Smooth Distances for Second Order Kinematic Robot Control

Vinicius Mariano Gonçalves, Anthony Tzes, Farshad Khorrami and Philippe Fraisse

I. Introduction

This is a supplementary material for the paper "Smooth Distances for Second Order Kinematic Robot Control", explaining how to compute $D^{\mathcal{A}}_{h,R}(p)$ for some 3D objects. Through this document, we will use $p = [x\ y\ z]^T$.

A. Removing the regularization parameter

In this document, we will explain how to compute a particular case of $D^{\mathcal{A}}_{h,R}(p)$, $\Pi^{\mathcal{A}}_{h,R}(p)$ when $R \rightarrow \infty$, henceforth denoted simply by $D^{\mathcal{A}}_{h}(p)$ and $\Pi^{\mathcal{A}}_{h}(p)$, respectively. This is because it turns out that we can compute these expressions for a generic R if we have procedures that can compute it for $R \rightarrow \infty$. Let $a_c = \textit{Cen}(\mathcal{A})$ (as it is the geometric center of objects, this can computed easily for objects as boxes, cylinders and spheres). We start by noting the following identity:

$$\frac{\|a-a_c\|^2}{R^2} + \frac{\|p-a\|^2}{h^2} = \frac{\|\hat{p}-a\|^2}{\eta^2} + \left(\frac{\|a_c\|^2}{R^2} + \frac{\|p\|^2}{h^2} - \frac{\|\hat{p}\|^2}{\eta^2}\right) \tag{1}$$

in which $\eta \triangleq (1/h^2 + 1/R^2)^{-1/2}$ and $\hat{p} \triangleq \eta^2(p/h^2 + a_c/R^2)$. Using this formula and the definition of $D^{\mathcal{A}}_{h,R}(p)$ and $D^{\mathcal{A}}_{h}(p)$ (the latter being the case in which $R \rightarrow \infty$ and thus $W^{\mathcal{A}}_{R}(a) = 1$ and $\textit{Vol}_R(\mathcal{A})$ is simply the volume of $\mathcal{A}$, $\textit{Vol}(\mathcal{A})$):

$$D^{\mathcal{A}}_{h,R}(p) = h^2 \log\left(\frac{\textit{Vol}_R(\mathcal{A})}{\textit{Vol}(\mathcal{A})}\right) + \frac{h^2}{2}\left(\frac{\|a_c\|^2}{R^2} + \frac{\|p\|^2}{h^2} - \frac{\|\hat{p}\|^2}{\eta^2}\right) + \frac{h^2}{\eta^2} D^{\mathcal{A}}_{\eta}(\hat{p}). \tag{2}$$

Furthermore, note that $h^2 \log\left(\frac{\textit{Vol}_R(\mathcal{A})}{\textit{Vol}(\mathcal{A})}\right) = -\frac{h^2}{R^2} D^{\mathcal{A}}_{R}(a_c)$. So:

$$D^{\mathcal{A}}_{h,R}(p) = -\frac{h^2}{R^2} D^{\mathcal{A}}_{R}(a_c) + \frac{h^2}{2}\left(\frac{\|a_c\|^2}{R^2} + \frac{\|p\|^2}{h^2} - \frac{\|\hat{p}\|^2}{\eta^2}\right) + \frac{h^2}{\eta^2} D^{\mathcal{A}}_{\eta}(\hat{p}). \tag{3}$$

The righthand side only depends on the computations of $D^{\mathcal{A}}_{g}(u)$ for the different values of g and points u. The formula for the projection is even simpler: taking the derivative of both sides with respect to p and using the fact that $\frac{\partial}{\partial p} D^{\mathcal{A}}_{h,R}(p) = p - \Pi^{\mathcal{A}}_{h,R}(p)$, $\frac{\partial}{\partial p} D^{\mathcal{A}}_{\eta}(p) = p - \Pi^{\mathcal{A}}_{\eta}(p)$ and also that $\frac{\partial \hat{p}}{\partial p} = (\eta^2/h^2)I$ (in which I is the identity matrix):

$$p - \Pi^{\mathcal{A}}_{h,R}(p) = (p - \hat{p}) + \left(\hat{p} - \Pi^{\mathcal{A}}_{\eta}(\hat{p})\right) \ \rightarrow\ \Pi^{\mathcal{A}}_{h,R}(p) = \Pi^{\mathcal{A}}_{\eta}(\hat{p}). \tag{4}$$

Consequently, henceforth, without loss of generality, we will work with the case in which $R \rightarrow \infty$.

