

# Differential Equations (Ext 1 & 2)

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*Booklet repository:*

<https://github.com/vuhung16au/math-olympiad-ml/>

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*"Talent is important, but how one develops and nurtures it is even more so."*

Terence Tao

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# 1 Introduction

## 1.1 Project Overview

This is a short, exam-focused booklet to help you revise differential equations with solid fundamentals, challenging enrichment problems, and useful appendices for HSC prep.

## 1.2 Where This Topic Sits in the NESA Pathway

This booklet helps you connect differential equations across **Mathematics Extension 1** and **Mathematics Extension 2**, especially for **Mechanics** in Extension 2.

## 1.3 Who This Booklet Is For

This resource is intended for:

- Extension 1 students who want a clear, focused differential equations guide,
- Extension 2 students who want stronger links between calculus, modelling, and mechanics,
- motivated learners who want an early look at how school calculus grows into university maths and engineering,
- teachers and tutors who want a concise reference plus enrichment material to stretch stronger students.

## 1.4 How to Use This Booklet

- Review the fundamentals.
- Try problems first without hints, then check and retry. Practice makes progress.
- Review yourself using the appendices.

## 1.5 Prerequisite Knowledge

To use this book well, make sure you:

- have a good understanding of derivatives and anti-derivatives,
- are comfortable with basic integrals,
- have seen some Year 10–12 physics ideas; if you are rusty, this booklet will refresh what you need as you go.

## 1.6 Extension 2 Mechanics Connections

These ideas show up a lot in **Mechanics** and **Calculus** in Extension 2, and many questions come down to setting up the right differential equation and substitution.

- **Exponential growth and decay:**  $\frac{dy}{dt} = ky$  leads to  $y = Ae^{kt}$ .
- **Resistive motion:** equations such as  $m\frac{dv}{dt} = mg - kv$  lead to terminal velocity and approach-to-limit behavior.

- **Escape velocity and gravitation:** the substitution  $a = v \frac{dv}{dx}$  turns force laws into equations in  $v$  and  $x$ .
- **Simple harmonic motion and spring motion:** equations such as  $\ddot{x} + n^2x = 0$  connect differential equations to oscillation and period.
- **Water tanks and Torricelli-style models:**  $\frac{dh}{dt} = -k\sqrt{h}$  is a standard separable form.

## 1.7 Choosing the Right Substitution

Choose the acceleration form that matches the data: use  $\frac{dv}{dt}$  for time-based equations,  $v \frac{dv}{dx}$  for velocity-displacement links, and  $\frac{d}{dx} \left( \frac{1}{2}v^2 \right)$  for energy-style methods. They are equivalent rewrites, so pick the one that makes integration cleaner.

## 1.8 Core Syllabus vs Enrichment

This booklet builds strong **core HSC differential equations skills** and then pushes you with **hard enrichment problems**, so you can aim for high marks now and be ready for higher maths later.

## 2 Fundamentals Review

This quick review covers the core language, notation, and standard forms you will use throughout the booklet.

### 2.1 Differential equations and solutions

A differential equation links an unknown function to its derivatives. Here we focus on first-order equations (highest derivative is  $y'$ ).

- A **general solution** contains an arbitrary constant such as  $C$ .
- A **particular solution** is found after using a condition such as  $y(0) = 3$ .
- An **initial condition** picks one curve from the family.

**Example:**  $\frac{dy}{dx} = 2x \Rightarrow y = x^2 + C$ , and  $y(0) = 3 \Rightarrow y = x^2 + 3$ .

### 2.2 Direction fields and equilibrium solutions

For  $\frac{dy}{dx} = F(x, y)$ , a slope field draws a short segment at each point  $(x, y)$  with slope  $F(x, y)$ .

- If the slope is positive, solution curves rise from left to right.
- If the slope is negative, solution curves fall from left to right.
- If the slope is zero along  $y = k$ , then  $y = k$  is an equilibrium solution.

**Example:**  $\frac{dy}{dx} = y - 2$  has equilibrium  $y = 2$ .

### 2.3 Separable structure

Common separable forms are:

$$\frac{dy}{dx} = f(x), \quad \frac{dy}{dx} = g(y), \quad \frac{dy}{dx} = f(x)g(y).$$

If variables can be separated, use this workflow:

1. Rearrange so all  $y$  terms are with  $dy$  and all  $x$  terms are with  $dx$ .
2. Integrate both sides.
3. Apply any initial condition.
4. Interpret the result in context.

**Example:**  $\frac{dy}{dx} = xy \Rightarrow \frac{1}{y} dy = x dx$ .

## 2.4 Logistic-style models

The logistic equation is

$$\frac{dy}{dx} = ky(M - y),$$

equivalently

$$\frac{dy}{dx} = ry \left(1 - \frac{y}{M}\right),$$

where  $M$  is the carrying capacity.

- $y = 0$  and  $y = M$  are equilibrium values.
- If  $0 < y < M$ , growth is positive.
- If  $y > M$ , the model predicts decay back toward the carrying capacity.

**Example:** If  $M = 100$  and  $y = 20$ , then  $\left(1 - \frac{y}{M}\right) = 0.8$  so growth is positive.

### 3 Differential Equations

The core language and notation were introduced in the Fundamentals Review. From here, the focus shifts to using those ideas in practice: reading behavior from slope fields, solving separable forms, and interpreting autonomous models such as logistic growth.

We begin with slope fields, where local gradient information helps us predict solution behavior before full symbolic solving.

## 4 Slope Fields

### 4.1 Concept Overview

A slope field (also called a direction field) is a visual map of the differential equation  $y' = F(x, y)$ . At each point, you draw a short segment with slope  $F(x, y)$ , and solution curves follow those local directions.

This lets you read behavior before doing heavy algebra. It is especially useful for spotting equilibrium solutions and predicting long-run trends.

### 4.2 Key Ideas

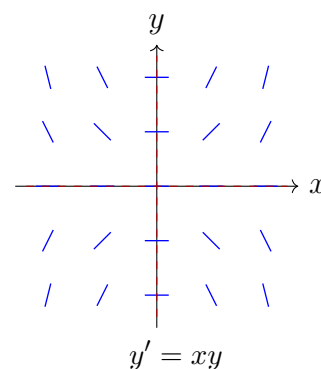
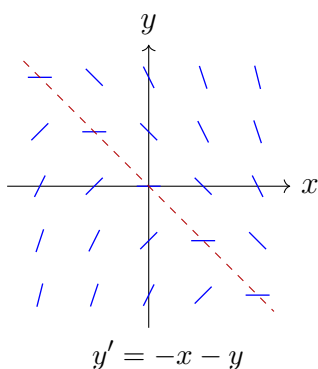
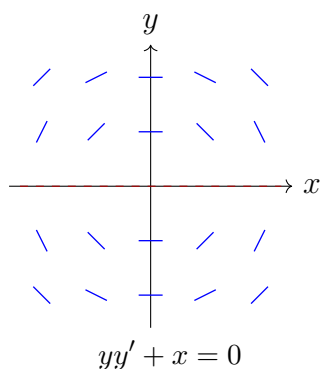
- If the segment is horizontal, then the slope is zero.
- If the segment tilts upward, then the solution increases locally.
- If the segment tilts downward, then the solution decreases locally.
- A constant solution  $y = k$  is an equilibrium if the slope is zero there.

### 4.3 Interpretation Notes

- A particular solution must follow the local tangent directions shown by the field.
- If nearby segments steer curves toward a horizontal line, that line behaves like a long-run level.
- A full answer may require both a sketch and a written description of behavior as  $x \rightarrow \infty$  or  $x \rightarrow -\infty$ .

### 4.4 Worked Slope-Field Examples

The mini-fields below show how different differential equations create different local patterns.



- For  $yy' + x = 0$ , the line  $y = 0$  is a singular row where the slope formula  $y' = -x/y$  is undefined.
- For  $y' = -x - y$ , the line  $y = -x$  is an isocline of zero slope.
- For  $y' = xy$ , both axes are zero-slope lines, and the sign of  $xy$  controls whether the local segments tilt up or down.

#### Enrichment

Slope fields are an early example of qualitative analysis. Even when an equation is hard to solve exactly, the field can still reveal equilibrium levels, monotonic behavior, and asymptotic trends.

## 5 Separable Differential Equations

### 5.1 Concept Overview

Many HSC differential equations are separable, which means you can move the  $y$  terms and  $x$  terms to opposite sides before integrating. This is one of the main algebraic techniques you need to master.

### 5.2 Canonical Forms

$$\frac{dy}{dx} = f(x), \quad \frac{dy}{dx} = g(y), \quad \frac{dy}{dx} = f(x)g(y).$$

For a separable equation, we aim for a structure like

$$A(y) dy = B(x) dx.$$

### 5.3 Interpretation Notes

- The form  $y' = f(x)$  means the slope depends only on  $x$ .
- The form  $y' = g(y)$  means the slope depends only on the current value of  $y$ .
- The form  $y' = f(x)g(y)$  mixes both influences and is often the richest case.

#### Enrichment

At university level, students later meet equations that are not separable and need different methods. Learning to recognize separable structure now builds the pattern-recognition needed for those later techniques.

## 6 Autonomous Equations, Equilibria, and the Logistic Model

### 6.1 Concept Overview

An autonomous differential equation has the form  $y' = g(y)$ , so the slope depends only on the current value of  $y$ . These equations often make equilibrium values and long-run behavior easier to see.

This section focuses on equilibrium and stability analysis for autonomous equations. The logistic equation is a key worked example because it combines growth with a limiting effect: a quantity grows when small, then slows as it approaches a carrying capacity.

## 6.2 Equilibria and Stability (Phase-Line Thinking)

For an autonomous equation  $y' = g(y)$ , equilibrium values occur where  $g(y) = 0$ . To classify behavior, check the sign of  $g(y)$  in intervals between equilibrium values:

- if  $g(y) > 0$ , solutions increase;
- if  $g(y) < 0$ , solutions decrease;
- if nearby arrows point toward an equilibrium, it is stable;
- if nearby arrows point away, it is unstable.

## 6.3 Key Forms

$$\frac{dy}{dx} = g(y)$$

$$\frac{dy}{dx} = ky(M - y) \quad \text{or} \quad \frac{dy}{dx} = ry \left(1 - \frac{y}{M}\right).$$

## 6.4 Interpretation Notes

- In the logistic model,  $y = 0$  and  $y = M$  are equilibrium values.
- If  $0 < y < M$ , growth is positive.
- If  $y > M$ , the rate is negative and the model pushes back toward  $M$ .

## 6.5 Other Common Autonomous Models

The logistic equation is one important model, but not the only one. Other standard autonomous forms include:

- Exponential growth/decay:  $\frac{dy}{dt} = ky$ .
- Newton cooling/approach-to-ambient:  $\frac{dT}{dt} = -k(T - T_a)$ .
- Logistic with harvesting:  $\frac{dy}{dt} = ry \left(1 - \frac{y}{M}\right) - h$ .

These can all be analysed with the same equilibrium-and-sign method used for logistic growth.

### Enrichment

Stable/unstable equilibrium language and phase-line analysis become very important in engineering, dynamical systems, and mathematical biology. The logistic equation is often a student's first encounter with these ideas, but the same framework extends to many other models.

## 7 Problems

### 7.1 Warm-Up

#### Problem 7.1: Verifying Solutions by Substitution

Verify by substitution that the given function is a solution to the differential equation for all values of the constants  $A$ ,  $B$ , and  $C$ .

