.43}$$

$$C_{22} = \frac{Mk^2(Pk') + (\frac{M}{2} - m_\psi)(Pk)(Pk') + (m_\psi - \frac{M}{2})M^2(kk') - M(Pk)(kk')}{M^2 k^2 - (P \cdot k)^2}, \tag{C.1.44}$$

C.2 NIR and LF projection

To solve the coupled system (C.1.40) for the invariant amplitudes, we let $\eta_1 = \eta_2 = \frac{1}{2}$ and assume the following NIR for $\phi_{1,2}$:

$$\phi_i(k^2, Pk) = i \int_{-1}^{1} \mathrm{d}z' \int_{0}^{+\infty} \mathrm{d}\gamma' \frac{g_i(\gamma', z')}{\left[(k + z'\frac{P}{2})^2 - (1 - z'^2)\kappa^2 - (z'\overline{m} - \Delta)^2 - \gamma' + i\varepsilon\right]^3}. \tag{C.2.1}$$

Plugging Eq. (C.2.1) into (C.1.40), we obtain:

$$-i\phi_i(k^2, Pk) = \int_{-1}^{1} \mathrm{d}z' \int_{0}^{+\infty} \mathrm{d}\gamma' \sum_{j=1,2} \widetilde{V}_{ij}(k^2, Pk, \gamma', z') g_j(\gamma', z'), \tag{C.2.2}$$

where ϕ_i on the LHS is expressed in terms of g_i through Eq. (C.2.1), and:

$$\begin{aligned}\widetilde{V}_{ij}(k^2, Pk, \gamma', z') \equiv & \frac{i\lambda_\psi \lambda_\phi}{\left[(\frac{P}{2} + k)^2 - m_\psi^2 + i\varepsilon\right]\left[(\frac{P}{2} - k)^2 - m_\phi^2 + i\varepsilon\right]} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{1}{[(k - k')^2 - \mu^2 + i\varepsilon]} \times \\ & \times \frac{C_{ij}(k, k', P)}{\left[(k' + z'\frac{P}{2})^2 - (1 - z'^2)\kappa^2 - (z'\overline{m} - \Delta)^2 - \gamma' + i\varepsilon\right]^3}.\end{aligned} \tag{C.2.3}$$

Applying the LF projection, i.e. integrating over k^-, we obtain:

$$\int_{-1}^{1} \mathrm{d}z' \int_{0}^{+\infty} \mathrm{d}\gamma'\, B(\gamma, z; \gamma', z') g_i(\gamma', z') = \int_{-1}^{1} \mathrm{d}z' \int_{0}^{+\infty} \mathrm{d}\gamma' \sum_{j=1}^{2} V_{ij}(\gamma', z'; \gamma', z') g_j(\gamma', z'), \tag{C.2.4}$$

where:

$$B(\gamma, z; \gamma', z') \equiv iP^+ \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^-}{2\pi} \frac{1}{\left[(k + z'\frac{P}{2})^2 - (1 - z'^2)\kappa^2 - (z'\overline{m} - \Delta)^2 - \gamma' + i\varepsilon\right]^3}, \tag{C.2.5}$$

$$V_{ij}(\gamma, z; \gamma', z') \equiv iP^+ \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^-}{2\pi} \widetilde{V}_{ij}(k^2, Pk, \gamma', z'), \tag{C.2.6}$$

and the factors iP^+ are introduced for later convenience. Noteworthy, the projected kernels B and V_{ij} depend only upon the LF variables defined by:

$$\gamma \equiv \mathbf{k}_\perp^2, \tag{C.2.7}$$

$$z \equiv -\frac{2k^+}{M} = 1 - 2\xi. \tag{C.2.8}$$

The kernel B in Eq. (C.2.5) has been already computed in Subsec. 3.2.4, and it is given by:

$$B(\gamma, z; \gamma', z') = \frac{\delta(z - z')}{[\gamma' + D_0(\gamma, z)]^2}. \tag{C.2.9}$$

Here the function:

$$D_0(\gamma, z) \equiv \gamma + (1 - z^2)\kappa^2 + (z\overline{m} - \Delta)^2. \tag{C.2.10}$$

The kernel V_{ij} requires considerably more work. We make the C_{ij}'s dependence upon k' explicit by writing:

$$C_{ij} = C_{ij}^{(0)} + C_{ij}^{(1)\mu} k'_\mu. \tag{C.2.11}$$

To perform the k' integral in Eq. C.2.3, we use the Feynman parametrization (see Subsec. C.5.2):

$$\frac{1}{A^3 B} = 3 \int_0^1 v^2 \, \mathrm{d}v \frac{1}{\{vA + (1-v)B\}^4}.$$

The evaluation of the k' integral is thus reduced to the evaluation of the two integrals:

$$I = \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{1}{\left\{ v \left[(k' + z' \frac{P}{2})^2 - (1 - z'^2)\kappa^2 - (z'\overline{m} - \Delta)^2 - \gamma' \right] + (1-v)\left[(k-k')^2 - \mu^2 \right] + i\varepsilon \right\}^4}, \tag{C.2.12}$$

$$I^\mu = \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{k'^\mu}{\left\{ v \left[(k' + z' \frac{P}{2})^2 - (1 - z'^2)\kappa^2 - (z'\overline{m} - \Delta)^2 - \gamma' \right] + (1-v)\left[(k-k')^2 - \mu^2 \right] + i\varepsilon \right\}^4}. \tag{C.2.13}$$

We rewrite the two denominators in (C.2.12) and (C.2.13) as:

$$\{ \cdots \}^4 \equiv \left\{ q^2 - s \right\}^4,$$

where:

$$q \equiv k' + \frac{1}{2}\left[v z' P - 2(1-v)k \right] \tag{C.2.14}$$

$$s \equiv v(\kappa^2 + \Delta^2 - 2z'\overline{m}\Delta + \gamma') + (1-v)\mu^2 + v^2 z'^2 \frac{M^2}{4} - v(1-v)k^2 - v(1-v)z' P \cdot k \tag{C.2.15}$$

Making this change of variables, we can evaluate (C.2.12) and (C.2.13) through the standard integrals given in Subsec. C.5.3. We obtain:

$$I = \frac{i}{6(4\pi)^2} \frac{1}{(s - i\varepsilon)^2}, \tag{C.2.16}$$

$$I^\mu = \frac{i}{6(4\pi)^2} \frac{t^\mu}{(s - i\varepsilon)^2}, \tag{C.2.17}$$

with:

$$t \equiv (1-v)k - \frac{1}{2} v z' P. \tag{C.2.18}$$

Coming back to Eq. (C.2.3), we have:

$$\widetilde{V}_{ij} = -\frac{\lambda_\psi \lambda_\phi}{2(4\pi)^2} \int_0^1 v^2 \mathrm{d}v \frac{1}{\left[(\frac{P}{2} + k)^2 - m_\psi^2 + i\varepsilon \right] \left[(\frac{P}{2} - k)^2 - m_\phi^2 + i\varepsilon \right]} \times$$
$$\times \frac{C_{ij}^{(0)} + C_{ij}^{(1)\mu} t_\mu}{\left[-v(\kappa^2 + \Delta^2 - 2z'\overline{m}\Delta + \gamma') - (1-v)\mu^2 - v^2 z'^2 \frac{M^2}{4} + v(1-v)k^2 + v(1-v)z' P \cdot k + i\varepsilon \right]^2}. \tag{C.2.19}$$

To simplify the following calculations, we boost to the rest frame of the bound state, where $\mathbf{P}_\perp = \mathbf{0}$ and $P^\pm = M$. Defining:

$$k_D^+ = v(1-v)\frac{M}{2}(z'-z), \tag{C.2.20}$$

$$l_D = -v(\kappa^2 + \Delta^2 - 2z'\overline{m}\Delta + \gamma') - (1-v)\mu^2 - v(1-v)(\gamma + \frac{M^2}{4}z'z) - v^2 z'^2\frac{M^2}{4}, \tag{C.2.21}$$

$$k_d^- = -\frac{M}{2} + \frac{2}{M(1-z)}(\gamma + m_\psi^2), \tag{C.2.22}$$

$$k_u^- = \frac{M}{2} - \frac{2}{M(1+z)}(\gamma + m_\phi^2), \tag{C.2.23}$$

and writing:

$$C_{ij}^{(0)} + C_{ij}^{(1)\mu}t_\mu = c_{ij}^{(0)} + c_{ij}^{(1)}k^-,$$

we have:

$$V_{ij}(\gamma, z; \gamma', z') = \frac{2i\lambda_\psi\lambda_\phi}{(4\pi)^2}\frac{1}{M}\int_0^1 v^2\mathrm{d}v \int_{-\infty}^{+\infty}\frac{\mathrm{d}k^-}{2\pi}\times$$
$$\times \frac{c_{ij}^{(0)} + c_{ij}^{(1)}k^-}{\Big[(1-z)(k^- - k_d^-) + i\varepsilon\Big]\Big[(1+z)(k^- - k_u^-) - i\varepsilon\Big]\Big[k^- k_D^+ + l_D + i\varepsilon\Big]^2}. \tag{C.2.24}$$