B. Computing projections numerically

In this document, we will show how to compute $D^{\mathcal{A}}_{h}(p)$ for some sets. However, we will also need to compute the *h-projections* $\Pi^{\mathcal{A}}_{h}(p)$. In that case, we note that $\Pi^{\mathcal{A}}(p) = p - \frac{\partial D^{\mathcal{A}}_{h}}{\partial p}(p)$. We suggest to compute $\frac{\partial D^{\mathcal{A}}_{h}}{\partial p}(p)$ numerically, that is:

$$\begin{aligned}
\frac{\partial D^{\mathcal{A}}_{h}}{\partial x}(p) &\approx \frac{D^{\mathcal{A}}_{h}(p+\epsilon e_x) - D^{\mathcal{A}}_{h}(p-\epsilon e_x)}{2\epsilon} \\
\frac{\partial D^{\mathcal{A}}_{h}}{\partial y}(p) &\approx \frac{D^{\mathcal{A}}_{h}(p+\epsilon e_y) - D^{\mathcal{A}}_{h}(p-\epsilon e_y)}{2\epsilon} \\
\frac{\partial D^{\mathcal{A}}_{h}}{\partial z}(p) &\approx \frac{D^{\mathcal{A}}_{h}(p+\epsilon e_z) - D^{\mathcal{A}}_{h}(p-\epsilon e_z)}{2\epsilon}
\end{aligned} \tag{5}$$

in which $e_x = [1\ 0\ 0]^T, e_y = [0\ 1\ 0]^T$, $e_z = [0\ 0\ 1]^T$ and ϵ is a small number (we suggest $\epsilon = 0.001$).

V. M. Gonçalves is with Center for Artificial Intelligence and Robotics (CAIR), New York University Abu Dhabi, United Arab Emirates
A. Tzes is with New York University Abu Dhabi, Electrical Engineering, Abu Dhabi 129188, United Arab Emirates
F. Khorrami is with New York University, Electrical & Computer Engineering Department, Brooklyn, NY 11201, USA.
P. Fraisse is with Laboratoire d'informatique, de robotique et de microelectronique de Montpellier (LIRMM) Montpellier, France.

C. Canonical objects

In this document, we will show how to compute $D_h^{\mathcal{A}}(p)$ (and thus $\Pi_h^{\mathcal{A}}(p)$, see the previous subsection) for some sets in a *canonical* pose. For example, for a box centered at $p = [0\ 0\ 0]^T$ of a reference frame with its sides aligned with the axis of this reference frame. For these objects in a general pose, other than the canonical one, we use the property derived in the paper: if $E(\cdot)$ is a rigid transformation and $E^{-1}(\cdot)$ is its inverse:

$$\begin{aligned} D_h^{E(\mathcal{A})}(p) &= D_h^{\mathcal{A}}\Big(E^{-1}(p)\Big) \\ \Pi_h^{E(\mathcal{A})}(p) &= E\Big(\Pi_h^{\mathcal{A}}\Big(E^{-1}(p)\Big)\Big). \end{aligned} \tag{6}$$

D. The Cartesian Product Property

Using the definition of $\mathcal{A}$, it is easy to see that if $\mathcal{A}_i$ are subsets of $\mathbb{R}^{n_i}$ for a $n_i \geq 1$, $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times ... \times \mathcal{A}_m$, $p^i \in \mathbb{R}^{n_i}$ and $p = [(p^1)^T\ (p^2)^T\ ...\ (p^m)^T]^T$ then:

$$D_h^{\mathcal{A}}(p) = \sum_{i=1}^{m} D_h^{\mathcal{A}_i}(p^i). \tag{7}$$

We can use this to compute the h-distance function for complex sets that are build as Cartesian product of simpler sets.