**a**  $y''' = 12$ ;  $y = 2x^3 + Ax^2 + Bx + C$

**b**  $y'' - 5y' + 6y = 0$ ;  $y = Ae^{2x} + Be^{3x}$

**c**  $y'' + 9y = 18$ ;  $y = A \cos(3x) + B \sin(3x) + 2$

**d**  $y'' - 4y' + 4y = 0$ ;  $y = (Ax + B)e^{2x}$

**Hint:**

- **General Strategy:** Do not attempt to integrate or "solve" the differential equation from scratch. Instead, calculate the necessary derivatives ( $y'$ ,  $y''$ ,  $y'''$ ) from the given solution  $y$ , and substitute them into the Left Hand Side (LHS) of the differential equation. Simplify the algebra to show it perfectly equals the Right Hand Side (RHS).
- **For (c):** Remember the chain rule for trigonometric functions:  $\frac{d}{dx}[\cos(kx)] = -k \sin(kx)$ .
- **For (d):** You must use the **Product Rule** ( $y' = u'v + uv'$ ) to differentiate  $y$ , treating  $(Ax + B)$  as  $u$  and  $e^{2x}$  as  $v$ . Factor out  $e^{2x}$  at each step to make the algebra manageable.

**Solution 7.1****(a)** Given  $y = 2x^3 + Ax^2 + Bx + C$ .Differentiating with respect to  $x$ :

$$y' = 6x^2 + 2Ax + B$$

$$y'' = 12x + 2A$$

$$y''' = 12$$

Substitute into the LHS of the differential equation:

$$\text{LHS} = y''' = 12.$$

Since LHS = RHS, the function is verified as a solution.  $\square$ **(b)** Given  $y = Ae^{2x} + Be^{3x}$ .

$$y' = 2Ae^{2x} + 3Be^{3x}$$

$$y'' = 4Ae^{2x} + 9Be^{3x}$$

Substitute into the LHS of the differential equation ( $y'' - 5y' + 6y$ ):

$$\begin{aligned} \text{LHS} &= (4Ae^{2x} + 9Be^{3x}) - 5(2Ae^{2x} + 3Be^{3x}) + 6(Ae^{2x} + Be^{3x}) \\ &= 4Ae^{2x} + 9Be^{3x} - 10Ae^{2x} - 15Be^{3x} + 6Ae^{2x} + 6Be^{3x} \\ &= e^{2x}(4A - 10A + 6A) + e^{3x}(9B - 15B + 6B) \\ &= e^{2x}(0) + e^{3x}(0) = 0 \end{aligned}$$

Since LHS = RHS, the function is verified as a solution.  $\square$ **(c)** Given  $y = A \cos(3x) + B \sin(3x) + 2$ .

$$y' = -3A \sin(3x) + 3B \cos(3x)$$

$$y'' = -9A \cos(3x) - 9B \sin(3x)$$

Substitute into the LHS of the differential equation ( $y'' + 9y$ ):

$$\begin{aligned} \text{LHS} &= (-9A \cos(3x) - 9B \sin(3x)) + 9(A \cos(3x) + B \sin(3x) + 2) \\ &= -9A \cos(3x) - 9B \sin(3x) + 9A \cos(3x) + 9B \sin(3x) + 18 \\ &= 18 \end{aligned}$$

Since LHS = RHS, the function is verified as a solution.  $\square$ **(d)** Given  $y = (Ax + B)e^{2x}$ .Using the product rule ( $u = Ax + B, v = e^{2x}$ ):

$$\begin{aligned} y' &= (A)(e^{2x}) + (Ax + B)(2e^{2x}) \\ &= e^{2x}(A + 2Ax + 2B) \end{aligned}$$

$$\begin{aligned} y'' &= (2e^{2x})(A + 2Ax + 2B) + (e^{2x})(2A) \\ &= e^{2x}(2A + 4Ax + 4B + 2A) \\ &= e^{2x}(4A + 4B + 4Ax) \end{aligned}$$

Substitute into the LHS of the differential equation ( $y'' - 4y' + 4y$ ):

$$\text{LHS} = e^{2x}(4A + 4B + 4Ax) - 4[e^{2x}(A + 2B + 2Ax)] + 4[(Ax + B)e^{2x}]$$

Factor out the common  $e^{2x}$  term:

$$\begin{aligned} \text{LHS} &= e^{2x} [(4A + 4B + 4Ax) - (4A + 8B + 8Ax) + (4Ax + 4B)] \\ &= e^{2x} [A(4 - 4) + B(4 - 8 + 4) + Ax(4 - 8 + 4)] \\ &= e^{2x}[0] = 0 \end{aligned}$$

Since LHS = RHS, the function is verified as a solution.  $\square$

**Takeaways 7.1**

- “Solving” a differential equation from scratch can be very complex, but “verifying” a solution is simply an exercise in standard differential calculus. If an exam question asks to “Show that  $y = \dots$  is a solution,” always use substitution.
- When verifying differential equations involving exponentials (like parts b and d), factoring out the exponential term ( $e^{kx}$ ) makes the algebraic cancellation much easier to spot and less prone to simple arithmetic errors.

**Problem 7.2: Solving Initial Value Problems by Direct Integration**

Solve these initial value problems. In each case, use integration to find the general solution, then use the initial condition to evaluate the constant.

**a**  $y' = -2; \quad y(1) = 4$

**b**  $y' = 4x + 1; \quad y(0) = -3$

**c**  $y' = 6x^2 - 2x + 5; \quad y(-1) = 2$

**d**  $y' = \cos(2x); \quad y\left(\frac{\pi}{4}\right) = 1$

**e**  $y' = 4e^{-x}; \quad y(0) = 2$

**f**  $y' = \frac{4}{x^2}; \quad y(2) = 1$

**Hint:**

- **The Two-Step Process:** Every differential equation of the form  $y' = f(x)$  can be solved by direct integration:  $y = \int f(x) dx$ . Do not forget to add the constant of integration  $(+C)$  immediately after integrating. This creates the **general solution**.
- **Evaluating  $C$ :** Once you have the general solution, substitute the given  $x$  and  $y$  values from the initial condition to solve for  $C$ . Finally, rewrite the complete equation with your newly found  $C$  value to get the **particular solution**.
- **For (f):** Rewrite  $\frac{4}{x^2}$  using negative index laws ( $4x^{-2}$ ) before attempting to apply the power rule for integration.

**Solution 7.2**

(a)  $y = -2x + C$ ,  $4 = -2(1) + C \Rightarrow C = 6$ , so  $y = -2x + 6$ .

(b)  $y = 2x^2 + x + C$ ,  $-3 = C$ , so  $y = 2x^2 + x - 3$ .

(c)  $y = 2x^3 - x^2 + 5x + C$ . Using  $y(-1) = 2$  gives  $2 = -2 - 1 - 5 + C$ , hence  $C = 10$  and

$$y = 2x^3 - x^2 + 5x + 10.$$

(d)  $y = \frac{1}{2} \sin(2x) + C$ . Since  $y(\frac{\pi}{4}) = 1$ ,

$$1 = \frac{1}{2} \sin \frac{\pi}{2} + C = \frac{1}{2} + C \Rightarrow C = \frac{1}{2},$$

so  $y = \frac{1}{2} \sin(2x) + \frac{1}{2}$ .

(e)  $y = -4e^{-x} + C$ ,  $2 = -4 + C \Rightarrow C = 6$ , so  $y = -4e^{-x} + 6$ .

(f)  $y = \int 4x^{-2} dx = -\frac{4}{x} + C$ . Using  $y(2) = 1$  gives  $1 = -2 + C$ , so  $C = 3$  and

$$y = -\frac{4}{x} + 3.$$

**Takeaways 7.2**

- **General vs. Particular:** An indefinite integral always produces a family of curves (the general solution). An initial value problem “pins down” exactly one of those curves (the particular solution) by forcing it to pass through a specific  $(x, y)$  coordinate.
- **Always write  $+C$  immediately:** A common student error is to integrate, evaluate the numbers, and *then* try to add  $C$ . The constant must be written on the exact same line the integral sign disappears.

**Problem 7.3: Implicit Differentiation and Conic Sections**

**a**

**i** Differentiate both sides of the ellipse equation  $4x^2 + 9y^2 = 36$  with respect to  $x$ , using the chain rule where necessary.

**ii** Hence show that  $\frac{dy}{dx} = -\frac{4x}{9y}$  at each point on the ellipse.

**iii** Part **ii** needs qualification. Where is the differential equation undefined and what does this mean geometrically for the curve?

**b** Likewise, find  $\frac{dy}{dx}$  for these curves:

**i** the parabola  $y^2 = 12x$

**ii** the hyperbola  $xy = 8$

**iii** the circle  $x^2 + y^2 = r^2$

**iv** the hyperbola  $y^2 - x^2 = 1$

**v** the curve  $x^2y + xy^2 = 10$

**vi** the astroid  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

**Hint:**

- **The Golden Rule of Implicit Differentiation:** Differentiate  $x$  terms normally. When you differentiate a  $y$  term with respect to  $x$ , differentiate it normally as if it were an  $x$ , but immediately multiply it by  $\frac{dy}{dx}$ . (This is just an application of the Chain Rule:  $\frac{d}{dx}[f(y)] = f'(y) \cdot \frac{dy}{dx}$ ).

- **For Parts b(ii) and b(v):** Be careful! Terms like  $xy$  and  $x^2y$  require the **Product Rule** ( $uv + uv'$ ). For example, to differentiate  $xy$ , let  $u = x$  and  $v = y$ .

- **For Part a(iii):** A fraction is mathematically undefined when its denominator is zero. Set the denominator to zero, find the  $y$ -value, and substitute it back into the original ellipse equation to find the corresponding  $x$ -coordinates.

**Solution 7.3**

**a(i)** Differentiating both sides with respect to  $x$ :

$$\begin{aligned}\frac{d}{dx}(4x^2) + \frac{d}{dx}(9y^2) &= \frac{d}{dx}(36) \\ 8x + 18y \frac{dy}{dx} &= 0\end{aligned}$$

**a(ii)** Rearranging to make  $\frac{dy}{dx}$  the subject:

$$\begin{aligned}18y \frac{dy}{dx} &= -8x \\ \frac{dy}{dx} &= -\frac{8x}{18y} = -\frac{4x}{9y} \quad \square\end{aligned}$$

**a(iii)** The differential equation is undefined when the denominator is 0, which occurs when  $9y = 0 \implies y = 0$ .

Substituting  $y = 0$  into the original equation:

$$4x^2 + 9(0)^2 = 36 \implies 4x^2 = 36 \implies x^2 = 9 \implies x = \pm 3$$

The derivative is undefined at the points  $(3, 0)$  and  $(-3, 0)$ . Geometrically, this is expected because these are the  $x$ -intercepts of the ellipse, where the curve has **vertical tangents** (infinite gradient).