Notice that, for $z^2 > 1$, the above integral vanishes, as can be seen by contour integration. For $z^2 \leq 1$ employing the J integrals of Subsec. C.5.4, we obtain:

$$V_{ij}(\gamma, z; \gamma', z') = \frac{\lambda_\psi\lambda_\phi}{2(4\pi)^2}\frac{1}{D_0(\gamma, z)}\int_0^1 v^2\mathrm{d}v\left(\theta(k_D^+)\frac{c_{ij}^{(0)} + c_{ij}^{(1)}k_u^-}{\Big[k_u^- k_D^+ + l_D\Big]^2} + \theta(-k_D^+)\frac{c_{ij}^{(0)} + c_{ij}^{(1)}k_d^-}{\Big[k_d^- k_D^+ + l_D\Big]^2}\right). \tag{C.2.25}$$

Defining the coupling constant by (see Appendix D):

$$\lambda_\phi\lambda_\psi = 8\pi m_\phi\alpha, \tag{C.2.26}$$

we finally obtain the integral equation satisfied by the Nakanishi weight functions:

$$\int_0^{+\infty} B(\gamma, z; \gamma')g_i(\gamma', z) = \alpha\int_{-1}^{+1}\mathrm{d}z'\int_0^\infty \mathrm{d}\gamma'\sum_{j=1}^2 A_{ij}(\gamma, z; \gamma', z')g_j(\gamma', z') \tag{C.2.27}$$

with the kernels:

$$B(\gamma, z; \gamma') = \frac{1}{[\gamma' + D_0(\gamma, z)]^2}, \tag{C.2.28}$$

$$A_{ij}(\gamma, z; \gamma', z') \equiv \frac{m_\phi}{4\pi D_0(\gamma, z)}\int_0^1 v^2\mathrm{d}v\left(\theta(z'-z)\frac{c_{ij}^{(0)} + c_{ij}^{(1)}k_u^-}{\Big[k_u^- k_D^+ + l_D\Big]^2} + \theta(z-z')\frac{c_{ij}^{(0)} + c_{ij}^{(1)}k_d^-}{\Big[k_d^- k_D^+ + l_D\Big]^2}\right). \tag{C.2.29}$$

The coefficients $c_{ij}^{(k)}$ are explicitly given by $c_{11}^{(1)} = c_{21}^{(1)} = c_{22}^{(1)} = 0$ and:

$$\begin{cases} c_{11}^{(0)} &= \frac{M}{2} + m_\psi, \\ c_{12}^{(0)} &= -(1-v)(\frac{z}{4}M + \frac{\gamma}{M}) - (\frac{M}{2} + m_\psi)\frac{vz'}{2}, \\ c_{21}^{(0)} &= M, \\ c_{22}^{(0)} &= (1-v)(m_\psi - \frac{M}{2}) - \frac{vz'}{2}M, \\ c_{12}^{(1)} &= \frac{1}{2}(1-v)(1-z). \end{cases} \tag{C.2.30}$$

C.3 Normalization condition

The normalization condition for the BS amplitude reads (cf. Subsecs. 2.1.3 and 2.2.3):

$$\lim_{P^0\to\omega(\mathbf{P})} i\overline{\chi}_r \frac{(G_0^{-1}-iK)}{P^2-M^2}\chi_s = \delta_{sr}, \tag{C.3.1}$$

where s and r denote the spin arguments. Explicitly:

$$\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\overline{\chi}_r(q;P)N(q,k;P)\chi_s(k;P) = -4iM^2\delta_{sr}, \tag{C.3.2}$$

where:

$$N \equiv 2P^\mu \left.\frac{\partial(G_0^{-1}-iK)}{\partial P^\mu}\right|_{P^0=\omega(\mathbf{P})}. \tag{C.3.3}$$

For ladder kernel, with free propagators in G_0, we have:

$$G_0^{-1}(q,k;P) = (2\pi)^4\delta^4(k-q)\left[\frac{\not{P}}{2}+\not{k}-m_\psi\right]\left[m_\phi^2+Pk-\frac{P^2}{4}-k^2\right], \tag{C.3.4}$$

$$N(q,k;P) = \left[m_\phi^2-\frac{3}{4}M^2-k^2+2Pk\right]\not{P}+(2Pk-M^2)(\not{k}-m_\psi) \tag{C.3.5}$$

Introducing the two parametrizations (C.1.11) and (C.1.24), we obtain:

$$\sum_{j,l=1,2} I_{jl}^{rs} = -4i\delta_{sr}M^2, \tag{C.3.6}$$

where:

$$I_{11}^{rs} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\phi_1(k^2,Pk)\phi_1(k^2,Pk)\overline{u}(P,r)N(k;P)u(P,s) \tag{C.3.7}$$

$$I_{12}^{rs} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\phi_1(k^2,Pk)\phi_2(k^2,Pk)\overline{u}(P,r)N(k;P)\frac{\not{k}}{M}u(P,s), \tag{C.3.8}$$

$$I_{21}^{rs} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\phi_2(k^2,Pk)\phi_1(k^2,Pk)\overline{u}(P,r)\frac{\not{k}}{M}N(k;P)u(P,s). \tag{C.3.9}$$

$$I_{22}^{rs} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\phi_2(k^2,Pk)\phi_2(k^2,Pk)\overline{u}(P,r)\frac{\not{k}}{M}N(k;P)\frac{\not{k}}{M}u(P,s). \tag{C.3.10}$$

Using $\not{P}u(P,s) = Mu(P,s)$ and:

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}F(k^2,Pk)\,k^\mu = P^\mu\int \frac{\mathrm{d}^4k}{(2\pi)^4}F(k^2,Pk)\,\frac{Pk}{M^2},$$

which follows from Lorentz invariance, we find:

$$I_{11}^{rs} = \delta_{rs}M\int \frac{\mathrm{d}^4k}{(2\pi)^4}\phi_1(k^2,Pk)\phi_1(k^2,Pk)\left\{Mm_\psi+m_\phi^2-\frac{3}{4}M^2+(1-\frac{2m_\psi}{M})Pk+\frac{2(Pk)^2}{M^2}-k^2\right\},$$

$$I_{12}^{rs} = I_{21}^{rs} = \delta_{rs}M\int \frac{\mathrm{d}^4k}{(2\pi)^4}\phi_1(k^2,Pk)\phi_2(k^2,Pk)\left\{\left[Mm_\psi+m_\phi^2-\frac{3}{4}M^2\right]\frac{Pk}{M^2}+\right.$$
$$\left.+(1-\frac{m_\psi}{M})\frac{2(Pk)^2}{M^2}-k^2+k^2\frac{Pk}{M^2}\right\}$$

$$I_{22}^{rs} = \delta_{rs}M\int \frac{\mathrm{d}^4k}{(2\pi)^4}\phi_2(k^2,Pk)\phi_2(k^2,Pk)\left\{(\frac{m_\phi^2}{M^2}-\frac{3}{4})\frac{2(Pk)^2}{M^2}+\frac{4(Pk)^3}{M^4}+(\frac{3}{4}+\frac{m_\psi}{M}-\frac{m_\phi^2}{M^2})k^2+\right.$$
$$\left.+(-3-2\frac{m_\psi}{M})k^2\frac{Pk}{M^2}+\frac{(k^2)^2}{M^2}\right\}.$$

These integrals can be evaluated as explained in Subsec. C.5.5. Omitting the common δ_{rs} factors, we have:

$$\frac{i}{4M^2}I_{11} = \left[m_\phi^2 + m_\psi M - \frac{3}{4}M^2\right]\frac{A_{11}}{4M} + (\frac{M^2}{2} - m_\psi M)\frac{2P_\mu}{M^2}\frac{A_{11}^\mu}{4M} + 2\frac{P_\mu P_\nu}{M^2}\frac{A_{11}^{\mu\nu}}{4M} - \frac{B_{11}}{4M},$$
$$\frac{i}{4M^2}I_{12} = \frac{1}{2}\left[m_\phi^2 + m_\psi M - \frac{3}{4}M^2\right]\frac{2P_\mu}{M^2}\frac{A_{12}^\mu}{4M} + (1 - \frac{m_\psi}{M})\frac{2P_\mu P_\nu}{M^2}\frac{A_{12}^{\mu\nu}}{4M} - \frac{B_{12}}{4M} + \frac{1}{2}\frac{2P_\mu}{M^2}\frac{B_{12}^\mu}{4M},$$
$$\frac{i}{4M^2}I_{22} = \left[\frac{m_\phi^2}{M^2} - \frac{3}{4}\right]\frac{2P_\mu P_\nu}{M^2}\frac{A_{22}^{\mu\nu}}{4M} + \frac{4P_\mu P_\nu P_\rho}{M^4}\frac{A_{22}^{\mu\nu\rho}}{4M} + (\frac{3}{4} + \frac{m_\psi}{M} - \frac{m_\phi^2}{M^2})\frac{B_{22}}{4M} +$$
$$(-\frac{3}{2} - \frac{m_\psi}{M})\frac{2P_\mu}{M^2}\frac{B_{22}^\mu}{4M} + \frac{1}{M^2}\frac{C_{22}}{4M},$$