E. The Error Function

The *Error Function* $\textit{Erf}(u) \triangleq \frac{2}{\sqrt{\pi}}\int_0^u e^{-t^2}dt$ appears often in the calculations of $D^{\mathcal{A}}(p)$ for some simple objects. It has no closed form in terms of a finite number elementary functions, but there are very good approximation for it in terms of elementary functions that will be used here. Let

$$\textit{J}(u) \triangleq \frac{a}{(a-1)\sqrt{\pi u^2} + \sqrt{\pi u^2 + a^2}}. \tag{8}$$

in which $a = 2.7889$. Then, $\textit{Erf}(u) \approx \textit{sign}(u)(1 - e^{-u^2}J(u))$. This approximation is excellent for all values of u. More details can be seen in [1].

A related function that will often appear in our calculations is, for $L \geq 0$ and $v \in \mathbb{R}$:

$$\begin{aligned} &\textit{Int}_h(v, L) \triangleq -h^2 \log\left(\frac{1}{2L}\int_{-L}^{L} e^{-\frac{(u-v)^2}{2h^2}}du\right) = \\ &-h^2 \log\left(\sqrt{\frac{\pi}{2}}\frac{h}{2L}\left(\textit{Erf}\left(\frac{L+v}{\sqrt{2}h}\right) + \textit{Erf}\left(\frac{L-v}{\sqrt{2}h}\right)\right)\right). \end{aligned} \tag{9}$$

Int stands for *interval* as it is $D_h^{\mathcal{A}}(p)$ for $\mathcal{A} = [-L, L]$, in which $p \in \mathbb{R}$.

Without a careful evaluation, this function can easily be problematic to be computed. When $|v| \geq 5\sqrt{2}h + L$ the sum of *Erf*'s inside the log is already very close to 0 in most naive implementations of the error function, generating $+\infty$ as a result. The approximation $\textit{Erf}(v) \approx \textit{sign}(v)(1 - e^{-v^2}J(v))$ allow us to solve this problem. We will consider two cases, $|v| \leq L$, in which the aforementioned problem do not happen, and when $|v| \geq L$, in which we need to rewrite the function to avoid underflows.

For $|v| \leq L$, both $u = (L+v)/(\sqrt{2}h)$ and $u = (L+v)/(\sqrt{2}h)$ are nonnegative and we can simply use $\textit{Erf}(u) \approx 1 - e^{-u^2}J(u)$. Then we have the following approximation for $\textit{Int}_h(v, L)$:

$$\begin{aligned} &\textit{Int}_h(v, L) \approx -h^2 \log\left(\sqrt{\frac{\pi}{2}}\frac{h}{2L}\left(2 - e^{-\frac{(L+v)^2}{2h^2}}J\Big(\frac{v+L}{\sqrt{2}h}\Big) - e^{-\frac{(L-v)^2}{2h^2}}J\Big(\frac{v-L}{\sqrt{2}h}\Big)\right)\right) \\ &\text{for } |v| \leq L. \end{aligned} \tag{10}$$

For $|v| \geq L$, we note that we can assume, without loss of generality, that $v \geq 0$, since $\textit{Int}_h(v, L)$ is an even function of v. Note than, in this case, $u_1 = (L+v)/(\sqrt{2}h) \geq 0$ and $u_2 = (L-v)/(\sqrt{2}h) \leq 0$. Using the approximations $\textit{Erf}(u_1) \approx (1 - e^{-u_1^2}J(u_1))$, $\textit{Erf}(u_2) \approx -(1 - e^{-u_2^2}J(u_2))$, and factoring out the term $-e^{-\frac{(L-v)^2}{2h^2}}$ out of the log, we can write:

$$\mathit{Int}_h(v,L) \approx \frac{(v-L)^2}{2} - h^2\log\left(\sqrt{\frac{\pi}{2}}\frac{h}{2L}\left(J\left(\frac{v-L}{\sqrt{2}h}\right) - J\left(\frac{v+L}{\sqrt{2}h}\right)e^{\frac{-2Lv}{h^2}}\right)\right)$$
for $v \geq L$. If $v \leq -L$, use $-v$ in the formula instead. (11)