**b(i) Parabola**  $y^2 = 12x$

$$2y \frac{dy}{dx} = 12 \implies \frac{dy}{dx} = \frac{12}{2y} = \frac{6}{y}$$

**b(ii) Hyperbola**  $xy = 8$  (Use Product Rule:  $u = x, v = y$ )

$$\begin{aligned}(1)(y) + (x) \left( \frac{dy}{dx} \right) &= 0 \\ x \frac{dy}{dx} &= -y \implies \frac{dy}{dx} = -\frac{y}{x}\end{aligned}$$

**b(iii) Circle**  $x^2 + y^2 = r^2$  (Note:  $r^2$  is a constant)

$$2x + 2y \frac{dy}{dx} = 0 \implies 2y \frac{dy}{dx} = -2x \implies \frac{dy}{dx} = -\frac{x}{y}$$

**b(iv) Hyperbola**  $y^2 - x^2 = 1$

$$2y \frac{dy}{dx} - 2x = 0 \implies 2y \frac{dy}{dx} = 2x \implies \frac{dy}{dx} = \frac{x}{y}$$

**b(v) Curve**  $x^2y + xy^2 = 10$  (Use Product Rule on both terms)

$$\begin{aligned}\left[ (2x)(y) + (x^2) \left( \frac{dy}{dx} \right) \right] + \left[ (1)(y^2) + (x) \left( 2y \frac{dy}{dx} \right) \right] &= 0 \\ 2xy + x^2 \frac{dy}{dx} + y^2 + 2xy \frac{dy}{dx} &= 0\end{aligned}$$

Factor out  $\frac{dy}{dx}$ :

$$\begin{aligned}\frac{dy}{dx}(x^2 + 2xy) &= -2xy - y^2 \\ \frac{dy}{dx} &= -\frac{2xy + y^2}{x^2 + 2xy} = -\frac{y(2x + y)}{x(x + 2y)}\end{aligned}$$

**b(vi) Astroid**  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$  (Note:  $a^{\frac{2}{3}}$  is a constant)

$$\begin{aligned}\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} \frac{dy}{dx} &= 0 \\ \frac{2}{3}y^{-\frac{1}{3}} \frac{dy}{dx} &= -\frac{2}{3}x^{-\frac{1}{3}} \\ \frac{dy}{dx} &= -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = -\sqrt[3]{\frac{y}{x}}\end{aligned}$$

**Takeaways 7.3**

- **Why use Implicit Differentiation?** For closed curves like circles and ellipses, making  $y$  the subject results in messy  $\pm$  square root equations. Implicit differentiation bypasses this, allowing you to find the gradient function directly from the original relation.
- **Coordinate Dependency:** Notice that almost all the gradient functions ( $\frac{dy}{dx}$ ) in this exercise contain *both*  $x$  and  $y$ . This means you cannot find the slope of a tangent just by knowing the  $x$ -value; you must know the full  $(x, y)$  coordinate point on the curve.

**Problem 7.4: Repeated Integration and Polynomial Families**

Consider the  $n$ -th order differential equation  $\frac{d^n y}{dx^n} = n!$ , where  $n$  is a positive integer.

- i By integrating repeatedly, find the general solution for  $n = 3$ .
- ii Hence, write down the general solution for the  $n$ -th order equation  $\frac{d^n y}{dx^n} = n!$ . Express your answer using summation notation ( $\sum$ ) with arbitrary constants  $c_k$ .
- iii For  $n = 3$ , a particular solution  $y = P(x)$  has a stationary point of inflection at  $(1, 2)$ . Find the explicit algebraic equation for  $P(x)$ .
- iv For a general  $n$ , suppose a solution  $y = Q(x)$  satisfies the initial conditions  $Q^{(k)}(a) = 0$  for all integers  $0 \leq k \leq n - 1$ . Prove that  $Q(x) = (x - a)^n$ .

**Hint:**

- **For i:** Think about the Factor Theorem and root multiplicity. If a polynomial and its first derivative are both zero at  $x = a$ , what does that mean for the root  $(x - a)$ ? Extend this logic up to the  $(n - 1)$ -th derivative. Alternatively, use the Taylor Series expansion centered at  $x = a$ .
- **For ii:** Look at the pattern of the terms from part i. A polynomial of degree  $n$  generated this way will have  $n$  arbitrary constants for the lower-degree terms.
- **For iii:** A "stationary point of inflection" at  $x = 1$  means both the first derivative  $y'(1) = 0$  and the second derivative  $y''(1) = 0$ . Substitute  $x = 1$  into your derivatives to solve for the constants sequentially.
- **For iv:** Think about the Factor Theorem and root multiplicity. If a polynomial and its first derivative are both zero at  $x = a$ , what does that mean for the root  $(x - a)$ ? Extend this logic up to the  $(n - 1)$ -th derivative. Alternatively, use the Taylor Series expansion centered at  $x = a$ .

**Solution 7.4**

**i** For  $n = 3$ , the differential equation is  $\frac{d^3y}{dx^3} = 3! = 6$ .  
Integrate once:

$$y'' = \int 6 dx = 6x + A$$

Integrate twice:

$$y' = \int (6x + A) dx = 3x^2 + Ax + B$$

Integrate a third time:

$$y = \int (3x^2 + Ax + B) dx = x^3 + \frac{A}{2}x^2 + Bx + C$$

To make the notation cleaner, let  $c_2 = \frac{A}{2}$ ,  $c_1 = B$ , and  $c_0 = C$ .

**General Solution:**  $y = x^3 + c_2x^2 + c_1x + c_0$

**ii** Following the pattern from part **i**, integrating  $n!$  exactly  $n$  times produces the term  $x^n$ . Each integration introduces a new constant that eventually becomes the coefficient for a lower power of  $x$ .

Thus, the general solution is a monic polynomial of degree  $n$ :

$$y = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \dots + c_1x + c_0$$

Using summation notation:

$$y = x^n + \sum_{k=0}^{n-1} c_k x^k$$

**iii** From part **i**,  $P(x) = x^3 + c_2x^2 + c_1x + c_0$ .

We are given a stationary point of inflection at  $(1, 2)$ .

This means  $P(1) = 2$ ,  $P'(1) = 0$ , and  $P''(1) = 0$ .

The derivatives are:

$$P'(x) = 3x^2 + 2c_2x + c_1$$

$$P''(x) = 6x + 2c_2$$

Use  $P''(1) = 0$ :

$$6(1) + 2c_2 = 0 \implies c_2 = -3$$

Use  $P'(1) = 0$ :

$$3(1)^2 + 2(-3)(1) + c_1 = 0 \implies 3 - 6 + c_1 = 0 \implies c_1 = 3$$

Use  $P(1) = 2$ :

$$1^3 - 3(1)^2 + 3(1) + c_0 = 2 \implies 1 - 3 + 3 + c_0 = 2 \implies c_0 = 1$$

Therefore, the explicit equation is:

$$P(x) = x^3 - 3x^2 + 3x + 1 \text{ (which factors elegantly to } P(x) = (x - 1)^3 + 2).$$

**iv Method 1 (Multiplicity of Roots):**

Since  $Q^{(n)}(x) = n!$ , integrating  $n$  times tells us  $Q(x)$  is a monic polynomial of degree  $n$  (from part **ii**).

We are given  $Q(a) = 0, Q'(a) = 0, \dots, Q^{(n-1)}(a) = 0$ .

By the properties of polynomials, if a polynomial and its first  $n - 1$  derivatives evaluate to 0 at  $x = a$ , then  $x = a$  is a root of multiplicity  $n$ .

Therefore,  $Q(x) = c(x - a)^n$ .

Since  $Q(x)$  is a monic polynomial (its leading term is  $1x^n$ ), the constant  $c$  must be 1.

Thus,  $Q(x) = (x - a)^n$ .  $\square$

**Method 2 (Taylor Series):**

Any polynomial of degree  $n$  can be expressed exactly as a Taylor expansion centered at  $x = a$ :

$$Q(x) = \sum_{k=0}^n \frac{Q^{(k)}(a)}{k!} (x - a)^k$$

We are given  $Q^{(k)}(a) = 0$  for all  $0 \leq k \leq n - 1$ . Therefore, all terms in the sum are zero except the final  $n$ -th term.

$$Q(x) = \frac{Q^{(n)}(a)}{n!} (x - a)^n$$

Since  $Q^{(n)}(x) = n!$  for all  $x$ ,  $Q^{(n)}(a) = n!$ .

$$Q(x) = \frac{n!}{n!} (x - a)^n = (x - a)^n. \quad \square$$

**Takeaways 7.4**

- **Connecting Integration to Polynomials:** Repeated integration of a constant mathematically generates the family of all polynomials. The constants of integration directly correspond to the polynomial’s coefficients.
- **Boundary Conditions as Geometry:** Initial value problems don’t just have to be written as  $y(0) = 1$ . They can be phrased geometrically, like “has a stationary point of inflection.”
- **The Foundation of Taylor Series:** Part iv is a sneak peek into Taylor Polynomials, a crucial university calculus concept where entire functions are constructed based purely on the values of their derivatives at a single point!

**Problem 7.5: Analyzing Solution Curves via Implicit Differentiation**

Consider the initial value problem  $y' = x^2 - y^2$  with the initial condition  $y(1) = 1$ . Suppose that the function  $y = f(x)$  is a valid solution to this equation.

- i Calculate the gradient of the curve  $y = f(x)$  at the point  $(1, 1)$ .
- ii Differentiate the differential equation implicitly with respect to  $x$  to show that  $y'' = 2x - 2x^2y + 2y^3$ .
- iii Hence, determine the concavity of the curve at  $(1, 1)$  and state whether this point is a local maximum, local minimum, or neither.
- iv Determine the exact value of  $f'''(1)$ .

**Hint:**

- **For i:** Substitute  $x = 1$  and  $y = 1$  directly into the given differential expression for  $y'$ .
- **For ii:** Use implicit differentiation. Remember that  $\frac{d}{dx}(y^2) = 2y \cdot y'$ . Then, substitute the original expression for  $y'$  back into your result and expand the brackets to get the required “show that” form.
- **For iii:** Use the value of  $y''$  at  $(1, 1)$  to determine concavity (positive = concave up, negative = concave down). Combine this with your gradient result from part i to classify the nature of the stationary point.
- **For iv:** Differentiate the expanded expression for  $y''$  from part ii with respect to  $x$ . You will need to use the **Product Rule** for the  $-2x^2y$  term.

**Solution 7.5**

i Given  $y' = x^2 - y^2$ .

At the point  $(1, 1)$ , substitute  $x = 1, y = 1$ :

$$y'(1) = 1^2 - 1^2 = 0$$

The gradient of the curve at  $(1, 1)$  is 0.

ii Differentiating both sides of  $y' = x^2 - y^2$  with respect to  $x$ :

$$y'' = \frac{d}{dx}(x^2) - \frac{d}{dx}(y^2)$$

$$y'' = 2x - 2y \cdot y'$$

Substitute the original differential equation  $y' = x^2 - y^2$  into this expression:

$$y'' = 2x - 2y(x^2 - y^2)$$

$$y'' = 2x - 2x^2y + 2y^3 \quad \square$$

iii To find the concavity, evaluate  $y''$  at  $x = 1, y = 1$ :

$$y''(1) = 2(1) - 2(1)^2(1) + 2(1)^3$$

$$y''(1) = 2 - 2 + 2 = 2$$

Since  $y''(1) > 0$ , the curve is **concave up** at this point.

Because the gradient  $y'(1) = 0$  (from part i) and the concavity is positive, the point  $(1, 1)$  is a **local minimum**.

iv We need to differentiate  $y'' = 2x - 2x^2y + 2y^3$  with respect to  $x$ .