where the integrals A, B and C are given in Subsec. C.5.5. The normalization condition reads:

$$\sum_{j,l=1,2}\frac{i}{4M^2}I_{jl} = 1.$$

Following Subsec. C.5.5, we write $X_{ij}^{\mu\nu\cdots} = \left(g_i|\mathcal{X}_{ij}^{\mu\nu\cdots}|g_j\right)$, where the $\mathcal{X}$ are given in (C.5.49)-(C.5.55), which we now rewrite in a more convenient form. Putting:

$$\zeta = \frac{1}{2}\left[(1-v)z'' + vz'\right], \tag{C.3.11}$$
$$s = v\gamma' + (1-v)\gamma'' + \kappa^2 + \Delta^2 - 4\zeta\overline{m}\Delta + \zeta^2 M^2, \tag{C.3.12}$$

we have:

$$\mathcal{A} = \frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}v\, v^2(1-v)^2\frac{1}{s^4}\left\{\frac{3}{2}\right\}, \tag{C.3.13}$$
$$\frac{2P_\mu}{M^2}\mathcal{A}^\mu = \frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}v\, v^2(1-v)^2\frac{1}{s^4}\left\{-3\zeta\right\} \tag{C.3.14}$$
$$\frac{2P_\mu P_\nu}{M^2}\mathcal{A}^{\mu\nu} = \frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}v\, v^2(1-v)^2\frac{1}{s^4}\left\{3\zeta^2M^2 - \frac{1}{2}s\right\}, \tag{C.3.15}$$
$$\frac{4P_\mu P_\nu P_\rho}{M^4}\mathcal{A}^{\mu\nu\rho} = \frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}v\, v^2(1-v)^2\frac{1}{s^4}\left\{-6\zeta^3M^2 + 3\zeta s\right\} \tag{C.3.16}$$
$$\mathcal{B} = \frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}v\, v^2(1-v)^2\frac{1}{s^4}\left\{\frac{3}{2}\zeta^2M^2 - s\right\}, \tag{C.3.17}$$
$$\frac{2P_\mu}{M^2}\mathcal{B}^\mu = \frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}v\, v^2(1-v)^2\frac{1}{s^4}\left\{-3\zeta^3M^2 + 3\zeta s\right\}, \tag{C.3.18}$$
$$\mathcal{C} = \frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}v\, v^2(1-v)^2\frac{1}{s^4}\left\{\frac{3}{2}\zeta^4M^4 - 3\zeta^2M^2s + \frac{3}{2}s^2\right\}. \tag{C.3.19}$$

We finally obtain the normalization condition in the form:

$$\sum_{ij=1,2}(g_i|\mathcal{F}_{ij}|g_j) = 1, \tag{C.3.20}$$

where:

$$(g_i|\mathcal{F}_{ij}|g_j) = \int_{-\infty}^{+\infty} \mathrm{d}\gamma'' \int_{-1}^{+1} \mathrm{d}z'' \int_{-\infty}^{+\infty} \mathrm{d}\gamma' \int_{-1}^{+1} \mathrm{d}z' g_i(\gamma'', z'') \mathcal{F}_{ij}(\gamma'', z'', \gamma', z') g_j(\gamma', z'), \tag{C.3.21}$$

$$\mathcal{F}_{ij} = \frac{1}{(8\pi)^2 M} \int_0^1 \mathrm{d}v\, v^2 (1-v)^2 \frac{f_{ij}}{s^4}, \tag{C.3.22}$$

$$f_{11} = \frac{3}{2}\left[m_\phi^2 + m_\psi M - \frac{3}{4}M^2\right] + (3m_\psi M - \frac{3M^2}{2})\zeta + \frac{3}{2}M^2\zeta^2 + \frac{s}{2} \tag{C.3.23}$$

$$f_{12} = f_{21} = -\frac{3}{2}\left[m_\phi^2 + m_\psi M - \frac{3}{4}M^2\right]\zeta + (\frac{1}{2} - \frac{m_\psi}{M})3\zeta^2 M^2 - \frac{3}{2}\zeta^3 M^2 + \frac{1}{2}(\frac{m_\psi}{M} + 1 + 3\zeta)s, \tag{C.3.24}$$

$$f_{22} = 3\left[\frac{m_\phi^2}{2M^2} - \frac{3}{8} + \frac{m_\psi}{2M}\right]\zeta^2 M^2 + 3(\frac{m_\psi}{M} - \frac{1}{2})\zeta^3 M^2 + \frac{3}{2}\zeta^4 M^2 + \\ + \left[\frac{m_\phi^2}{2M^2} - \frac{3}{8} - \frac{m_\psi}{M} - 3(\frac{1}{2} + \frac{m_\psi}{M})\zeta - 3\zeta^2 + \frac{3s}{2M^2}\right] s \tag{C.3.25}$$

C.4 Wick-rotated BSE

We discuss in this Section the Wick-rotation of the BSE for the scalar exchange ladder kernel. This is obtained from Eq. (C.1.40) by the formal substitutions $k^0 = ik^4$, $k'^0 = ik'^4$ in the rest frame $\mathbf{P} = \mathbf{0}$:

$$\hat{\phi}_i(k) = \frac{\lambda_\psi \lambda_\phi}{\left[m_\psi^2 + \mathbf{k}^2 + (k^4 - i\eta_1 M)^2\right]\left[m_\phi^2 + \mathbf{k}^2 + (k^4 + i\eta_2 M)^2\right]} \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{\sum_{j=1,2} \hat{C}_{ij}(k, k', P)\hat{\phi}_j(k')}{(k^4 - k'^4)^2 + (\mathbf{k} - \mathbf{k}')^2 + \mu^2}, \tag{C.4.1}$$

where the Wick-rotated amplitudes are $\hat{\phi}_j(k) = \phi_j(ik^4, \mathbf{k})$ and the coefficients are:

$$\hat{C}_{11} = (\eta_1 + \frac{m_\psi}{M})M, \tag{C.4.2}$$

$$\hat{C}_{12} = \frac{-k^2(\mathbf{k}\cdot\mathbf{k}') + i(\eta_1 + \frac{m_\psi}{M})M\mathbf{k}^2 k'^4 + i(\eta_1 - \frac{m_\psi}{M})Mk^4(\mathbf{k}\cdot\mathbf{k}')}{M\mathbf{k}^2} \tag{C.4.3}$$

$$\hat{C}_{21} = M, \tag{C.4.4}$$

$$\hat{C}_{22} = \frac{(\frac{m_\psi}{M} - \eta_1)M\mathbf{k}\cdot\mathbf{k}' + i\mathbf{k}^2 k'^4 - ik^4(\mathbf{k}\cdot\mathbf{k}')}{\mathbf{k}^2}. \tag{C.4.5}$$

Notice that, in Eq. (C.4.1), we have reintroduced the two coefficients $\eta_{1,2}$, for reasons to be explained below.

Since the amplitudes $\hat{\phi}_j$ depend on k only through the invariants $k^2 = -\mathbf{k}^2 - (k^4)^2$ and $Pk = iMk^4$, they can be regarded as functions of $x = k^4$ and $y = |\mathbf{k}|$. Therefore, we can immediately perform the angular integrations in:

$$\mathrm{d}^4 k' = \mathrm{d}x' y'^2 \mathrm{d}y' \mathrm{d}\omega \mathrm{d}\varphi,$$

where $\omega = \cos\theta = \frac{\mathbf{k}\cdot\mathbf{k}'}{|\mathbf{k}||\mathbf{k}'|}$. Decomposing the coefficients as:

$$\hat{C}_{ij} = \hat{c}_{ij}^{(0)} + \hat{c}_{ij}^{(1)} \mathbf{k}\cdot\mathbf{k}',$$

and using the two integrals:

$$\int_{-1}^{1} \mathrm{d}\omega \, \frac{1}{a - b\omega} = \frac{1}{b} \ln \frac{a+b}{a-b},$$
$$\int_{-1}^{1} \mathrm{d}\omega \, \frac{\omega}{a - b\omega} = -\frac{2}{b} + \frac{a}{b^2} \ln \frac{a+b}{a-b},$$

we obtain:

$$\int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{\sum_{j=1,2} \hat{C}_{ij}(k,k',P)\hat{\phi}_j(x',y')}{(k^4 - k'^4)^2 + (\mathbf{k}-\mathbf{k}')^2 + \mu^2} = \int \frac{\mathrm{d}x'\mathrm{d}y'}{(2\pi)^3} \sum_{j=1,2} \hat{\phi}_j(x',y') \left\{ \frac{b}{4y^2} \left[\hat{c}_{ij}^{(0)} + \frac{a}{2} \hat{c}_{ij}^{(1)} \right] \ln \frac{a+b}{a-b} - y'^2 \hat{c}_{ij}^{(1)} \right\},$$