Here is the C code:

```
#include <math.h>


#define PI 3.1415926
#define SQRTHALFPI 1.2533141
#define SQRT2 1.4142135
#define CONSTJA 2.7889


double fun_J(double u)
{
   return CONSTJA/((CONSTJA−1)∗sqrt(PI∗u∗u) + sqrt(PI∗u∗u+CONSTJA∗CONSTJA));
}

double Int(double v, double h, double L)
{
   if( abs(v) <= L)
   {
      double A1 = exp(−(L−v)∗(L−v)/(2∗h∗h))∗fun_J((v−L)/(SQRT2∗h));
      double A2 = exp(−(L+v)∗(L+v)/(2∗h∗h))∗fun_J((v+L)/(SQRT2∗h));
      return −h∗h∗log(SQRTHALFPI∗(h/(2∗L))∗(2−A1−A2));
   }
   else
   {
      //The function is even
      v = abs(v);

      double A1 = fun_J((v−L)/(SQRT2∗h));
      double A2 = exp(−2∗L∗v/(h∗h))∗fun_J((v+L)/(SQRT2∗h));
      return 0.5∗(v−L)∗(v−L) −h∗h∗log(SQRTHALFPI∗(h/(2∗L))∗(A1−A2));
   }
}
```

F. The Modified Bessel Function of the First Kind

Another function that will often appear is the *Modified Bessel Function of the First Kind* of order 0, defined as $I_0(u) \triangleq \frac{1}{2\pi}\int_0^{2\pi} e^{u\cos(\theta)}d\theta$.

Let $\hat{I}_0(u) = \cosh(u)^{-1}I_0(u)$. Then we define, for $R \geq 0$ and $v \in \mathbb{R}^+$, the following function that will appear in our calculations:

$$\mathit{Cir}_h(v,R) \triangleq -h^2\log\left(\frac{1}{R^2}\int_0^R r\left(e^{-\frac{(r-v)^2}{2h^2}} + e^{-\frac{(r+v)^2}{2h^2}}\right)\hat{I}_0\left(\frac{rv}{h^2}\right)dr\right). \tag{12}$$

Cir stands for *circle* as it is related $D_h^{\mathcal{A}}(p)$ when $\mathcal{A}$ is a circle (with interior) in $\mathbb{R}^2$, centered at the origin and with radius R. More precisely, since the distance function in this case will be radially symmetric, $D_h^{\mathcal{A}}(p) = \mathit{Cir}_h(\|p\|, R)$.

It is beneficial to study an scaled version of this function, in which v, r and R are scaled by $1/h$. Making the change of variables $\rho = r/h$ in the integral and considering $\nu = v/h$, $P = R/h$ we obtain that

$$\mathit{Cir}_h(h\nu, hP) = -h^2\log\left(\frac{1}{P^2}\int_0^P \rho\left(e^{-\frac{1}{2}(\rho-\nu)^2} + e^{-\frac{1}{2}(\rho+\nu)^2}\right)\hat{I}_0(\rho\nu)\,d\rho\right). \tag{13}$$

Now, if we graph the function $f(\rho,\nu) \triangleq \rho\left(e^{-\frac{1}{2}(\rho-\nu)^2} + e^{-\frac{1}{2}(\rho+\nu)^2}\right)\hat{I}_0(\rho\nu)$ on ρ for fixed values of ν ($\nu \geq 0$), we will see that the maximum of $f(\rho,\nu)$ is approximatelly at $r^*(\nu) = \sqrt{1+\nu^2}$, and it is practically zero for $r \leq r^*(\nu) - 3$ and $r \geq r^*(\nu) + 3$. Therefore, let $\underline{F}(\nu,P) \triangleq \max(0, r^*(\nu) - 3)$ and $\overline{F}(\nu,P) \triangleq \min(P, r^*(\nu) + 3)$

$$\int_0^{R/h} f(\rho,\nu)d\rho \approx \int_{\underline{F}(\nu,P)}^{\overline{F}(\nu,P)} f(\rho,\nu)d\rho. \tag{14}$$

We can integrate the integral at the right numerically using, for example, Gaussian quadrature. For that, let $\rho = \underline{F}(\nu,P) + \frac{\overline{F}(\nu,P) - \underline{F}(\nu,P)}{2}(g+1)$. Then the integral becomes:

$$\int_0^{R/h} f(\rho,\nu)d\rho \approx \left(\frac{\overline{F}-\underline{F}}{2}\right)\int_{-1}^{1} f\left(\underline{F}+\left(\frac{\overline{F}-\underline{F}}{2}\right)(g+1),\nu\right)dg. \tag{15}$$

and thus Gauss-Legendre quadrature can be applied. This integral only make sense if $\overline{F}(\nu,P)\geq\underline{F}(\nu,P)$. This holds if $\nu\leq P$, which, returning to the original variables, implies $v\leq R$.