Using the product rule on the middle term ( $u = 2x^2, v = y$ ):

$$y''' = 2 - [4x \cdot y + 2x^2 \cdot y'] + 6y^2 \cdot y'$$

$$y''' = 2 - 4xy - 2x^2y' + 6y^2y'$$

Now, evaluate at  $x = 1, y = 1$ , and crucially, substitute  $y'(1) = 0$ :

$$y'''(1) = 2 - 4(1)(1) - 2(1)^2(0) + 6(1)^2(0)$$

$$y'''(1) = 2 - 4 - 0 + 0 = -2$$

**Takeaways 7.5**

- **Extracting Geometry from DEs:** Differential equations don't just ask us to "find  $y$ "; they contain a wealth of geometric information. We can find gradients, concavities, and classify turning points without ever knowing the explicit algebraic equation of the curve!
- **Strategic Substitution:** When evaluating higher-order derivatives at a specific point (like in part iv), substitute the known values of the lower-order derivatives (e.g.,  $y' = 0$ ) *immediately* after differentiating. Trying to simplify the algebra before substituting the numbers often leads to unnecessary errors.

**Problem 7.6: Non-linear Slope Fields and Isoclines**

Some differential equations cannot be solved algebraically. Instead, we must rely on geometric analysis (slope fields) and numerical approximations to understand their solution curves.

Consider the differential equation  $\frac{dy}{dx} = x - y^2$  and the particular solution  $y = f(x)$  that passes through the origin  $(0, 0)$ .

- i** An *isocline* is a curve along which the gradient of the solution curves is constant. Find the Cartesian equation of the zero-isocline (where  $\frac{dy}{dx} = 0$ ). Sketch this isocline and clearly shade the region in the plane where the solution curves are strictly increasing.
- ii** By differentiating the differential equation implicitly with respect to  $x$ , find an expression for  $\frac{d^2y}{dx^2}$ .
- iii** Hence, prove that any stationary point on *any* solution curve for this differential equation must be a local minimum.
- iv** Since the exact algebraic function  $y = f(x)$  cannot be found, use Euler’s method with a step size of  $h = 0.5$  to estimate the value of  $f(1)$ .

- **For iv:** Set up an Euler’s method table with columns for  $x_n, y_n, y'_n$ , and  $y_{n+1} = y_n + h \cdot y'_n$ . Start your first row at  $(x_0, y_0) = (0, 0)$ .
- **For iii:** At a stationary point, what is the value of  $\frac{d^2y}{dx^2}$ ? Substitute this into your expression from part ii to determine the concavity.
- **For ii:** Use implicit differentiation. Remember that  $\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$ .
- **For i:** Set  $\frac{dy}{dx} = 0$  to find the equation of the isocline. For the shaded region, consider the inequality  $x - y^2 > 0$ .

**Hint:**

**Solution 7.6**

i To find the zero-isocline, set the gradient to zero:

$$\frac{dy}{dx} = 0 \implies x - y^2 = 0 \implies x = y^2$$

This is a sideways parabola opening to the right, with its vertex at the origin. The solution curves are strictly increasing when their gradient is positive:

$$\frac{dy}{dx} > 0 \implies x - y^2 > 0 \implies x > y^2$$

Therefore, the curves are increasing in the region “inside” the sideways parabola (to the right of the boundary  $x = y^2$ ).

ii Differentiating  $\frac{dy}{dx} = x - y^2$  implicitly with respect to  $x$ :

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}(x) - \frac{d}{dx}(y^2) \\ \frac{d^2y}{dx^2} &= 1 - 2y \frac{dy}{dx} \quad \square \end{aligned}$$

iii At any stationary point on a solution curve, the gradient is zero ( $\frac{dy}{dx} = 0$ ). Substituting this into the second derivative from part ii:

$$\frac{d^2y}{dx^2} = 1 - 2y(0) = 1$$

Because  $\frac{d^2y}{dx^2} = 1 > 0$  constantly at all stationary points, the curve is strictly **concave up** whenever it flattens out. Therefore, any turning point must be a **local minimum**.  $\square$

iv Using Euler’s method with  $h = 0.5$ , starting at  $(x_0, y_0) = (0, 0)$ :

Step 1:

$$\begin{aligned} y'_0 &= x_0 - (y_0)^2 = 0 - 0^2 = 0 \\ y_1 &= y_0 + h(y'_0) = 0 + 0.5(0) = 0 \end{aligned}$$

The new point is  $(x_1, y_1) = (0.5, 0)$ .

Step 2:

$$\begin{aligned} y'_1 &= x_1 - (y_1)^2 = 0.5 - 0^2 = 0.5 \\ y_2 &= y_1 + h(y'_1) = 0 + 0.5(0.5) = 0.25 \end{aligned}$$

The new point is  $(x_2, y_2) = (1.0, 0.25)$ .

Therefore, the estimated value is  $f(1) \approx 0.25$ .

**Takeaways 7.6**

- **Geometric Power:** Even when integration totally fails us, calculus still provides a roadmap. We found the exact nature of the turning points without ever knowing the curve’s equation!
- **Isoclines vs. Solution Curves:** Don’t confuse them. The parabola  $x = y^2$  is *not* a solution curve itself; it is simply the invisible “ridge” where all the actual solution curves hit their local minimums as they pass through it.

**Problem 7.7: The Geometry of Isoclines and Invariant Solutions**

An *isocline* is a curve along which every solution to a differential equation has the same gradient  $c$ . That is, it is the contour line defined by  $\frac{dy}{dx} = c$ . Prove the following statements regarding the geometry of isoclines.

- i** Prove that for any differential equation of the form  $\frac{dy}{dx} = f(x^2 + y^2)$ , the isoclines form a family of concentric circles centered at the origin.
- ii** Prove that for any differential equation of the form  $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$ , the isoclines form a family of straight lines passing through the origin (excluding the origin itself).
- iii** Suppose a straight line  $y = mx + b$  is an isocline corresponding to a gradient  $c$  for a given differential equation. Prove that this line is *also* a solution curve to the differential equation if and only if  $m = c$ .
- iv** Consider the differential equation  $\frac{dy}{dx} = \frac{x-y}{x+y}$ . Using the results from parts **ii** and **iii**, find the exact Cartesian equations of the two straight lines that are both isoclines and solution curves for this differential equation.

**Hint:**

- **For i & ii:** Start by setting  $\frac{dy}{dx} = c$ . You are looking to show that the resulting equation defines a circle (for part i) or a line through the origin (for part ii). Treat  $c$  as a constant, meaning  $f^{-1}(c)$  or  $g^{-1}(c)$  is just another constant  $k$ .
- **For iii:** If a line is a solution curve, its actual derivative  $\left(\frac{dy}{dx}(mx + b)\right)$  must perfectly match the gradient prescribed by the differential equation ( $c$ ) everywhere on that line.
- **For iv:** Divide the numerator and the denominator of the DE by  $x$  to show it fits the form from part **ii**. Since its isoclines are  $y = mx$ , substitute  $y = mx$  into the DE to find the gradient  $c$  of the slope field along that line. Finally, apply part **iii** to find the specific values of  $m$ .

**Solution 7.7**

**i** Let the isocline correspond to the gradient  $c$ . Then:

$$\frac{dy}{dx} = c \implies f(x^2 + y^2) = c$$

Let  $k$  be a constant such that  $f(k) = c$ . Then:

$$x^2 + y^2 = k$$

For  $k > 0$ , this is the standard Cartesian equation for a circle centered at the origin with radius  $\sqrt{k}$ . Thus, the isoclines form a family of concentric circles.  $\square$

**ii** Let the isocline correspond to the gradient  $c$ . Then:

$$\frac{dy}{dx} = c \implies g\left(\frac{y}{x}\right) = c$$

Let  $m$  be a constant such that  $g(m) = c$ . Then:

$$\frac{y}{x} = m \implies y = mx \quad (x \neq 0)$$

This is the standard equation of a straight line passing through the origin. Thus, the isoclines form a family of radial lines.  $\square$

**iii** Let the differential equation be  $\frac{dy}{dx} = F(x, y)$ .

We are given that  $y = mx + b$  is an isocline for gradient  $c$ . This means  $F(x, mx + b) = c$  for all  $x$ . For the line  $y = mx + b$  to also be a solution curve, it must satisfy the differential equation:

$$\begin{aligned} \frac{d}{dx}(mx + b) &= F(x, mx + b) \\ m &= c \end{aligned}$$

Therefore, the line is a solution curve if and only if its physical slope ( $m$ ) equals the gradient prescribed by the DE ( $c$ ).  $\square$

**iv** Given  $\frac{dy}{dx} = \frac{x-y}{x+y}$ .

Divide the numerator and denominator by  $x$ :

$$\frac{dy}{dx} = \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}}$$

This fits the form  $g\left(\frac{y}{x}\right)$  from part **ii**, so its isoclines are straight lines of the form  $y = mx$ . Substitute  $y = mx$  into the DE to find the gradient  $c$  along this isocline:

$$c = \frac{x - mx}{x + mx} = \frac{1 - m}{1 + m}$$

By part **iii**, for this isocline to also be a solution curve, its slope  $m$  must equal  $c$ :

$$\begin{aligned} m &= \frac{1 - m}{1 + m} \\ m(1 + m) &= 1 - m \\ m^2 + m &= 1 - m \\ m^2 + 2m - 1 &= 0 \end{aligned}$$

Using the quadratic formula:

$$m = \frac{-2 \pm \sqrt{4 - 4(1)(-1)}}{2} = \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

Therefore, the two straight lines that are both isoclines and solution curves are:

$$y = (-1 + \sqrt{2})x \quad \text{and} \quad y = (-1 - \sqrt{2})x$$

**Takeaways 7.7**

- **The “Skeleton” of a Slope Field:** Recognizing algebraic forms like  $f(x^2 + y^2)$  or  $g(y/x)$  immediately tells you the geometric layout (circles or radial lines) of the slope field without calculating a single vector.
- **Invariant Lines:** Finding straight-line solutions to complex differential equations is a common high-level technique. It represents the perfect intersection where the geometry of the curve perfectly aligns with the underlying “current” of the differential equation.

**Problem 7.8: Trigonometric Curves and Auxiliary Angles**

A curve  $y = f(x)$  passes through the origin  $(0, 0)$ . The gradient function of the curve is given by the differential equation  $\frac{dy}{dx} = \sin x - \cos x$ .

- i Find the exact coordinates of the stationary point in the interval  $0 \leq x \leq \pi$ , and determine its nature.
- ii Find the explicit equation of the curve  $y = f(x)$ .
- iii By using the auxiliary angle method, express the equation of the curve in the form  $y = C - \sqrt{2} \sin(x + \alpha)$ , where  $\alpha$  is an acute angle.
- iv Hence, or otherwise, find the exact global maximum value of  $y$  and the first positive value of  $x$  at which it occurs.

• **For i:** A stationary point occurs where  $\frac{dy}{dx} = 0$ . Solve  $\sin x - \cos x = 0$  by dividing both sides by  $\cos x$  to create a  $\tan x$  equation. Use the second derivative to test its nature.

• **For ii:** Integrate the gradient function. Be careful with signs:  $\int \sin x \, dx = -\cos x$  and  $\int -\cos x \, dx = -\sin x$ . Substitute  $(0, 0)$  to find the constant  $C$ .

• **For iii:** Take the  $-\cos x - \sin x$  part of your equation and factor out a negative. Then, expand  $\sin(x + \alpha) = \sin x \cos \alpha + \cos x \sin \alpha$  to find the matching angle  $\alpha$ .

• **For iv:** Look at your equation from part iii. For  $y$  to be a maximum, the term being subtracted ( $\sqrt{2} \sin(x + \alpha)$ ) must be at its absolute minimum. What is the minimum value of a sine function?