where:

$$a = (x - x')^2 + y^2 + y'^2 + \mu^2,$$
$$b = 2yy'.$$

By writing $\lambda_\phi \lambda_\psi = 8\pi m_\phi \alpha$, the Wick-rotated BSE (C.4.1) becomes:

$$\left[m_\psi^2 + y^2 + (x - i\eta_1 M)^2 \right] \left[m_\phi^2 + y^2 + (x + i\eta_2 M)^2 \right] \hat{\phi}_i(x,y) = \frac{m_\phi \alpha}{(2\pi)^2} \cdot \frac{1}{y^2} \int \mathrm{d}x'\mathrm{d}y' \sum_{j=1,2} \hat{\phi}_j(x',y') \times$$
$$\times \left\{ b \left(\hat{c}_{ij}^{(0)} + \frac{a}{2} \hat{c}_{ij}^{(1)} \right) \ln \frac{a+b}{a-b} - b^2 \hat{c}_{ij}^{(1)} \right\}, \tag{C.4.6}$$

or, in a more compact notation:

$$\hat{\phi}_i(x,y) = \alpha \int \mathrm{d}x'\mathrm{d}y' \sum_{j=1,2} \hat{V}_{ij}(x,y;x,y') \hat{\phi}_j(x',y'), \tag{C.4.7}$$

$$\hat{V}_{ij}(x,y;x',y') = \frac{\frac{m_\phi}{(2\pi)^2} \left\{ b \left(\hat{c}_{ij}^{(0)} + \frac{a}{2} \hat{c}_{ij}^{(1)} \right) \ln \frac{a+b}{a-b} - b^2 \hat{c}_{ij}^{(1)} \right\}}{y^2 \left[m_\psi^2 + y^2 + (x - i\eta_1 M)^2 \right] \left[m_\phi^2 + y^2 + (x + i\eta_2 M)^2 \right]}. \tag{C.4.8}$$

where the coefficients are given explicitly by:

$$\begin{cases} \hat{c}_{11}^{(0)} &= (\eta_1 + \frac{m_\psi}{M})M, \\ \hat{c}_{12}^{(0)} &= i(\eta_1 + \frac{m_\psi}{M})x', \\ \hat{c}_{21}^{(0)} &= M, \\ \hat{c}_{22}^{(0)} &= ix', \end{cases} \qquad \begin{cases} \hat{c}_{11}^{(1)} &= 0, \\ \hat{c}_{12}^{(1)} &= \frac{-x^2 - y^2 + i(\eta_1 - \frac{m_\psi}{M})Mx}{My^2}, \\ \hat{c}_{21}^{(1)} &= 0, \\ \hat{c}_{22}^{(1)} &= \frac{(\frac{m_\psi}{M} - \eta_1)M - ix}{y^2}. \end{cases} \tag{C.4.9}$$

To solve the system (C.4.7), we first express the two unbounded variables x and y in terms of two bounded ones $\overline{x}, \overline{y}$:

$$x = \frac{\overline{x}}{1 - |\overline{x}|}, \qquad (-1 < \overline{x} < 1), \tag{C.4.10}$$

$$y = \frac{\overline{y}}{1 - \overline{y}}, \qquad (0 \leq \overline{y} < 1), \tag{C.4.11}$$

and define the new amplitudes:

$$f_i(\overline{x}, \overline{y}) \equiv y^2 \hat{\phi}_i(x,y) \tag{C.4.12}$$

which satisfy the integral equation:

$$f_i(\overline{x},\overline{y}) = \alpha \int_{-1}^{1} \mathrm{d}\overline{x}' \int_{0}^{1} \mathrm{d}\overline{y}' \sum_{j=1,2} W_{ij}(\overline{x},\overline{y};\overline{x}',\overline{y}') f_j(\overline{x}',\overline{y}'), \tag{C.4.13}$$

$$W_{ij}(\overline{x},\overline{y};\overline{x}',\overline{y}') \equiv \frac{1}{(1-|\overline{x}'|)^2(1-\overline{y}')^2} \frac{y^2}{y'^2} \hat{V}_{ij}(x,y;x',y'). \tag{C.4.14}$$

Also, we have treated the singularities of $\frac{y^2}{y'^2}\hat{V}_{ij}$ by doing a Taylor expansion around $\frac{b}{a} \approx 0$[4]:

$$\frac{y^2}{y'^2}\hat{V}_{ij}(x,y;x,y') = \frac{\frac{m_\phi}{(2\pi)^2}\left\{\frac{8y^2}{a}(1+\frac{b^2}{3a^2})\hat{c}_{ij}^{(0)} + \frac{4b^2}{3a^2}y^2\hat{c}_{ij}^{(1)}\right\}}{\left[m_\psi^2 + y^2 + (x - i\eta_1 M)^2\right]\left[m_\phi^2 + y^2 + (x + i\eta_2 M)^2\right]} + O^6(\frac{b}{a}). \tag{C.4.15}$$

Finally, concerning the choice of the arbitrary constants $\eta_{1,2}$, we have found that, for $M \gg |m_\phi - m_\psi|$, the choice:

$$\eta_1 = \frac{M^2 + m_\psi^2 - m_\phi^2}{2M^2}, \qquad \eta_2 = \frac{M^2 + m_\phi^2 - m_\psi^2}{2M^2}, \tag{C.4.16}$$

which is motivated by the analytic solution of the Wick-Cutkosky model [26], improves somewhat the stability of the numerical solution. Notice that, with this choice:

$$m_\psi^2 - \eta_1^2 M^2 = m_\phi^2 - \eta_2^2 M^2 = f^2 \overline{m}^2,$$

where:

$$f^2 = \frac{\left[(m_\psi + m_\phi)^2 - M^2\right]\left[M^2 - (m_\psi - m_\phi)^2\right]}{4M^2} = \frac{\kappa^2(M^2 - 4\Delta^2)}{M^2},$$

so that the denominators in the propagators (see Eq. (C.4.8)) always have a non negative real part for $M > |m_\phi - m_\psi|$. When $M \lesssim |m_\phi - m_\psi|$, we have used:

$$\eta_1 = \frac{m_\psi}{m_\phi + m_\psi}, \qquad \eta_2 = \frac{m_\phi}{m_\phi + m_\psi},$$

which also leads to non-vanishing denominators, but appears to make the numerical solution less stable.

C.5 Mathematical tools

C.5.1 Trace identities

We collect here some useful trace identities for obtaining the BSE for the invariant amplitudes ϕ_i. Recall that:

$$\begin{aligned} \mathrm{Tr}(\not{A}\not{B}) &= 4AB, \\ \mathrm{Tr}(\not{A}\not{B}\not{C}\not{D}) &= 4\left[(AB)(CD) - (AC)(BD) + (AD)(BC)\right], \end{aligned}$$

and that the trace of an odd number of γ matrices vanishes. Using these facts, one easily computes:

[4] Notice that the ratio $\frac{b}{a}$ vanishes in the limits $y(y') \to 0$ and $y(y') \to \infty$, with $y'(y)$ fixed.

$$\mathscr{T}_1(k,k',P) \equiv \mathrm{Tr}\left\{(\eta_1 \not{P} + \not{k} + m_\psi)(\phi_1 + \phi_2 \frac{\not{k}'}{M})\frac{\not{P}+M}{2M}\right\} \tag{C.5.1}$$

$$= 2\phi_1\left[(\eta_1 + \frac{m_\psi}{M})M + \frac{Pk}{M}\right] + 2\phi_2\left[(\eta_1 + \frac{m_\psi}{M})\frac{Pk'}{M} + \frac{kk'}{M}\right]$$

$$\mathscr{T}_2(k,k',P) \equiv \mathrm{Tr}\left\{\frac{\not{k}}{M}(\eta_1 \not{P} + \not{k} + m_\psi)(\phi_1 + \phi_2 \frac{\not{k}'}{M})\frac{\not{P}+M}{2M}\right\} \tag{C.5.2}$$

$$= 2\phi_1\left[(\eta_1 + \frac{m_\psi}{M})\frac{Pk}{M} + \frac{k^2}{M}\right] + 2\phi_2\left[2\eta_1\frac{(Pk)(Pk')}{M^3} + \frac{k^2(Pk')}{M^3} + (\frac{m_\psi}{M} - \eta_1)\frac{kk'}{M}\right]$$