If $v\geq R$, we will integrate in the whole interval from 0 to $P=R/h$ in the Gauss Legendre quadrature rule. However, we need to be careful to avoid underflows.

Let $g_i\in[-1,1]$ be the N points in the Gauss-Legendre quadrature, in an increasing order, with associated weights w_i. Thus, $g_N\leq 1$ is the greatest of the weights and the mapped point in the interval 0 to $P=R/h$ is $\tilde{\rho}\triangleq 0+0.5(R/h-0)(g_N+1)=0.5(R/h)(g_N+1)$. Define the function

$$\hat{f}(\rho,\nu,\tilde{\rho})\triangleq e^{\frac{1}{2}(\tilde{\rho}-\nu)^2}f(\rho,\nu)=\rho\left(e^{-\frac{1}{2}(\rho-\nu)^2+\frac{1}{2}(\tilde{\rho}-\nu)^2}+e^{-\frac{1}{2}(\rho+\nu)^2+\frac{1}{2}(\tilde{\rho}-\nu)^2}\right)\hat{I}_0\left(\rho\nu\right). \tag{16}$$

Thus, for $v\geq R$, we compute

$$\mathit{Cir}_h(v,R)=\frac{(v-h\tilde{\rho})^2}{2}-h^2\log\left(\frac{h^2}{R^2}\int_0^{R/h}\hat{f}(\rho,vh,\tilde{\rho})d\rho\right) \tag{17}$$

in which the integral inside is approximated using Gauss-Legendre quadrature in the interval $[0,R/h]$.

Note that we need to compute the values of the Bessel Function. If it is not readily available, we can use the excellent approximation given by (see [2]):

$$I_0(u)\approx\frac{\cosh(u)}{(1+0.25u^2)^{1/4}}\frac{1+0.24273u^2}{1+0.43023u^2}. \tag{18}$$

And thus:

$$\hat{I}_0(u)\approx\frac{1}{(1+0.25u^2)^{1/4}}\frac{1+0.24273u^2}{1+0.43023u^2}. \tag{19}$$

Here we provide the codes in C. Note that we use Gauss-Legendre quadrature of 7^{th} order, which seems good enough, but the code is easily modifiable if one wants to use higher-order quadratures.

```
#include <math.h>


#define PI 3.1415926;
#define SQRTHALFPI 1.2533141;
#define SQRT2 1.4142135;


double fun_I0_hat(double u)
{
    return pow(1+0.25*u*u,-0.25)*(1 + 0.24273*u*u)/(1 + 0.43023*u*u);
}


double fun_f(double nu, double rho)
{
    double A1 = exp(-0.5*(rho-nu)*(rho-nu));
    double A2 = exp(-0.5*(rho+nu)*(rho+nu));
    return rho*(A1+A2)*fun_I0_hat(rho*nu);
}

double fun_f_hat(double nu, double rho, double rhobar)
{
    double A1 = exp(-0.5*(rho-nu)*(rho-nu) + 0.5*(rhobar-nu)*(rhobar-nu));
    double A2 = exp(-0.5*(rho+nu)*(rho+nu) + 0.5*(rhobar-nu)*(rhobar-nu));
    return rho*(A1+A2)*fun_I0_hat(rho*nu);
}

double max(double a, double b)
{
    if (a >= b)
    {
        return a;
    }
    else
    {
```

```
        return b;
    }
}

double min(double a, double b)
{
    if (a >= b)
    {
        return b;
    }
    else
    {
        return a;
    }
}

double Cir(double v, double h, double R)
{
    //The function should be called only for v >= 0
    v = abs(v);

    //Change here the Gauss-Legendre quadrature
    int N=7;
    double node[N] = {-0.94910, -0.74153, -0.40584, 0,  0.40584, 0.74153, 0.94910};
    double weight[N]= {0.12948,0.27970,0.38183,0.4179,0.38183,0.27970,0.12948};
    //end

    double F_low,F_up,delta,rhobar,y;


    if( v <= R)
    {
       F_low = max(0,sqrt((v/h)*(v/h)+1)-3);
       F_up  = min(R/h,sqrt((v/h)*(v/h)+1)+3);
       delta  = 0.5*(F_up-F_low);

       y=0;
       for(int i=0; i<N; i++)
       {
         y = y + weight[i]*fun_f(v/h,F_low + delta*(node[i]+1));
       }
       y = delta*y;
       return -h*h*log(y*(h/R)*(h/R));
    }
    else
    {
       F_low = 0;
       F_up  = R/h;
       delta  = 0.5*(F_up-F_low);
       rhobar = F_low + delta*(node[N-1]+1);

       y=0;
       for(int i=0; i<N; i++)
       {
         y = y + weight[i]*fun_f_hat(v/h,F_low + delta*(node[i]+1),rhobar);
       }
       y = delta*y;
       return 0.5*(v-h*rhobar)*(v-h*rhobar)-h*h*log(y*(h/R)*(h/R));
    }
}
```