**Hint:**

**Solution 7.8**

i Stationary points occur when  $\frac{dy}{dx} = 0$ :

$$\begin{aligned}\sin x - \cos x &= 0 \\ \sin x &= \cos x\end{aligned}$$

Divide by  $\cos x$  (assuming  $\cos x \neq 0$ ):

$$\tan x = 1$$

In the interval  $0 \leq x \leq \pi$ ,  $x = \frac{\pi}{4}$ .

To find the  $y$ -coordinate, we need to integrate first (or we can state the nature first). Let's find the nature:

$$\frac{d^2y}{dx^2} = \cos x + \sin x$$

When  $x = \frac{\pi}{4}$ :

$$\frac{d^2y}{dx^2} = \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

Since  $\frac{d^2y}{dx^2} > 0$ , the curve has a **local minimum** at  $x = \frac{\pi}{4}$ . (We will find the exact  $y$ -coordinate in part ii).

ii Integrate to find  $y$ :

$$\begin{aligned}y &= \int (\sin x - \cos x) dx \\ y &= -\cos x - \sin x + C\end{aligned}$$

Substitute the initial condition  $x = 0, y = 0$ :

$$\begin{aligned}0 &= -\cos(0) - \sin(0) + C \\ 0 &= -1 - 0 + C \implies C = 1\end{aligned}$$

Therefore, the equation of the curve is  $y = 1 - \cos x - \sin x$ .

(Returning to part i, the coordinates of the minimum are  $(\frac{\pi}{4}, 1 - \sqrt{2})$ ).

iii We want to express  $y = 1 - (\sin x + \cos x)$  in the form  $y = C - \sqrt{2} \sin(x + \alpha)$ .

Expand the target form:

$$\sqrt{2} \sin(x + \alpha) = \sqrt{2}(\sin x \cos \alpha + \cos x \sin \alpha)$$

Equating this to  $\sin x + \cos x$ :

$$\begin{aligned}\sqrt{2} \cos \alpha &= 1 \implies \cos \alpha = \frac{1}{\sqrt{2}} \\ \sqrt{2} \sin \alpha &= 1 \implies \sin \alpha = \frac{1}{\sqrt{2}}\end{aligned}$$

Since both sine and cosine are positive,  $\alpha$  is in the first quadrant:  $\alpha = \frac{\pi}{4}$ .

Therefore, the equation can be written as:

$$y = 1 - \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$$

iv Using the form  $y = 1 - \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$ :

The maximum value of  $y$  occurs when the sine term is at its minimum, i.e., when  $\sin\left(x + \frac{\pi}{4}\right) = -1$ .

Maximum value:  $y = 1 - \sqrt{2}(-1) = 1 + \sqrt{2}$ .

This occurs when:

$$\begin{aligned}x + \frac{\pi}{4} &= \frac{3\pi}{2} \\ x &= \frac{3\pi}{2} - \frac{\pi}{4} = \frac{6\pi}{4} - \frac{\pi}{4} = \frac{5\pi}{4}\end{aligned}$$

**Takeaways 7.8**

- **The Auxiliary Angle:** Whenever a differential equation yields a mix of  $\sin x$  and  $\cos x$  with the same frequency, combining them into a single trigonometric wave is the best way to determine amplitude, max/min values, and phase shifts.
- **Sign Traps:** Integrating  $\sin x$  to  $-\cos x$  and differentiating  $\sin x$  to  $\cos x$  is a classic source of careless errors. Always double-check your signs!

**Problem 7.9: Separable Equations and Exponentials**

Consider the differential equation  $\frac{dy}{dx} = (x - 1)(y + 1)$ .

- i By separating the variables, show that the general solution can be written in the form  $y = Ae^{\frac{x^2}{2}-x} - 1$ , where  $A$  is an arbitrary constant.
- ii Find the particular solution given the curve passes through the point  $(1, 2)$ . Show that your solution can be written as  $y = 3e^{\frac{1}{2}(x-1)^2} - 1$ .
- iii Find the coordinates of the turning point of this particular solution and determine its nature.
- iv For this particular solution, describe the behavior of  $y$  as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$ . Does the curve have any horizontal asymptotes?

**Hint:**

- **For i:** Move the  $(y + 1)$  term to the left with  $dy$  and leave the  $(x - 1)$  term on the right with  $dx$ . Integrate both sides. Remember that  $e^{C+D} = e^C \cdot e^D = A \cdot e^{\dots}$ .
- **For ii:** Substitute  $x = 1$  and  $y = 2$  into your general solution to solve for  $A$ . Once you substitute  $A$  back in, use index laws to combine the powers of  $e$ , and complete the square in the exponent.
- **For iii:** You don't need to differentiate your ugly exponential function! Look back at the original differential equation. A turning point occurs when  $\frac{dy}{dx} = 0$ .
- **For iv:** Look at the exponent  $\frac{1}{2}(x - 1)^2$ . Because it is squared, what happens to this value when  $x$  becomes a massive positive number OR a massive negative number?

**Solution 7.9**

i Given  $\frac{dy}{dx} = (x-1)(y+1)$ .  
Separate the variables:

$$\int \frac{1}{y+1} dy = \int (x-1) dx$$

$$\ln |y+1| = \frac{x^2}{2} - x + C$$

Convert to exponential form:

$$y+1 = e^{\frac{x^2}{2}-x+C} = e^C e^{\frac{x^2}{2}-x}$$

Let  $A = \pm e^C$  (allowing for positive or negative branches):

$$y = Ae^{\frac{x^2}{2}-x} - 1 \quad \square$$

ii Substitute the point  $x=1, y=2$ :

$$2 = Ae^{\frac{1^2}{2}-1} - 1$$

$$3 = Ae^{-1/2}$$

Multiply both sides by  $e^{1/2}$ :

$$A = 3e^{1/2}$$

Substitute  $A$  back into the general solution:

$$y = 3e^{1/2} e^{\frac{x^2}{2}-x} - 1$$

Using index laws to combine the base  $e$  terms:

$$y = 3e^{\frac{x^2}{2}-x+\frac{1}{2}} - 1$$

Factor out the  $\frac{1}{2}$  in the exponent:

$$y = 3e^{\frac{1}{2}(x^2-2x+1)} - 1$$

$$y = 3e^{\frac{1}{2}(x-1)^2} - 1 \quad \square$$

iii A turning point occurs where  $\frac{dy}{dx} = 0$ .

From the original DE:  $(x-1)(y+1) = 0$ .

So either  $x=1$  or  $y=-1$ .

For our particular solution,  $y = 3e^{\dots} - 1$ , which means  $y > -1$  always. Therefore, the turning point must be at  $x=1$ .

When  $x=1$ , we already know  $y=2$  (from our initial condition).

To determine its nature, look at the derivative  $\frac{dy}{dx} = (x-1)(y+1)$ .

Since  $y+1$  is always positive for this curve:

- If  $x < 1$ ,  $(x-1)$  is negative, so  $\frac{dy}{dx} < 0$ .
- If  $x > 1$ ,  $(x-1)$  is positive, so  $\frac{dy}{dx} > 0$ .

Therefore, the point  $(1, 2)$  is a local minimum.

iv Look at the exponent in  $y = 3e^{\frac{1}{2}(x-1)^2} - 1$ .

As  $x \rightarrow \infty$ ,  $(x-1)^2 \rightarrow \infty$ , so  $e^{\frac{1}{2}(x-1)^2} \rightarrow \infty$ , meaning  $y \rightarrow \infty$ .

As  $x \rightarrow -\infty$ ,  $(x-1)^2 \rightarrow \infty$  (since a negative squared becomes positive), so  $e^{\frac{1}{2}(x-1)^2} \rightarrow \infty$ , meaning  $y \rightarrow \infty$ .

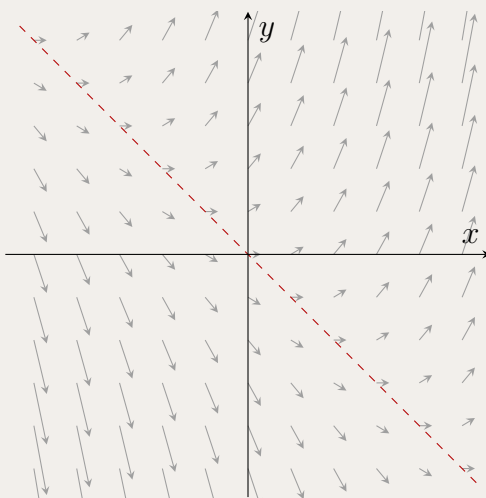
Because the curve shoots off to infinity in both directions, **it does not have any horizontal asymptotes.**

**Takeaways 7.9**

- **The Power of the Original DE:** Notice in part **iii** how much faster it is to find the stationary point using the un-integrated differential equation rather than trying to use the chain rule on  $3e^{\frac{1}{2}(x-1)^2}$ .
- **Completing the Square:** Grouping terms in an exponent to form a perfect square is a fantastic way to instantly determine a function's symmetry and behavior at infinity.

**Problem 7.10: Identifying the Vector Flow**

Examine the slope field shown below and decide which differential equation it represents.



- (A)  $\frac{dy}{dx} = x - y$
- (B)  $\frac{dy}{dx} = x + y$
- (C)  $\frac{dy}{dx} = xy$
- (D)  $\frac{dy}{dx} = y^2$

**Hint:**

- Find the line where the segments are horizontal. That is where  $\frac{dy}{dx} = 0$ .
- Check a simple point such as  $(1, 0)$  and compare the visible slope with each option.
- Eliminate any equation whose sign pattern does not match the first and third quadrants.

**Solution 7.10**

The correct choice is **(B)**. The field has zero slope along the diagonal  $y = -x$ , and only

$$x + y = 0 \iff y = -x$$

matches that observation. It also gives slope 1 at  $(1, 0)$ , consistent with the diagram. The other options fail quickly:

$$x - y = 0 \Rightarrow y = x, \quad xy = 0 \Rightarrow x = 0 \text{ or } y = 0, \quad y^2 \geq 0.$$

Hence the slope field represents

$$\frac{dy}{dx} = x + y.$$

**Takeaways 7.10**

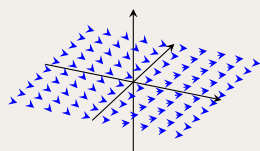
- The fastest slope-field test is usually the **zero-slope line**. It removes most wrong options immediately.
- A differential equation in  $x$  and  $y$  creates a geometric fingerprint: nullclines, sign regions, and steepness patterns.

**Problem 7.11: The Periodic Flow**

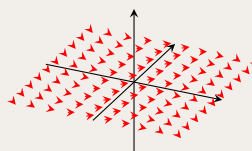
Which of the following slope fields best represents

$$\frac{dy}{dx} = \sin x?$$

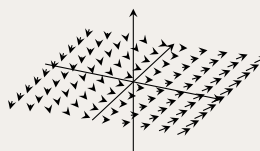
(A)



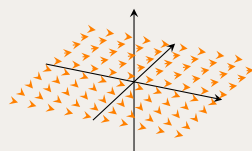
(B)



(C)



(D)



**Hint:**

- Because  $\sin x$  depends only on  $x$ , every vertical line should have identical slope segments.
- At  $x = 0$ , the slope must be 0, so the  $y$ -axis should consist of horizontal dashes.
- The pattern should repeat periodically as  $x$  moves left or right.