We compute the linear combinations:

$$L_1 \equiv \frac{1}{2}\frac{M^2k^2\mathscr{T}_1 - M^2(Pk)\mathscr{T}_2}{M^2k^2 - (P\cdot k)^2} = \sum_{j=1}^{2} C_{1j}(k,k',P)\phi_j, \tag{C.5.3}$$

$$L_2 \equiv \frac{1}{2}\frac{M^4\mathscr{T}_2 - M^2(Pk)\,\mathscr{T}_1}{M^2k^2 - (P\cdot k)^2} = \sum_{j=1}^{2} C_{2j}(k,k',P)\phi_j, \tag{C.5.4}$$

The coefficients C are given by:

$$C_{11} = \eta_1 M + m_\psi, \tag{C.5.5}$$

$$C_{12} = \frac{(\eta_1 M + m_\psi)k^2(Pk') + (\eta_1 M - m_\psi)(Pk)(kk') + Mk^2(kk') - \frac{2\eta_1}{M}(Pk)^2(Pk') - \frac{1}{M}k^2(Pk)(Pk')}{M^2k^2 - (Pk)^2}, \tag{C.5.6}$$

$$C_{21} = M, \tag{C.5.7}$$

$$C_{22} = \frac{Mk^2(Pk') + (\eta_1 M - m_\psi)(Pk)(Pk') + (m_\psi - \eta_1 M)M^2(kk') - M(Pk)(kk')}{M^2k^2 - (P\cdot k)^2}. \tag{C.5.8}$$

C.5.2 Feynman's parametrization

Let $z_1, z_2, \ldots, z_n$ be complex numbers with $z_i \neq 0$, and let $p_1, \ldots, p_n$ be positive integers. Then [10, Eq. (7-1)]:

$$\frac{1}{z_1^{p_1}z_2^{p_2}\cdots z_n^{p_n}} = \frac{\Gamma(\sum_{k=1}^n p_k)}{\prod_{k=1}^n \Gamma(p_k)}\prod_{k=1}^{n}\left(\int_0^1 \alpha_k^{p_k-1}\mathrm{d}\alpha_k\right)\frac{\delta(1-\sum_{k=1}^n \alpha_k)}{[\sum_{k=1}^n \alpha_k z_k]^{\sum_{k=1}^n p_k}}. \tag{C.5.9}$$

Special cases:

$$\frac{1}{z_1 z_2 \cdots z_n} = (n-1)!\prod_{k=1}^{n}\left(\int_0^1 \mathrm{d}\alpha_k\right)\frac{\delta(1-\sum_{k=1}^n \alpha_k)}{[\sum_{k=1}^n \alpha_k z_k]^n}, \tag{C.5.10}$$

$$\frac{1}{z^p w^q} = \frac{(p+q-1)!}{(p-1)!(q-1)!}\int_0^1 \alpha^{p-1}\mathrm{d}\alpha \int_0^1 \beta^{q-1}\mathrm{d}\beta \frac{\delta(1-\alpha-\beta)}{[\alpha z + \beta w]^{p+q}}. \tag{C.5.11}$$

C.5.3 Standard 4-dimensional integrals

Let s be a real number and n a positive integer. Then [58, Appendix A]:

$$I_n(s) = \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{1}{(q^2 - s + i\varepsilon)^n} = i\frac{(-1)^n}{(4\pi)^2} \frac{1}{(n-1)(n-2)} \frac{1}{(s - i\varepsilon)^{n-2}} \qquad (n \geq 3), \tag{C.5.12}$$

$$I_n^{\mu\nu}(s) = \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{q^\mu q^\nu}{(q^2 - s + i\varepsilon)^n} = \frac{i}{2}\frac{(-1)^{n-1}}{(4\pi)^2} \frac{1}{(n-1)(n-2)(n-3)} \frac{g^{\mu\nu}}{(s - i\varepsilon)^{n-3}} \qquad (n \geq 4), \tag{C.5.13}$$

and all integrals with an odd number of q factors in the numerator vanish.

C.5.4 J integrals

Define the integrals:

$$J_n = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^-}{2\pi} \frac{1}{\Big[(1-z)(k^- - k_d^-) + i\varepsilon\Big]\Big[(1+z)(k^- - k_u^-) - i\varepsilon\Big]\Big[k^- k_D^+ + l_D + i\varepsilon\Big]^n}, \quad (n \geq 1) \tag{C.5.14}$$

$$J_0 = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^-}{2\pi} \frac{\mathrm{Ln}(k^- k_D^+ + l_D + i\varepsilon)}{\Big[(1-z)(k^- - k_d^-) + i\varepsilon\Big]\Big[(1+z)(k^- - k_u^-) - i\varepsilon\Big]}, \tag{C.5.15}$$

where:

$$\mathrm{ImLn}(\zeta) \in [0, 2\pi)\,. \tag{C.5.16}$$

We prove the formulas:

$$J_n = \frac{M}{4}\frac{1}{iD_0(\gamma,z)}\left\{\frac{\theta(k_D^+)}{\Big[k_u^- k_D^+ + l_D\Big]^n} + \frac{\theta(-k_D^+)}{\Big[k_d^- k_D^+ + l_D\Big]^n}\right\}, \qquad (n \geq 1) \tag{C.5.17}$$

$$J_0 = \frac{M}{4}\frac{1}{iD_0(\gamma,z)}\Big\{\theta(k_D^+)\mathrm{Ln}(k_u^- k_D^+ + l_D) + \theta(-k_D^+)\mathrm{Ln}(k_d^- k_D^+ + l_D)\Big\}, \tag{C.5.18}$$

with

$$D_0(\gamma,z) = \frac{M}{4}(1-z^2)(k_d^- - k_u^-) \tag{C.5.19}$$

$$= \gamma + (z\overline{m} - \Delta)^2 + (1-z^2)\kappa^2 \tag{C.5.20}$$

and where it can be shown that:

$$k_u^- k_D^+ + l_D > 0 \quad (k_D^+ > 0)$$
$$k_d^- k_D^+ + l_D > 0 \quad (k_D^+ < 0)$$

Before proving Eqs. (C.5.17) and (C.5.18), let us note that these imply, for the integrals defined by:

$$J_1^1 \equiv \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^-}{2\pi} \frac{k^-}{\Big[(1-z)(k^- - k_d^-) + i\varepsilon\Big]\Big[(1+z)(k^- - k_u^-) - i\varepsilon\Big]\Big[k^- k_D^+ + l_D + i\varepsilon\Big]}, \tag{C.5.21}$$

$$J_2^1 \equiv \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^-}{2\pi} \frac{k^-}{\Big[(1-z)(k^- - k_d^-) + i\varepsilon\Big]\Big[(1+z)(k^- - k_u^-) - i\varepsilon\Big]\Big[k^- k_D^+ + l_D + i\varepsilon\Big]^2}, \tag{C.5.22}$$

$$J_2^2 \equiv \int_{-\infty}^{+\infty} \frac{\mathrm{d}k^-}{2\pi} \frac{(k^-)^2}{\Big[(1-z)(k^- - k_d^-) + i\varepsilon\Big]\Big[(1+z)(k^- - k_u^-) - i\varepsilon\Big]\Big[k^- k_D^+ + l_D + i\varepsilon\Big]^2}, \tag{C.5.23}$$

the explicit formulas:

$$J_1^1 = \frac{M}{4}\frac{1}{iD_0(\gamma,z)}\left\{\frac{k_u^-\theta(k_D^+)}{k_u^-k_D^+ + l_D} + \frac{k_d^-\theta(-k_D^+)}{k_d^-k_D^+ + l_D}\right\} \tag{C.5.24}$$

$$J_2^1 = \frac{M}{4}\frac{1}{iD_0(\gamma,z)}\left\{\frac{k_u^-\theta(k_D^+)}{\left[k_u^-k_D^+ + l_D\right]^2} + \frac{k_d^-\theta(-k_D^+)}{\left[k_d^-k_D^+ + l_D\right]^2}\right\} \tag{C.5.25}$$

$$J_2^2 = \frac{M}{4}\frac{1}{iD_0(\gamma,z)}\left\{\frac{(k_u^-)^2\theta(k_D^+)}{\left[k_u^-k_D^+ + l_D\right]^2} + \frac{(k_d^-)^2\theta(-k_D^+)}{\left[k_d^-k_D^+ + l_D\right]^2}\right\} - \frac{i}{1-z^2}\frac{\delta(k_D^+)}{l_D} \tag{C.5.26}$$

Eqs. (C.5.21) and (C.5.22) are obtained by differentiating Eqs. (C.5.18) and (C.5.17) (for $n = 1$) with respect to k_D^+. Eq. (C.5.23) is obtained in the same way from Eq. (C.5.21).