II. FORMULAES FOR OBJECTS

A. Sphere

For a sphere of radius R centered at $p = [0\ 0\ 0]^T$ (see Figure 1), clearly $D_h^{\mathcal{A}}(p)$ is radially symmetric, that is, $D_h^{\mathcal{A}}(p)$ depends only on $\|p\|$. Then, without loss of generality, we can assume that $p = [0\ 0\ \|p\|]^T$.

Using spherical coordinates, $a_x = r\cos(\phi)\sin(\theta)$, $a_y = r\sin(\phi)\sin(\theta)$ and $a_z = r\cos(\theta)$, with $dV = r^2\sin(\theta) r d\theta dr d\phi$. Now, since we have that $\|p - a\|^2 = r^2 - 2r\|p\|cos(\theta) + \|p\|^2$ we can conclude that

$$D_h^{\mathcal{A}}(p) = -h^2 \log\left(\frac{3}{4\pi R^3}\int_0^{2\pi}\int_0^R\int_0^\pi e^{-\frac{r^2-2r\|p\|cos(\theta)+\|p\|^2}{2h^2}} r^2\sin(\theta) r d\theta dr d\phi\right). \tag{20}$$

This can be rewritten as:

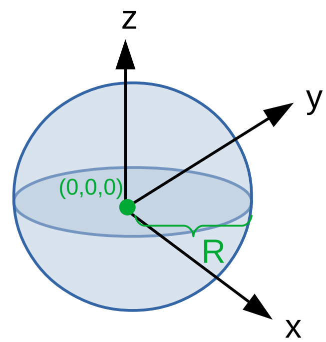

Fig. 1. Sphere in the canonical pose.

$$D_h^{\mathcal{A}}(p) = -h^2 \log\left(\frac{3}{4\pi R^3}\int_0^{2\pi}\int_0^R r^2 e^{-\frac{r^2+\|p\|^2}{2h^2}}\left(\int_0^{\pi} e^{\frac{r\|p\|cos(\theta)}{h^2}}\sin(\theta)d\theta\right)drd\phi\right). \tag{21}$$

The inner integral can be easily computed with the change of variables $v = r\|p\|\cos(\theta)/h^2$, resulting in:

$$D_h^{\mathcal{A}}(p) = -h^2 \log\left(\frac{3h^2}{4\pi R^3\|p\|}\int_0^{2\pi}\int_0^R r e^{-\frac{r^2+\|p\|^2}{2h^2}}\left(e^{r\|p\|/h^2} - e^{-r\|p\|/h^2}\right)drd\phi\right). \tag{22}$$

Using the fact that $e^{-\frac{r^2+\|p\|^2}{2h^2}}e^{r\|p\|/h^2} = e^{-\frac{(r-\|p\|)^2}{2h^2}}$, $e^{-\frac{r^2+\|p\|^2}{2h^2}}e^{-r\|p\|/h^2} = e^{-\frac{(r+\|p\|)^2}{2h^2}}$ and the fact that the integrand does not depend on ϕ, we can obtain

$$D_h^{\mathcal{A}}(p) = -h^2 \log\left(\frac{3h^2}{2R^3\|p\|}\int_0^R r\left(e^{-\frac{(r-\|p\|)^2}{2h^2}} - e^{-\frac{(r+\|p\|)^2}{2h^2}}\right)dr\right). \tag{23}$$

Thus, if we define:

$$\textit{Sph}_h(v,R) \triangleq -h^2 \log\left(\frac{3h^2}{2R^3 v}\int_0^R r\left(e^{-\frac{(r-v)^2}{2h^2}} - e^{-\frac{(r+v)^2}{2h^2}}\right)dr\right) \tag{24}$$

then $D_h^{\mathcal{A}}(p) = \textit{Sph}_h(\|p\|,R)$. *Sph* stands for *Sphere*.