**Solution 7.11**

The correct field is **(A)**. Since  $\frac{dy}{dx} = \sin x$  depends only on  $x$ , the slope is constant along each vertical line, which rules out **(D)**. Also,

$$\sin 0 = 0,$$

so the segments on the line  $x = 0$  must be horizontal; this excludes **(B)**. Field **(A)** also shows the expected periodic rise-and-fall pattern for the antiderivative

$$y = -\cos x + C.$$

Therefore the best match is

(A).

**Takeaways 7.11**

- If  $y' = f(x)$  only, then slope patterns repeat along **vertical** lines. If  $y' = g(y)$  only, they repeat along **horizontal** lines.
- Matching a slope field to a DE is often about structure first, not detailed algebra.

**Problem 7.12: Conic Sections from Differential Equations**

i Find the general solution to

$$\frac{dy}{dx} = -\frac{2x}{y}.$$

ii Find the particular solution through  $(1, 2)$ , identify the curve, and state its intercepts.

**Hint:**

- Separate the variables first, then integrate.
- After finding the constant, divide by the right number to compare with the standard ellipse form.
- Intercepts come from setting one variable to zero at a time.

**Solution 7.12**

(i) Separating variables,

$$y \, dy = -2x \, dx$$

so

$$\frac{y^2}{2} = -x^2 + C \implies \boxed{2x^2 + y^2 = C_1}.$$

(ii) Using (1, 2) gives

$$2(1)^2 + (2)^2 = C_1 \implies C_1 = 6.$$

Hence

$$2x^2 + y^2 = 6 \implies \frac{x^2}{3} + \frac{y^2}{6} = 1,$$

so the curve is an **ellipse**. Its intercepts are

$$y = 0 \implies x = \pm\sqrt{3}, \quad x = 0 \implies y = \pm\sqrt{6}.$$

**Takeaways 7.12**

- Some separable DEs generate whole families of conic sections rather than explicit functions.
- When  $y = 0$ , the original DE is undefined, which matches the vertical tangents at the ellipse's  $x$ -intercepts.

**Problem 7.13: Direction Field Features and Hyperbolic Solutions**

Consider

$$\frac{dy}{dx} = \frac{2x}{y-1}.$$

- Identify where the slopes are zero, undefined, positive, and negative.
- Solve the DE and identify the family of curves.

- Separate variables, integrate, then complete the square in  $y$ .
- A fraction is zero when its numerator is zero and undefined when its denominator is zero.

**Hint:**

**Solution 7.13**

(i) Zero slope occurs at  $x = 0$ , and the field is undefined at  $y = 1$ . The slope is positive when  $x$  and  $y - 1$  have the same sign, and negative otherwise.

(ii) Separating variables,

$$(y - 1) dy = 2x dx.$$

Integrating gives

$$\frac{1}{2}y^2 - y = x^2 + C,$$

so

$$(y - 1)^2 - 2x^2 = K.$$

Thus

$$\boxed{(y - 1)^2 - 2x^2 = K},$$

a family of hyperbolas centred at  $(0, 1)$ .

**Takeaways 7.13**

- Numerator-zero and denominator-zero lines are the quickest way to sketch a direction field.
- Completing the square after separation often reveals the hidden geometry of the solutions.

**Problem 7.14: Initial Value Problem with a Square-Root Branch**

Consider

$$\frac{dy}{dx} = \frac{x}{2y}.$$

- Find the general solution.
- Find the particular solution satisfying  $y(2) = -\sqrt{3}$ .
- Determine its maximal domain of validity.

- Separate variables first.
- The initial condition chooses the sign of the square root.
- Check both the explicit formula and the original DE when deciding the domain.

**Hint:**

**Solution 7.14**

(i) From

$$2y \, dy = x \, dx$$

we get

$$y^2 = \frac{x^2}{2} + C.$$

(ii) Using  $(2, -\sqrt{3})$  gives  $3 = 2 + C$ , so  $C = 1$ . Hence

$$y^2 = \frac{x^2}{2} + 1,$$

and the negative branch is required:

$$y = -\sqrt{\frac{x^2}{2} + 1}.$$

(iii) Since  $\frac{x^2}{2} + 1 > 0$  and  $y \neq 0$  for all  $x$ , the DE is defined for every real  $x$ . Thus the maximal domain is

$$(-\infty, \infty).$$

**Takeaways 7.14**

- A general implicit solution can split into multiple explicit branches.
- The initial condition selects the correct branch.

**Problem 7.15: Separable Equation and Horizontal Asymptote**

Consider

$$\frac{dy}{dx} = xy^2 e^{-x^2/2}, \quad y(0) = \frac{1}{2}.$$

- i Find the particular solution.
- ii Determine  $\lim_{x \rightarrow \infty} y(x)$ .

**Hint:**

- Integrate  $y^{-2}$  on the left.
- On the right, use  $n = -\frac{1}{2}x^2$ .

**Solution 7.15**

Separating variables gives

$$y^{-2} dy = xe^{-x^2/2} dx.$$

Hence

$$-\frac{1}{y} = -e^{-x^2/2} + C.$$

Using  $y(0) = \frac{1}{2}$  gives  $2 = 1 - C$ , so  $C = -1$ . Therefore

$$y(x) = \frac{1}{1 + e^{-x^2/2}}.$$

As  $x \rightarrow \infty$ ,  $e^{-x^2/2} \rightarrow 0$ , so

$$\lim_{x \rightarrow \infty} y(x) = 1.$$

Thus the curve approaches the horizontal asymptote  $y = 1$ .

**Takeaways 7.15**

- A separable equation can still require substitution inside one of the integrals.
- Long-run limits expose horizontal asymptotes quickly.

**Problem 7.16: Domain and Range from an Exponential DE**

Find the exact domain and range of the solution through the origin for

$$\frac{dy}{dx} = e^{2x+y}.$$

- Write  $e^{2x+y} = e^{2x}e^y$  and separate variables.
- The logarithm in the final answer controls the domain.

**Hint:**

**Solution 7.16**

Separating variables,

$$e^{-y} dy = e^{2x} dx.$$

Integrating gives

$$-e^{-y} = \frac{1}{2}e^{2x} + C.$$

Using  $(0, 0)$  gives  $C = -\frac{3}{2}$ , so

$$e^{-y} = \frac{3 - e^{2x}}{2}$$

and

$$y = \ln 2 - \ln(3 - e^{2x}).$$

For this to exist we need  $3 - e^{2x} > 0$ , so

$$x < \frac{1}{2} \ln 3.$$

As  $x \rightarrow -\infty$ ,  $y \rightarrow \ln(2/3)$ ; as  $x \rightarrow (\frac{1}{2} \ln 3)^-$ ,  $y \rightarrow \infty$ . Therefore

$$\text{Domain } \left(-\infty, \frac{1}{2} \ln 3\right), \quad \text{Range } \left(\ln \frac{2}{3}, \infty\right).$$

**Takeaways 7.16**

- Solving the DE is only part of the job; logarithms can restrict the domain afterwards.
- Endpoint limits often reveal both vertical and horizontal asymptotes.

**Problem 7.17: Reconstructing a Polynomial from a Direction Field**

An unknown differential equation has the form  $y' = P(x)$ , where  $P(x)$  is monic. Its slope field has horizontal tangents only at  $x = \pm 2$ , is negative on  $(-2, 2)$ , and positive elsewhere.

- Find the simplest such polynomial  $P(x)$ .
- Find the solution satisfying  $f(0) = 4$ .
- Determine the  $x$ -coordinates of all local extrema using only the field information.

- The sign pattern determines whether the parabola opens upward or downward.
- Roots of  $P$  come from the horizontal-tangent lines.

**Hint:**

**Solution 7.17**

The roots are  $\pm 2$ , and the sign pattern is positive outside and negative inside, so

$$P(x) = (x - 2)(x + 2) = x^2 - 4.$$

Integrating,

$$y = \frac{x^3}{3} - 4x + C.$$

Using  $f(0) = 4$  gives

$$f(x) = \frac{x^3}{3} - 4x + 4.$$

Since  $y'$  changes from positive to negative at  $x = -2$ , every solution has a local maximum there. Since  $y'$  changes from negative to positive at  $x = 2$ , every solution has a local minimum there.

**Takeaways 7.17**

- A direction field can determine a differential equation qualitatively before any integration happens.
- Sign changes in  $y'$  are enough to classify extrema.

**Problem 7.18: Implicit Curves and Domain of an IVP**

Consider

$$\frac{dy}{dx} = \frac{5}{y}.$$

- i Find the general solution and identify the family of curves.
- ii Find the particular solution satisfying  $f(0) = -4$ .
- iii Determine its maximal domain of validity.

**Hint:**

- Separate variables.
- The initial condition chooses one square-root branch.
- The DE fails when  $y = 0$ .

**Solution 7.18**

Separating variables,

$$y \, dy = 5 \, dx$$

gives

$$y^2 = 10x + C,$$

a family of right-opening parabolas. Using  $f(0) = -4$  gives  $C = 16$ , so

$$f(x) = -\sqrt{10x + 16}.$$

The square root needs  $10x + 16 \geq 0$ , but the DE requires  $y \neq 0$ . Since  $y = 0$  occurs at  $x = -\frac{8}{5}$ , the maximal domain containing 0 is

$$\left(-\frac{8}{5}, \infty\right).$$

**Takeaways 7.18**

- The domain of an IVP is controlled by the original DE, not only the explicit formula.
- A vertical tangent often marks the endpoint of the valid interval.

**Problem 7.19: Abstract Related Rates and Sign Analysis**

Suppose

$$\frac{dR}{dt} = -k^2, \quad \frac{dP}{dt} = -l^2 \frac{dR}{dt}, \quad \frac{dP}{dt} = m^2 \frac{dQ}{dt},$$

where  $k, l, m \neq 0$ .

- Express  $\frac{dP}{dt}$  and  $\frac{dQ}{dt}$  in terms of  $k, l, m$ .
- State whether  $P$  and  $Q$  are increasing or decreasing.
- Find  $\frac{dQ}{dR}$ .

- Use  $\frac{\partial P}{\partial R} \frac{dR}{dt} = \frac{dP}{dt}$ .
- Squares of non-zero reals are positive.
- Substitute sequentially.

**Hint:**

**Solution 7.19**

From  $\frac{dR}{dt} = -k^2$ ,

$$\frac{dP}{dt} = -l^2(-k^2) = k^2l^2.$$

Then

$$\frac{dQ}{dt} = \frac{1}{m^2} \frac{dP}{dt} = \frac{k^2l^2}{m^2}.$$

Since  $k^2, l^2, m^2 > 0$ , both  $\frac{dP}{dt}$  and  $\frac{dQ}{dt}$  are positive, so  $P$  and  $Q$  are increasing. Finally,

$$\frac{dQ}{dR} = \frac{dQ/dt}{dR/dt} = \frac{k^2l^2/m^2}{-k^2} = \boxed{-\frac{l^2}{m^2}}.$$

**Takeaways 7.19**

- Related-rate chains can often be solved by algebra before any heavier calculus is needed.
- Sign analysis is just as important as the exact magnitude.

**7.2 Medium****Problem 7.20: Transformations and Orthogonal Trajectories**

A family of curves can be represented by a single differential equation independent of its defining parameter. Furthermore, geometric transformations on the curves correspond to precise algebraic transformations on their differential equations.