Consider first the integral (C.5.14). If $k_D^+ > 0$, we can evaluate J_n by closing the contour in the upper plane:

$$J_n(k_D^+ > 0) = \frac{M}{4}\frac{1}{iD_0(\gamma,z)}\frac{1}{\left[k_u^-k_D^+ + l_D\right]^n},$$

Similarly, if $k_D^+ < 0$:

$$J_n(k_D^+ < 0) = \frac{M}{4}\frac{1}{iD_0(\gamma,z)}\frac{1}{\left[k_d^-k_D^+ + l_D\right]^n}.$$

If $k_D^+ = 0$ but $z^2 \neq 1$, the integral can, again, be evaluated as contour integral, with the result:

$$J_n(k_D^+ = 0) = \frac{M}{4}\frac{1}{iD_0(\gamma,z)l_D^n} = \lim_{k_D^+\to 0^\pm} J_n(k_D^+).$$

To conclude, suppose $z = 1$ and $k_D^+ = 0$. Then, using the identities:

$$\frac{1}{x - i\varepsilon} = \mathrm{P}\frac{1}{x} + i\pi\delta(x), \qquad \mathrm{P}\int_{-\infty}^{+\infty}\mathrm{d}x\,\frac{1}{x} = 0,$$

we obtain:

$$J_n(z = 1, k_D^+ = 0) = \frac{M}{4}\frac{1}{iD_0(\gamma,1)l_D^n} = \lim_{k_D^+\to 0^\pm} J_n(z = 1, k_D^+).$$

Similarly, for $z = -1$, one finds the correct limit of J_n, and Eq. (C.5.17) is proved.

The integral J_0 requires a little more attention. Notice that the (monodrome) function Ln, with the determination (C.5.16), has a cut along the positive x axis. When we apply the residue theorem to compute (C.5.15), we must pay attention to the argument $\zeta = k^-k_D + l_D + i\varepsilon$ of Ln. Indeed, when $k_D^+ > 0$, we compute the integral by closing the arc in the upper k^- plane, so $\mathrm{Im}\zeta > 0$. When $k_D^+ < 0$, we compute the integral by closing in the lower k^- plane, so $\mathrm{Im}\zeta > 0$ again. In any case, we do never cross the cut, so that we may regard $\mathrm{Ln}\zeta$ as an analytic function and apply the residue theorem. The singular cases $k_D^+ = 0$ and/or $z^2 = 1$ can be treated as above.

C.5.5 Integrals for the normalization condition

In Sec. C.3, we encountered the following integrals:

$$A_{ij} = i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\phi_i(k^2,Pk)\phi_j(k^2,Pk), \tag{C.5.27}$$

$$A^{\mu}_{ij} = i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\phi_i(k^2,Pk)\phi_j(k^2,Pk)k^{\mu}, \tag{C.5.28}$$

$$A^{\mu\nu}_{ij} = i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\phi_i(k^2,Pk)\phi_j(k^2,Pk)k^{\mu}k^{\nu}, \tag{C.5.29}$$

$$A^{\mu\nu\rho}_{ij} = i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\phi_i(k^2,Pk)\phi_j(k^2,Pk)k^{\mu}k^{\nu}k^{\rho}, \tag{C.5.30}$$

$$B_{ij} = i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\phi_i(k^2,Pk)\phi_j(k^2,Pk)k^2, \tag{C.5.31}$$

$$B^{\mu}_{ij} = i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\phi_i(k^2,Pk)\phi_j(k^2,Pk)k^2k^{\mu}, \tag{C.5.32}$$

$$C_{ij} = i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\phi_i(k^2,Pk)\phi_j(k^2,Pk)(k^2)^2. \tag{C.5.33}$$

Introducing the NIR of the scalar amplitudes ϕ_j and denoting by $X^{\mu\nu\rho\tau\cdots}_{ij}$ any one of the integrals (C.5.27)-(C.5.33), we have:

$$X^{\mu\nu\rho\tau\cdots}_{ij} = \int_0^{\infty}\mathrm{d}\gamma''\int_{-1}^{1}\mathrm{d}z''\int_0^{\infty}\mathrm{d}\gamma'\int_{-1}^{1}\mathrm{d}z' g_i(\gamma'',z'')\mathcal{X}^{\mu\nu\rho\tau}(z'',\gamma'',z',\gamma')g_j(\gamma',z') \equiv (g_i|\mathcal{X}^{\mu\nu\rho\tau}|g_j) \tag{C.5.34}$$

where:

$$\mathcal{X}^{\mu\nu\rho\tau\cdots}(z'',\gamma'',z',\gamma') \equiv \frac{1}{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^{\mu}k^{\nu}k^{\rho}k^{\tau}\cdots}{\left[D_1(\gamma'',z'')+i\varepsilon\right]^3\left[D_1(\gamma',z')+i\varepsilon\right]^3},$$
$$D_1(\gamma,z) \equiv k^2 + zP\cdot k - \kappa^2 - \Delta^2 + 2z\overline{m}\Delta - \gamma$$

Also, by introducing a Feynman parametrization, we have:

$$\mathcal{X}^{\mu\nu\rho\tau\cdots}(z'',\gamma'',z',\gamma') = 30\int_0^1 \mathrm{d}v\, v^2(1-v)^2 x^{\mu\nu\rho\tau\cdots},$$

with:

$$x^{\mu\nu\rho\tau\cdots} = \frac{1}{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^{\mu}k^{\nu}k^{\rho}k^{\tau}\cdots}{\left[D_2(\gamma'',z'',\gamma',z',v)+i\varepsilon\right]^6},$$
$$D_2 \equiv k^2 + \left[(1-v)z''+vz'\right]P\cdot k - \kappa^2 - (1-v)\gamma'' - v\gamma' - \Delta^2 + 2\left[(1-v)z''+vz'\right]\overline{m}\Delta$$

Letting:

$$t^{\mu} = -\zeta P^{\mu},$$
$$\zeta = \frac{1}{2}\left[(1-v)z''+vz'\right]$$
$$s = v\gamma' + (1-v)\gamma'' + \kappa^2 + \Delta^2 - 4\zeta\overline{m}\Delta + \zeta^2M^2,$$

and making the shift $k^\mu \to k^\mu + t^\mu$, we have:

$$x^{\mu\nu\rho\tau\cdots} = \frac{1}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k^\mu+t^\mu)(k^\nu+t^\nu)(k^\rho+t^\rho)(k^\tau+t^\tau)\cdots}{\left[k^2-s+i\varepsilon\right]^6}.$$

Notice that $s>0$, since $0\le\zeta\le\frac{1}{2}$ and:

$$\kappa^2+\Delta^2-4\zeta\overline{m}\Delta+\zeta^2M^2 = (1-4\zeta^2)\kappa^2+(\Delta-2\zeta\overline{m})^2.$$

In particular, we need to evaluate:

$$a = \frac{1}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{\left[k^2-s+i\varepsilon\right]^6}, \tag{C.5.35}$$

$$a^\mu = \frac{1}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^\mu+t^\mu}{\left[k^2-s+i\varepsilon\right]^6}, \tag{C.5.36}$$

$$a^{\mu\nu} = \frac{1}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k^\mu+t^\mu)(k^\nu+t^\nu)}{\left[k^2-s+i\varepsilon\right]^6}, \tag{C.5.37}$$

$$a^{\mu\nu\rho} = \frac{1}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k^\mu+t^\mu)(k^\nu+t^\nu)(k^\rho+t^\rho)}{\left[k^2-s+i\varepsilon\right]^6}, \tag{C.5.38}$$

$$b = \frac{1}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k+t)^2}{\left[k^2-s+i\varepsilon\right]^6}, \tag{C.5.39}$$

$$b^\mu = \frac{1}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k+t)^2(k^\mu+t^\mu)}{\left[k^2-s+i\varepsilon\right]^6}, \tag{C.5.40}$$

$$c = \frac{1}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k+t)^4}{\left[k^2-s+i\varepsilon\right]^6}. \tag{C.5.41}$$

Let us define the basic integrals:

$$\alpha = \frac{1}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{\left[k^2-s+i\varepsilon\right]^6}$$

$$\beta = \frac{1}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^2}{\left[k^2-s+i\varepsilon\right]^6}$$

$$\gamma = \frac{1}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^4}{\left[k^2-s+i\varepsilon\right]^6}$$

In terms of these, we have:

$$\begin{aligned}
a &= \alpha\\
a^\mu &= \alpha t^\mu,\\
a^{\mu\nu} &= \alpha t^\mu t^\nu + \frac{\beta}{4}g^{\mu\nu},\\
a^{\mu\nu\rho} &= \alpha t^\mu t^\nu t^\rho + \frac{\beta}{4}(g^{\mu\nu}t^\rho + g^{\nu\rho}t^\mu + g^{\rho\mu}t^\nu),\\
b &= \alpha t^2 + \beta,\\
b^\mu &= \alpha t^2 t^\mu + \frac{3}{2}\beta t^\mu,\\
c &= \alpha t^4 + 3\beta t^2 + \gamma.
\end{aligned}$$

To solve for α, β and γ, set:

$$\alpha_n(\theta) = \frac{1}{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{\left[\theta k^2 - s + i\varepsilon\right]^n} = \theta^{-2}\alpha_n(1).$$