Now, note that:

$$\begin{aligned}
&\int_0^R r e^{-\frac{(r+v)^2}{2h^2}}dr = \int_0^R (r+v-v)e^{-\frac{(r+v)^2}{2h^2}}dr = \\
&\int_0^R (r+v)e^{-\frac{(r+v)^2}{2h^2}}dr - v\int_0^R e^{-\frac{(r+v)^2}{2h^2}}dr = \\
&h^2\left(e^{-\frac{v^2}{2h^2}} - e^{-\frac{(R+v)^2}{2h^2}}\right) - v\sqrt{\frac{\pi}{2}}h\left(\textit{Erf}\left(\frac{R+v}{\sqrt{2}h}\right) - \textit{Erf}\left(\frac{v}{\sqrt{2}h}\right)\right).
\end{aligned}$$

Analogously:

$$\begin{aligned}
&\int_0^R r e^{-\frac{(r-v)^2}{2h^2}}dr = \\
&h^2\left(e^{-\frac{v^2}{2h^2}} - e^{-\frac{(R-v)^2}{2h^2}}\right) + v\sqrt{\frac{\pi}{2}}h\left(\textit{Erf}\left(\frac{R-v}{\sqrt{2}h}\right) + \textit{Erf}\left(\frac{v}{\sqrt{2}h}\right)\right).
\end{aligned}$$

Then:

$$D_h^{\mathcal{A}}(p) = -h^2 \log\left(\frac{3h^2}{2R^3}\left(h^2\left(\frac{e^{-\frac{(R+v)^2}{2h^2}} - e^{-\frac{(R-v)^2}{2h^2}}}{v}\right) + 2Re^{-\textit{Int}_h(v,R)/h^2}\right)\right). \tag{25}$$

This formula provides no problems if $v \leq R$ if we use the approximation for $\mathit{Int}_h(v, L)$ shown in Subsection I-E. However, for $v \geq R$ there can be numerical issues. In this case, we factor out $e^{-\frac{(R-v)^2}{2h^2}}$ to rewrite it as:

$$\frac{(v-R)^2}{2} - h^2 \log\left(\frac{3h^2}{2R^3}\left(h^2\left(\frac{e^{-\frac{2Rv}{h^2}}-1}{v}\right) + 2Re^{-\widehat{\mathit{Int}}_h(v,R)/h^2}\right)\right) \tag{26}$$

in which $\widehat{\mathit{Int}}_h(v, L) \triangleq \mathit{Int}_h(v, L) - \frac{(v-L)^2}{2}$. Note that, when $v = 0$, we need the limit

$$\lim_{v\to 0}\left(\frac{e^{-\frac{(R+v)^2}{2h^2}} - e^{-\frac{(R-v)^2}{2h^2}}}{v}\right) = -\frac{2R}{h^2}e^{-\frac{R^2}{2h^2}}. \tag{27}$$

Here is the C code:

```
double Sph(double v, double h, double R)
{

  //The function should be called only for v >= 0
   v = abs(v);

   double C = 3*(h*h)/(2*R*R*R);
   double A1, A2;
    if( v <= R)
   {
     if(v==0)
     {
        return -h*h*log(C*(-2*R*exp(-(R*R)/(2*h*h)) + 2*R*exp(-Int(0,h,R)/(h*h))));
  }
  else
  {
        A1 = exp(-((R+v)*(R+v)/(2*h*h)));
        A2 = exp(-((R-v)*(R-v)/(2*h*h)));
        return -h*h*log(C*(h*h*(A1-A2)/v + 2*R*exp(-Int(v,h,R)/(h*h))));
  }
   }
   else
   {
        A1 = exp(-(2*R*v/(h*h)));
        A2 = 1;
        return 0.5*(v-R)*(v-R)-h*h*log(C*(h*h*(A1-A2)/v + 2*R*exp((0.5*(v-R)*(v-R)-Int(v,h,R))/(h*h))));
   }
}
```

B. Box

For a box centered at $p = [0\ 0\ 0]^T$ with sides ℓ_x, ℓ_y and ℓ_z aligned with the x, y and z axis, respectively (see Figure 2), we have that $\mathcal{A} = [-\frac{\ell_x}{2}, \frac{\ell_x}{2}] \times [-\frac{\ell_y}{2}, \frac{\ell_y}{2}] \times [-\frac{\ell_z}{2}, \frac{\ell_z}{2}]$.