- By differentiating implicitly, show that the family of rectangular hyperbolas  $x^2 - y^2 = C$  (where  $C$  is an arbitrary constant) satisfies the differential equation  $\frac{dy}{dx} = \frac{x}{y}$ .
- Orthogonal trajectories* are a second family of curves that intersect the original family at right angles. Because their gradients are the negative reciprocals of the original curves, their differential equation is found by replacing  $\frac{dy}{dx}$  with  $-\frac{dx}{dy}$ . Find the differential equation for the orthogonal trajectories of  $x^2 - y^2 = C$ , and solve it to find their Cartesian equations. Describe this new family of curves.
- Suppose the original family  $x^2 - y^2 = C$  is translated  $h$  units to the right and  $k$  units up. Write down the Cartesian equation of the translated family and show that it satisfies the differential equation  $\frac{dy}{dx} = \frac{x-h}{y-k}$ .
- Instead of a translation, suppose a curve satisfying the general DE  $\frac{dy}{dx} = f(x, y)$  is dilated horizontally by a factor of  $\alpha$  and vertically by a factor of  $\beta$  (i.e., replacing  $x$  with  $\frac{x}{\alpha}$  and  $y$  with  $\frac{y}{\beta}$  in its Cartesian equation). Prove that the differential equation of this new dilated curve is  $\frac{dy}{dx} = \frac{\beta}{\alpha} f\left(\frac{x}{\alpha}, \frac{y}{\beta}\right)$ .

- **For i:** Differentiate both sides of  $x^2 - y^2 = C$  with respect to  $x$ . Remember that the derivative of an arbitrary constant  $C$  is 0, which is how the parameter disappears.
- **For ii:** The gradient of the original family is  $m_1 = \frac{y}{x}$ . The gradient of the orthogonal family must be  $m_2 = -\frac{x}{y}$ . Set  $\frac{dy}{dx} = -\frac{x}{y}$  and solve this new separable differential equation.
- **For iii:** Apply the standard translation transformations to  $x$  and  $y$  in the Cartesian equation first (replace  $x$  with  $(x - h)$ , etc.), then differentiate implicitly just like in part i.
- **For iv:** Let the coordinates of the new dilated curve be  $X$  and  $Y$ , where  $X = \alpha x$  and  $Y = \beta y$ . Use differentials to find  $\frac{dY}{dX}$ . Substitute  $\frac{dy}{dx} = f(x, y)$  and then replace the old  $x$  and  $y$  variables with their  $X$  and  $Y$  equivalents.

**Hint:**

**Solution 7.20**

i Given the family of curves  $x^2 - y^2 = C$ .

Differentiating implicitly with respect to  $x$ :

$$\begin{aligned}\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) &= \frac{d}{dx}(C) \\ 2x - 2y \frac{dy}{dx} &= 0 \\ 2y \frac{dy}{dx} &= 2x \implies \frac{dy}{dx} = \frac{x}{y} \quad \square\end{aligned}$$

ii The differential equation for the original family is  $\frac{dy}{dx} = \frac{x}{y}$ .

For the orthogonal trajectories, the gradient is the negative reciprocal:

$$\frac{dy}{dx} = -\frac{y}{x}$$

This is a separable differential equation. Rearranging:

$$\begin{aligned}\int \frac{1}{y} dy &= \int -\frac{1}{x} dx \\ \ln |y| &= -\ln |x| + C_1 \\ \ln |y| + \ln |x| &= C_1 \implies \ln |xy| = C_1 \\ xy &= \pm e^{C_1}\end{aligned}$$

Let  $A = \pm e^{C_1}$ , which is just another arbitrary constant.

**Solution:**  $xy = A$ .

**Description:** This new family of curves is also a set of rectangular hyperbolas, but they are rotated by  $45^\circ$  so that the  $x$  and  $y$  axes act as their asymptotes.

iii Translating  $x^2 - y^2 = C$  by  $h$  right and  $k$  up gives:

**Cartesian equation:**  $(x - h)^2 - (y - k)^2 = C$

Differentiating implicitly with respect to  $x$ :

$$\begin{aligned}2(x - h) \cdot 1 - 2(y - k) \frac{dy}{dx} &= 0 \\ 2(y - k) \frac{dy}{dx} &= 2(x - h) \\ \frac{dy}{dx} &= \frac{x - h}{y - k} \quad \square\end{aligned}$$

iv Let a point on the original curve be  $(x, y)$ , satisfying  $\frac{dy}{dx} = f(x, y)$ .

Let a point on the dilated curve be  $(X, Y)$ .

The dilation maps  $X = \alpha x$  and  $Y = \beta y$ .

Therefore,  $x = \frac{X}{\alpha}$  and  $y = \frac{Y}{\beta}$ .

We want to find the gradient of the new curve,  $\frac{dY}{dX}$ .

Using differentials:

$$\begin{aligned}dY &= d(\beta y) = \beta dy \\ dX &= d(\alpha x) = \alpha dx\end{aligned}$$

Therefore:

$$\frac{dY}{dX} = \frac{\beta dy}{\alpha dx} = \frac{\beta}{\alpha} \left( \frac{dy}{dx} \right)$$

Since the original curve satisfies  $\frac{dy}{dx} = f(x, y)$ , we substitute this in:

$$\frac{dY}{dX} = \frac{\beta}{\alpha} f(x, y)$$

Finally, write the equation purely in terms of the new variables  $X$  and  $Y$ :

$$\frac{dY}{dX} = \frac{\beta}{\alpha} f\left(\frac{X}{\alpha}, \frac{Y}{\beta}\right)$$

Replacing the dummy variables  $X$  and  $Y$  back with  $x$  and  $y$  yields the required result:

$$\frac{dy}{dx} = \frac{\beta}{\alpha} f\left(\frac{x}{\alpha}, \frac{y}{\beta}\right) \quad \square$$

**Takeaways 7.20**

- **Orthogonal Trajectories:** This is a powerful application of DEs. If you know the “flow lines” of an electric field ( $x^2 - y^2 = C$ ), the orthogonal trajectories ( $xy = A$ ) perfectly map the equipotential lines where voltage is constant!
- **The Geometry of Calculus:** Part iv proves a beautiful, generalized rule. If you stretch a graph vertically by 3, all its slopes get 3 times steeper (the  $\frac{\beta}{\alpha}$  multiplier), but those slopes now occur at stretched coordinates (the  $\frac{x}{\alpha}, \frac{y}{\beta}$  inputs).

**Problem 7.21: Separable Equations, Conics, and Domains**

Consider the differential equation  $\frac{dy}{dx} = \frac{4-2x}{y+3}$ .

- i By separating the variables, integrate to find the general solution in implicit form.
- ii By completing the square, show that the general solution represents a family of ellipses, and state the coordinates of their common centre.
- iii Find the explicit particular solution,  $y = f(x)$ , that satisfies the initial condition  $y(2) = 1$ .
- iv State the maximal domain of the particular solution found in part iii, and explain geometrically why the domain is restricted.

**Hint:**

- **For i:** For a real-valued function, the expression inside a square root must be non-negative ( $\geq 0$ ). Solve this inequality for  $x$ . Geometrically, think about the shape of an ellipse and vertical tangent lines.
- **For ii:** Move the  $x^2$  and  $x$  terms to the same side as the  $y$  terms. Factor out any coefficients in front of  $x^2$ , and use the “completing the square” method ( $x^2 + bx + (\frac{b}{2})^2$ ). Remember to balance the equation by adding the equivalent value to the constant  $C$  on the right-hand side.
- **For iii:** Substitute  $x = 2$  and  $y = 1$  into your implicit equation from part ii to find the exact value of your constant. Then, rearrange the equation to make  $y$  the subject. You will get a  $\pm$  square root; test the initial condition (2, 1) again to determine whether to keep the positive or negative branch.
- **For iv:** For a real-valued function, the expression inside a square root must be non-negative ( $\geq 0$ ). Solve this inequality for  $x$ . Geometrically, think about the shape of an ellipse and vertical tangent lines.

**Solution 7.21**

i Given  $\frac{dy}{dx} = \frac{4-2x}{y+3}$ .

Separate the variables:

$$(y + 3) dy = (4 - 2x) dx$$

Integrate both sides:

$$\int (y + 3) dy = \int (4 - 2x) dx$$

$$\frac{1}{2}(y + 3)^2 = 4x - x^2 + C_1$$

Multiply by 2 to clear the fraction (let  $C_2 = 2C_1$ ):

$$(y + 3)^2 = 8x - 2x^2 + C_2$$

ii Rearrange to group the variables on the left side:

$$2x^2 - 8x + (y + 3)^2 = C_2$$

Factor out the 2 from the  $x$  terms to prepare for completing the square:

$$2(x^2 - 4x) + (y + 3)^2 = C_2$$

Complete the square inside the bracket by adding  $(\frac{-4}{2})^2 = 4$ . Since this 4 is multiplied by the 2 outside the bracket, we must add 8 to the right side to maintain balance:

$$2(x^2 - 4x + 4) + (y + 3)^2 = C_2 + 8$$

$$2(x - 2)^2 + (y + 3)^2 = K \quad (\text{where } K = C_2 + 8)$$

Dividing by  $K$  gives the standard form  $\frac{(x-2)^2}{K/2} + \frac{(y+3)^2}{K} = 1$ , which represents a family of **ellipses** with a common centre at  $(2, -3)$ .

iii We are given the initial condition  $y = 1$  when  $x = 2$ .

Substitute into the general implicit equation to find  $K$ :

$$2(2 - 2)^2 + (1 + 3)^2 = K$$

$$2(0)^2 + (4)^2 = K \implies K = 16$$

The implicit particular solution is  $2(x - 2)^2 + (y + 3)^2 = 16$ .

Now, solve explicitly for  $y$ :

$$(y + 3)^2 = 16 - 2(x - 2)^2$$

$$y + 3 = \pm \sqrt{16 - 2(x - 2)^2}$$

$$y = -3 \pm \sqrt{16 - 2(x - 2)^2}$$

We must choose the correct branch. Since  $y(2) = 1$ :

$$y(2) = -3 \pm \sqrt{16 - 0} = -3 \pm 4.$$

To get +1, we must take the positive root.

Therefore, the explicit particular solution is:

$$y = -3 + \sqrt{16 - 2(x - 2)^2}$$

iv For the function to be defined over the real numbers, the expression under the square root must be non-negative:

$$16 - 2(x - 2)^2 \geq 0$$

$$2(x - 2)^2 \leq 16$$

$$(x - 2)^2 \leq 8$$

$$-\sqrt{8} \leq x - 2 \leq \sqrt{8}$$

$$2 - 2\sqrt{2} \leq x \leq 2 + 2\sqrt{2}$$

Therefore, the maximal domain is  $[2 - 2\sqrt{2}, 2 + 2\sqrt{2}]$ .

*Geometric explanation:* The integral curve is an ellipse. An explicit function  $y = f(x)$  must pass the vertical line test. The positive square root represents only the “top half” of the ellipse. The domain is restricted because the curve ceases to exist beyond its left and right vertices, where the ellipse has vertical tangents.

**Takeaways 7.21**

- **The Power of Implicit Curves:** First-order differential equations of the form  $\frac{dy}{dx} = \frac{ax+b}{cy+d}$  almost always map to conic sections (circles, ellipses, hyperbolas, or parabolas).
- **Functions vs. Relations:** A differential equation describes the *entire* geometric shape (the relation). However, when you solve an Initial Value Problem to find  $y = f(x)$ , you are often carving out a single, valid *functional branch* (like the top half of an ellipse) that obeys the vertical line test.