We have:

$$\begin{aligned}
\alpha &= \alpha_6(1),\\
\beta &= -\frac{1}{5}\left.\frac{\partial}{\partial\theta}\right|_{\theta=1}\alpha_5(\theta) = \frac{2}{5}\alpha_5(1),\\
\gamma &= \frac{1}{20}\left.\frac{\partial^2}{\partial\theta^2}\right|_{\theta=1}\alpha_4(\theta) = \frac{3}{10}\alpha_4(1).
\end{aligned}$$

From Subsec. C.5.3, we have:

$$\alpha_n(1) = \frac{1}{(4\pi)^2}\frac{(-1)^n}{(n-1)(n-2)}\frac{1}{s^{n-2}}.$$

Hence:

$$\begin{aligned}
\alpha &= \frac{1}{(4\pi)^2}\frac{\frac{1}{20}}{s^4},\\
\beta &= \frac{1}{(4\pi)^2}\frac{-\frac{1}{30}s}{s^4},\\
\gamma &= \frac{1}{(4\pi)^2}\frac{\frac{1}{20}s^2}{s^4},
\end{aligned}$$

and:

$$a = \frac{1}{(4\pi)^2}\frac{1}{s^4}\left\{\frac{1}{20}\right\} \tag{C.5.42}$$

$$a^\mu = \frac{1}{(4\pi)^2}\frac{1}{s^4}\left\{\frac{1}{20}t^\mu\right\} \tag{C.5.43}$$

$$a^{\mu\nu} = \frac{1}{(4\pi)^2}\frac{1}{s^4}\left\{\frac{1}{20}t^\mu t^\nu - \frac{1}{120}g^{\mu\nu}s\right\} \tag{C.5.44}$$

$$a^{\mu\nu\rho} = \frac{1}{(4\pi)^2}\frac{1}{s^4}\left\{\frac{1}{20}t^\mu t^\nu t^\rho - \frac{1}{120}(g^{\mu\nu}t^\rho + g^{\nu\rho}t^\mu + g^{\rho\mu}t^\nu)s\right\} \tag{C.5.45}$$

$$b = \frac{1}{(4\pi)^2}\frac{1}{s^4}\left\{\frac{1}{20}t^2 - \frac{1}{30}s\right\} \tag{C.5.46}$$

$$b^\mu = \frac{1}{(4\pi)^2}\frac{1}{s^4}\left\{\frac{1}{20}t^2t^\mu - \frac{1}{20}st^\mu\right\} \tag{C.5.47}$$

$$c = \frac{1}{(4\pi)^2}\frac{1}{s^4}\left\{\frac{1}{20}t^4 - \frac{1}{10}t^2s + \frac{1}{20}s^2\right\} \tag{C.5.48}$$

Finally, recalling that $t^\mu = -\zeta P^\mu$:

$$\mathcal{A} = \frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}v\, v^2(1-v)^2\frac{1}{s^4}\left\{\frac{3}{2}\right\}, \tag{C.5.49}$$

$$\mathcal{A}^\mu = \frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}v\, v^2(1-v)^2\frac{1}{s^4}\left\{-\frac{3}{2}\zeta P^\mu\right\} \tag{C.5.50}$$

$$\mathcal{A}^{\mu\nu} = \frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}v\, v^2(1-v)^2\frac{1}{s^4}\left\{\frac{3}{2}\zeta^2 P^\mu P^\nu - \frac{1}{4}g^{\mu\nu}s\right\}, \tag{C.5.51}$$

$$\mathcal{A}^{\mu\nu\rho} = \frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}v\, v^2(1-v)^2\frac{1}{s^4}\left\{-\frac{3}{2}\zeta^3 P^\mu P^\nu P^\rho + \frac{\zeta}{4}(g^{\mu\nu}P^\rho + g^{\nu\rho}P^\mu + g^{\rho\mu}P^\nu)s\right\} \tag{C.5.52}$$

$$\mathcal{B} = \frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}v\, v^2(1-v)^2\frac{1}{s^4}\left\{\frac{3}{2}\zeta^2 M^2 - s\right\}, \tag{C.5.53}$$

$$\mathcal{B}^\mu = \frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}v\, v^2(1-v)^2\frac{1}{s^4}\left\{-\frac{3}{2}\zeta^3 M^2 P^\mu + \frac{3}{2}\zeta P^\mu s\right\}, \tag{C.5.54}$$

$$\mathcal{C} = \frac{1}{(4\pi)^2}\int_0^1 \mathrm{d}v\, v^2(1-v)^2\frac{1}{s^4}\left\{\frac{3}{2}\zeta^4 M^4 - 3\zeta^2 M^2 s + \frac{3}{2}s^2\right\}. \tag{C.5.55}$$

C.5.6 LF integrals

In this Subsection we prove a distributional identity, the so-called LF integral, which is particularly useful for LF projection calculations.

Theorem. *Let $n \geq 1$ and suppose $Im z \neq 0$. Then:*

$$\int_{-\infty}^{+\infty}\frac{d\beta}{2\pi}\frac{1}{[\beta x + z]^{n+1}} = \begin{cases} -\frac{i}{nz^n}\delta(x) & \text{if } Im z > 0, \\ +\frac{i}{nz^n}\delta(x) & \text{if } Im z < 0. \end{cases} \tag{C.5.56}$$

The distributional identity (C.5.56) *is valid for any test function which is bounded and continuous at $x = 0$.*

Proof. Define:

$$D^{(n)}_{z,B}(x) \equiv \int_{-B}^{+B}\mathrm{d}\beta\frac{1}{[\beta x + z]^{n+1}} = \frac{2B}{n}\sum_{k=0}^{n-1}(z+Bx)^{-(n-k)}(z-Bx)^{-(k+1)}.$$

If $f(x)$ is as stated above, then:

$$\begin{aligned}\left\langle D^{(n)}_{z,B}, f\right\rangle &= \frac{2B}{n}\int_{-\infty}^{+\infty}\mathrm{d}x\, f(x)\sum_{k=0}^{n-1}(z+Bx)^{-(n-k)}(z-Bx)^{-(k+1)}, \\ &= \frac{2}{n}\int_{-\infty}^{+\infty}\mathrm{d}u\, f(\frac{u}{B})\sum_{k=0}^{n-1}(z+u)^{-(n-k)}(z-u)^{-(1+k)}.\end{aligned}$$

Since the integrand goes at worst as $\frac{1}{u^2}$ for $u \to \infty$, and it is bounded in any compact interval, it admits an L^1 majorant, of the form Cu^{-2}, say. We can therefore apply the dominated convergence theorem:

$$\lim_{B\to\infty} \left\langle D^{(n)}_{z,B}, f \right\rangle = \frac{2}{n} f(0) \sum_{k=0}^{n-1} I^{(n)}_k,$$

where:

$$I^{(n)}_k = (-1)^{k+1} \int_{-\infty}^{+\infty} \mathrm{d}u \, (z+u)^{-(n-k)} (u-z)^{-(1+k)}.$$

Assume $\mathrm{Im} z > 0$ and close the contour in the upper plane:

$$I^{(n)}_k = -2\pi i (-1)^k R^{(n)}_k$$

The residue of the integrand of $I^{(n)}_k$ at $z = u$ is:

$$R^{(n)}_k = \frac{1}{k!} \left[\frac{\partial^k}{\partial u^k} \right]_{u=z} (z+u)^{-(n-k)} = (-1)^k \binom{n-1}{k} 2^{-n} z^{-n}.$$

Therefore:

$$I^{(n)}_k = -2\pi i \binom{n-1}{k} 2^{-n} z^{-n},$$

$$\sum_{k=0}^{n-1} I^{(n)}_k = -\pi i z^{-n},$$

$$\lim_{B\to\infty} \left\langle D^{(n)}_{z,B}, f \right\rangle = -\frac{2\pi i}{n} \frac{f(0)}{z^n},$$

as was to be proved. The case $\mathrm{Im} z < 0$ is treated similarly. □

Appendix D

Extreme non-relativistic limit of the Yukawa interaction

D.1 Effective Yukawa potential

Consider a fermion field ψ and a scalar field ϕ interacting through the exchange of a scalar field φ:

$$\mathscr{L} = \varphi \left\{ \lambda_\psi \overline{\psi}\psi + \lambda_\phi \phi^\dagger \phi \right\}. \tag{D.1.1}$$