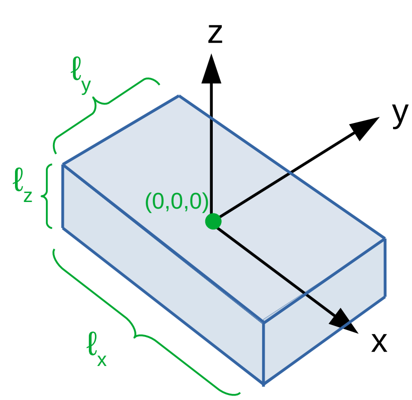

Fig. 2. Box in the canonical pose.

Thus, using the Cartesian product property (Subsection I-D) and the fact that for $\mathcal{A}_i = [-\frac{L_i}{2}, \frac{L_i}{2}]$ and $p^i \in \mathbb{R}$, $D_h^{\mathcal{A}_i}(p^i) = \mathit{Int}_h\left(p^i, \frac{L_i}{2}\right)$, we have that

$$D_h^{\mathcal{A}}(p) = \textit{Int}_h\left(x, \frac{\ell_x}{2}\right) + \textit{Int}_h\left(y, \frac{\ell_y}{2}\right) + \textit{Int}_h\left(z, \frac{\ell_z}{2}\right). \tag{28}$$

We can use the approximation for $\textit{Int}_h(v, L)$ shown in Subsection I-E.

C. Cylinder

For a cylinder centered at $p = [0\ 0\ 0]^T$ with radius R and height H (see Figure Figure 3), we use the fact that $\mathcal{A} = \mathcal{C}(R) \times [-H/2, H/2]$, in which $\mathcal{C}(R)$ is a circle centered at the origin of $\mathbb{R}^2$ with radius R.

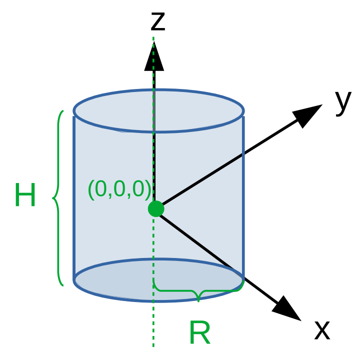

Fig. 3. Cylinder in the canonical pose.

We first compute $D_h^{\mathcal{C}(R)}(p_{xy})$, in which $p_{xy} = [x\ y]^T$. We can exploit the fact that the distance function for $\mathcal{C}(R)$ is radially symmetric in the variables p_{xy}, that is, the distance depends only on $\sqrt{x^2+y^2}$. Thus, without loss of generality, we can assume $p_{xy} = [\sqrt{x^2+y^2}\ \ 0]^T$. Plugging this into the integral definition for $D_h^{\mathcal{C}(R)}(p_{xy})$, using polar coordinates, the definition of the modified Bessel function of the first kind of order 0 and the results in Subsection I-D, we can see that $D_h^{\mathcal{C}(R)}(p_{xy}) = \textit{Cir}_h(\sqrt{x^2+y^2}, R)$.

Thus, using the Euclidean product property (Subsection I-D), we have that:

$$D_h^{\mathcal{A}}(p) = \textit{Cir}_h(\sqrt{x^2+y^2}, R) + \textit{Int}_h\left(z, \frac{H}{2}\right). \tag{29}$$

We can then use the approximation for $\textit{Int}_h(v, L)$ and $\textit{Cir}_h(v, R)$ shown in Subsections I-E and I-D, respectively.

References

[1] C. Ren and A. R. MacKenzie, "Closed-form approximations to the error and complementary error functions and their applications in atmospheric science," *Atmospheric Science Letters*, vol. 8, no. 3, pp. 70–73, 2007. [Online]. Available: https://rmets.onlinelibrary.wiley.com/doi/abs/10.1002/asl.154
[2] J. Olivares, P. Martin, and E. Valero, "A simple approximation for the modified bessel function of zero order i0(x)," vol. 1043, p. 012003, jun 2018.