In the non-relativistic limit, this interaction determines an instantaneous potential between the two particles ψ and ϕ. To derive the form of the potential, let us compute the S-matrix element for the elastic scattering process:

$$\psi(p,s)\phi(k) \to \psi(p',s')\phi(k')$$

to lowest order in the coupling constants $\lambda_{\psi,\phi}$. This is given by the one-boson exchange diagram of Figure D.1.1, and evaluates to:

$$\langle \psi(p',s')\phi(k')|S|\psi(p,s)\phi(k)\rangle = (2\pi)^4\delta^4(p'+k'-p-k)\sqrt{\frac{m_\psi}{(2\pi)^3 p'^0}}\sqrt{\frac{m_\psi}{(2\pi)^3 p^0}}\frac{1}{\sqrt{(2\pi)^3 2k^0}}\frac{1}{\sqrt{(2\pi)^3 2k'^0}}\times$$
$$\times\, \overline{u}(p',s')u(p,s)\frac{i\lambda_\phi\lambda_\psi}{\mu^2+\mathbf{q}^2-(q^0)^2}, \tag{D.1.2}$$

where $q = p' - p$. In the above formula, the external states are normalized according to:

$$\langle \psi(p',s')\phi(k')|\psi(p,s)\phi(k)\rangle = \delta^3(\mathbf{p}-\mathbf{p}')\delta^3(\mathbf{k}-\mathbf{k}')\delta_{ss'} \tag{D.1.3}$$

in order to facilitate the non-relativistic limit. The spinors are normalized according to:

$$\sum_{s=\pm\frac{1}{2}} u(p,s)\overline{u}(p,s) = \frac{\not{p}+m_\psi}{2m_\psi}. \tag{D.1.4}$$

In the extreme non-relativistic limit, the S-matrix element (D.1.2)reduces to:

$$\langle \psi(p',s')\phi(k')|S|\psi(p,s)\phi(k)\rangle = (2\pi)^{-2}\delta^4(p'+k'-p-k)\chi^\dagger_{s'}\chi_s\frac{i\lambda_\phi\lambda_\psi}{2m_\phi}\frac{1}{\mu^2+\mathbf{q}^2}+O(\beta^2) \tag{D.1.5}$$

where $\chi_{s,s'}$are the two components spinors of the initial and final fermions, and β stands for the speed of any (real) particle involved in the process.

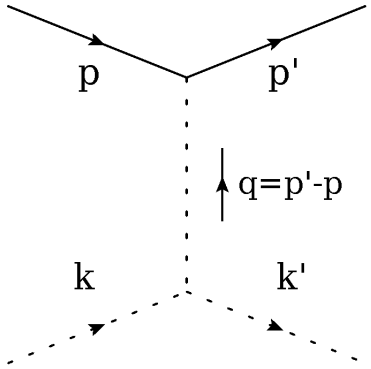

Figure D.1.1. One scalar exchange diagram for ψ-ϕ elastic scattering (cf. Eq. (D.1.2)).

Notice that the relativistic corrections are of second order in β. In particular, the spin of the fermion enters the matrix element (D.1.2) only through the coupling $\overline{u}(p',s')u(p,s)$. From the expansion:

$$\overline{u}(p',s')u(p,s) = \chi^\dagger_{s'}(1 + \frac{\mathbf{q}^2}{8m_\psi^2} - i\frac{\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{p})}{4m_\psi^2})\chi_s + O(\beta^4) \tag{D.1.6}$$

we see that the corrections due to spin are enhanced when m_ψ is smaller or, equivalently, when $r = \frac{m_\psi}{m_\phi}$ is smaller for fixed $\overline{m} = \frac{m_\psi + m_\phi}{2}$.

Equation (D.1.5) should be compared with the S-matrix element obtained from non-relativistic potential scattering in the Born approximation. Assuming a spin-independent potential V (as suggested by Eq. (D.1.5)), this is given by (see, e.g., [59]):

$$\langle\psi(p',s')\phi(k')|S|\psi(p,s)\phi(k)\rangle_{\text{Born}} = -2\pi i\delta^4(p'+k'-p-k)\chi^\dagger_{s'}\chi_s V(\mathbf{q}), \tag{D.1.7}$$

where:

$$V(\mathbf{q}) \equiv \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}}V(\mathbf{r}). \tag{D.1.8}$$

Comparing Eqs. (D.1.5) and (D.1.7), we see that:

$$\int \mathrm{d}^3\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}}V(\mathbf{r}) = -\frac{\lambda_\phi\lambda_\psi}{2m_\phi}\frac{1}{\mu^2+\mathbf{q}^2},$$

or:

$$(2\pi)^3 V(\mathbf{r}) = -\frac{\lambda_\phi\lambda_\psi}{2m_\phi}\int \mathrm{d}^3\mathbf{q}\, e^{i\mathbf{q}\cdot\mathbf{r}}\frac{1}{\mu^2+\mathbf{q}^2}. \tag{D.1.9}$$

The integral can be evaluated as follows:

$$\begin{aligned}
\int \mathrm{d}^3\mathbf{q}\, e^{i\mathbf{q}\cdot\mathbf{r}}\frac{1}{\mu^2+\mathbf{q}^2} &= 2\pi\int_0^\infty \mathrm{d}q\frac{q^2}{\mu^2+q^2}\int_{-1}^{+1}\mathrm{d}\cos\theta\, e^{iqr\cos\theta} \\
&= \frac{2\pi}{ir}\int_0^\infty \mathrm{d}q\frac{q(e^{iqr}-e^{-iqr})}{\mu^2+q^2} \\
&= \frac{2\pi}{ir}\int_{-\infty}^\infty \mathrm{d}q\frac{qe^{iqr}}{(q+i\mu)(q-i\mu)} \\
&= \frac{2\pi^2 e^{-\mu r}}{r}.
\end{aligned}$$

The potential in Eq. (D.1.9) is thus explicitly given by:

$$V(\mathbf{r}) = -\frac{\lambda_\phi \lambda_\psi}{8\pi m_\phi}\frac{e^{-\mu r}}{r}, \tag{D.1.10}$$

Hence, the interaction density (D.1.1) determines, in the non-relativistic limit, a binding potential of the Yukawa type:

$$V(\mathbf{r}) = -\frac{\alpha e^{-\mu r}}{r}, \tag{D.1.11}$$

where the effective dimensionless coupling constant is given by:

$$\alpha \equiv \frac{\lambda_\phi \lambda_\psi}{8\pi m_\phi}. \tag{D.1.12}$$

The spin-orbit correction, which comes from the expansion (D.1.6), has the expression:

$$V_{\text{so}} = -\frac{1}{2m_\psi^2 r}\frac{\mathrm{d}V}{\mathrm{d}r}\mathbf{L}\cdot\mathbf{S}, \tag{D.1.13}$$

and has the *opposite sign* of the corresponding term in QED, due to the scalar coupling $\overline{\psi}\psi$ of the fermion.

D.2 Unitarily equivalent Yukawa systems

In the non-relativistic limit, the binding energies of the fermion-scalar bound states are found by solving the time-independent Schrödinger equation for the Hamiltonian operator:

$$H(\alpha, m, \mu) = \frac{\mathbf{p}^2}{2m} - \frac{\alpha e^{-\mu r}}{r}, \tag{D.2.1}$$

where $m \equiv \frac{m_\phi m_\psi}{m_\phi + m_\psi}$ denotes the reduced mass of the fermion-scalar system.

Observe that:

$$U^\dagger(\theta) H(\alpha, m, \mu) U(\theta) = H(\theta^{-1}\alpha, \theta^2 m, \theta\mu), \tag{D.2.2}$$

where the unitary operator $U(\theta)$ is defined by:

$$U(\theta) \equiv \exp\left[-i\theta\frac{\mathbf{p}\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{p}}{2}\right]. \tag{D.2.3}$$

Since unitarily equivalent operators have the same spectrum, the levels of the three parameters family of Hamiltonians (D.2.1) satisfy:

$$E(\alpha, m, \mu) = E(\theta^{-1}\alpha, \theta^2 m, \theta\mu). \tag{D.2.4}$$

Notice that the binding energy of the ground state is $B = -E_0$, where E_0 is the ground state level of H. If we fix B, m and μ and denote by $\alpha(B, m, \mu)$ the corresponding coupling constant, we see that Eq. (D.2.4) implies:

$$\theta\alpha(B, \theta^2 m, \theta\mu) = \alpha(B, m, \mu). \tag{D.2.5}$$

In particular, choosing $\theta = \sqrt{\frac{m'}{m}}$, we have[1]:

$$\sqrt{\frac{m'}{m}}\alpha(B, m', \sqrt{\frac{m'}{m}}\mu) = \alpha(B, m, \mu) \tag{D.2.6}$$

[1]Observe that, in the $\mu = 0$ case, Eq. (D.2.6) follows directly from Bohr's formula for hydrogen energy levels:

$$E_n(m, \alpha, \mu = 0) = -\frac{m\alpha^2}{2n^2}.$$

Since the coupling constant is dimensionless, in turn it can be written as a function of the dimensionless quantities $\hat{B} \equiv B/\overline{m}$, $r \equiv \frac{m_\psi}{m_\phi}$ and $\hat{\mu} \equiv \mu/\overline{m}$, where $\overline{m} = \frac{m_\phi + m_\psi}{2}$. From the previous equation, we obtain:

$$\lambda(r)\alpha(\hat{B}, r, \lambda(r)\hat{\mu}) = \alpha(\hat{B}, r = 1, \hat{\mu}), \tag{D.2.7}$$

where:

$$\lambda(r) \equiv \frac{2\sqrt{r}}{1+r}. \tag{D.2.8}$$