**Problem 7.22: Singular Points and Self-Intersecting Curves**

Normally, a solution curve to a first-order differential equation cannot cross itself. If it did, it would have two different gradients at the intersection point, contradicting the nature of a function. However, this rule breaks down at a **singular point**, where the gradient expression becomes undefined in the form  $\frac{0}{0}$ .

Consider the curve defined by the equation  $y^2 = x^2 - x^4$ .

- i By differentiating implicitly, show that the differential equation for this curve is  $\frac{dy}{dx} = \frac{x(1-2x^2)}{y}$ .
- ii Find the coordinates of all points where the curve has a horizontal tangent, and all points where it has a vertical tangent. (Exclude the origin for now).
- iii The origin (0, 0) is a singular point. Substitute  $y = mx$  into the original Cartesian equation and analyze the behavior as  $x \rightarrow 0$  to find the two exact gradients at which the curve crosses itself at the origin.
- iv The curve can be parameterized by  $x = \sin(t)$ . Find the corresponding parametric equation for  $y$  in terms of  $t$ . Hence, verify your two gradients from part **iii** by calculating  $\frac{dy}{dt} \div \frac{dx}{dt}$  at the two different  $t$ -values that map to the origin.

- **For iv:** If  $x = \sin(t)$ , substitute this into  $x^2 - x^4$ . Factor it out and use a trigonometric identity to find a simple expression for  $y^2$ . Be careful with signs when taking the square root. What values of  $t$  (in the interval  $[-\pi, \pi]$ ) give  $x = 0$ ?
- **For iii:** If you substitute  $y = mx$  into  $y^2 = x^2 - x^4$ , you are essentially finding the intersection of the curve with a straight line through the origin. Divide by  $x^2$  and see what happens to  $m^2$  as  $x$  approaches 0.
- **For ii:** Horizontal tangents occur where the numerator of the gradient is 0 (and the denominator is not). Vertical tangents occur where the denominator is 0 (and the numerator is not). Once you find the  $x$ -values, substitute them back into the *original equation* to find the  $y$ -values!
- **For i:** Remember the chain rule for  $y^2$ . Differentiate both sides with respect to  $x$ .

**Hint:**

**Solution 7.22**

**i** Given  $y^2 = x^2 - x^4$ .

Differentiating both sides implicitly with respect to  $x$ :

$$2y \frac{dy}{dx} = 2x - 4x^3$$

$$\frac{dy}{dx} = \frac{2x - 4x^3}{2y} = \frac{x(1 - 2x^2)}{y} \quad \square$$

**ii Horizontal Tangents** ( $\frac{dy}{dx} = 0$ ):

The numerator must be 0:  $x(1 - 2x^2) = 0$ .

Since we are excluding the origin ( $x = 0$ ), we have  $1 - 2x^2 = 0 \implies x^2 = \frac{1}{2} \implies x = \pm \frac{1}{\sqrt{2}}$ .

Substitute these  $x$ -values back into the original equation:

$$y^2 = \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^4 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \implies y = \pm \frac{1}{2}$$

This gives four points with horizontal tangents:  $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{2}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$ .

**Vertical Tangents** ( $\frac{dx}{dy} = 0$ ):

The denominator must be 0:  $y = 0$ .

Substitute  $y = 0$  into the original equation:

$$0 = x^2 - x^4 = x^2(1 - x^2)$$

Excluding  $x = 0$ , we have  $1 - x^2 = 0 \implies x = \pm 1$ .

This gives two points with vertical tangents:  $(1, 0)$  **and**  $(-1, 0)$ .

**iii** To find the crossing gradients at the origin, substitute the line  $y = mx$  into  $y^2 = x^2 - x^4$ :

$$(mx)^2 = x^2 - x^4$$

$$m^2 x^2 = x^2(1 - x^2)$$

For  $x \neq 0$ , we can divide by  $x^2$ :

$$m^2 = 1 - x^2$$

As we approach the origin ( $x \rightarrow 0$ ), the gradients of the intersecting lines approach the gradients of the tangents:

$$m^2 = \lim_{x \rightarrow 0} (1 - x^2) = 1$$

$$m = \pm 1$$

Therefore, the curve crosses itself at the origin with gradients of **1 and -1**.

**iv** Given  $x = \sin(t)$ .

Substitute into the RHS of the original equation:

$$x^2 - x^4 = \sin^2(t) - \sin^4(t) = \sin^2(t)[1 - \sin^2(t)] = \sin^2(t) \cos^2(t)$$

Therefore,  $y^2 = (\sin(t) \cos(t))^2$ , so we can parameterize  $y$  as  $y = \sin(t) \cos(t)$ .

(Note: Using the double angle formula, this is  $y = \frac{1}{2} \sin(2t)$ ).

To find the gradients using parameters:

$$\frac{dx}{dt} = \cos(t)$$

$$\frac{dy}{dt} = \cos(2t)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos(2t)}{\cos(t)}$$

The curve passes through the origin ( $x = 0, y = 0$ ) when  $\sin(t) = 0$ .

In the domain  $[-\pi, \pi]$ , this occurs at  $t = 0, t = \pi$ , and  $t = -\pi$ .

Let's check the gradients at the two main "passes" through the origin:

- At  $t = 0$ :  $\frac{dy}{dx} = \frac{\cos(0)}{\cos(0)} = \frac{1}{1} = \mathbf{1}$ .
- At  $t = \pi$ :  $\frac{dy}{dx} = \frac{\cos(2\pi)}{\cos(\pi)} = \frac{1}{-1} = \mathbf{-1}$ .

This perfectly verifies the gradients found algebraically in part **iii**.

**Takeaways 7.22**

- **The Anatomy of  $\frac{0}{0}$ :** When both the numerator and denominator of a differential equation are zero, it doesn't mean the curve stops existing. It means the geometry is complex—usually a self-intersection, a cusp, or an isolated point.
- **Parametrization as a Loophole:** Implicit equations are notoriously difficult to analyze at singular points because  $y$  is not a true function of  $x$  (it fails the vertical line test). Parametric equations solve this by making  $x$  and  $y$  functions of time ( $t$ ), allowing the “particle” tracing the curve to peacefully pass through the origin twice at different times!

**Problem 7.23: Homogeneous Differential Equations and Domain Boundaries**

Consider the non-linear differential equation  $x \frac{dy}{dx} = y + \sqrt{x^2 - y^2}$  defined for  $x > 0$ .

- i By using the substitution  $y = ux$ , where  $u$  is an unknown function of  $x$ , show that the differential equation transforms into the separable equation  $x \frac{du}{dx} = \sqrt{1 - u^2}$ .
- ii Solve this separable differential equation to find the general solution for  $y$  in implicit form.
- iii Find the explicit particular solution  $y = f(x)$  given the initial condition  $y(1) = 0$ .
- iv Determine the maximal domain of the particular solution found in part iii. Explain geometrically what happens to the solution curve at the endpoints of this domain.

**Hint:**

- **For i:** Use the product rule on  $y = ux$  to find  $\frac{dy}{dx}$ . Substitute both  $y$  and  $\frac{dy}{dx}$  into the original equation. Factor  $x^2$  out of the square root, noting that since  $x > 0$ ,  $\sqrt{x^2} = x$ .
- **For ii:** Rearrange the equation to group all  $u$  terms with  $du$  and all  $x$  terms with  $dx$ . You will need to recall the standard integral  $\int \frac{1}{\sqrt{a^2 - x^2}} dx$ .
- **For iii:** Substitute  $x = 1$  and  $y = 0$  into your implicit equation to find the constant of integration, then rearrange to make  $y$  the subject.
- **For iv:** Look closely at your integration step involving the inverse trigonometric function. What is the range of  $\arcsin(\theta)$ ? Apply this inequality to the right-hand side of your equation to find the permissible values of  $x$ .

**Solution 7.23**

**i** Let  $y = ux$ .

Differentiating with respect to  $x$  using the product rule:

$$\frac{dy}{dx} = u \cdot (1) + x \cdot \frac{du}{dx} = u + x \frac{du}{dx}$$

Substitute  $y$  and  $\frac{dy}{dx}$  into the original differential equation:

$$\begin{aligned} x \left( u + x \frac{du}{dx} \right) &= ux + \sqrt{x^2 - (ux)^2} \\ xu + x^2 \frac{du}{dx} &= ux + \sqrt{x^2(1 - u^2)} \end{aligned}$$

Since  $x > 0$ , we can simplify  $\sqrt{x^2}$  to exactly  $x$ :

$$xu + x^2 \frac{du}{dx} = xu + x\sqrt{1 - u^2}$$

Subtract  $xu$  from both sides:

$$x^2 \frac{du}{dx} = x\sqrt{1 - u^2}$$

Divide by  $x$  (since  $x > 0$ ,  $x \neq 0$ ):

$$x \frac{du}{dx} = \sqrt{1 - u^2} \quad \square$$

**ii** Separate the variables  $u$  and  $x$ :

$$\frac{1}{\sqrt{1 - u^2}} du = \frac{1}{x} dx$$

Integrate both sides:

$$\int \frac{1}{\sqrt{1 - u^2}} du = \int \frac{1}{x} dx$$

$$\arcsin(u) = \ln(x) + C \quad (\text{Note: absolute value on } x \text{ is not needed as } x > 0)$$

Substitute  $u = \frac{y}{x}$  back into the equation:

$$\arcsin\left(\frac{y}{x}\right) = \ln(x) + C$$

**iii** Apply the initial condition  $y(1) = 0$ :

$$\begin{aligned} \arcsin\left(\frac{0}{1}\right) &= \ln(1) + C \\ 0 &= 0 + C \implies C = 0 \end{aligned}$$

The implicit equation is  $\arcsin\left(\frac{y}{x}\right) = \ln(x)$ .

Taking the sine of both sides to make  $y$  the subject:

$$\begin{aligned} \frac{y}{x} &= \sin(\ln x) \\ \mathbf{y} &= \mathbf{x} \sin(\ln \mathbf{x}) \end{aligned}$$

**iv** From the integration step, we have the relation  $\arcsin\left(\frac{y}{x}\right) = \ln(x)$ .

The principal range of the arcsine function is  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ . Therefore, the right-hand side must satisfy this restriction for the equation to hold:

$$-\frac{\pi}{2} \leq \ln(x) \leq \frac{\pi}{2}$$

Exponentiating all parts of the inequality (base  $e$ ):

$$e^{-\pi/2} \leq x \leq e^{\pi/2}$$

The maximal domain is  $[e^{-\pi/2}, e^{\pi/2}]$ .

*Geometric Explanation:* The original differential equation contains the term  $\sqrt{x^2 - y^2}$ , which requires  $x^2 \geq y^2$ , meaning the solution must be trapped in the region between the lines  $y = x$  and  $y = -x$ . At the endpoints of the domain ( $x = e^{\pm\pi/2}$ ),  $\ln(x) = \pm\frac{\pi}{2}$ , causing  $\sin(\ln x) = \pm 1$ . This means  $\frac{y}{x} = \pm 1 \implies y = \pm x$ . The solution curve terminates precisely because it collides with these boundary lines, where it becomes tangent to them.

**Takeaways 7.23**

- **Homogeneous Equations:** Differential equations where every term can be written as a function of  $(y/x)$  are called *homogeneous* first-order DEs. The substitution  $y = ux$  is the universal key to transforming them into simple separable equations.
- **Hidden Domain Restrictions:** Algebraic manipulation can sometimes hide restrictions. Even though the final function  $y = x \sin(\ln x)$  seems like it should be valid for all  $x > 0$ , the integration step involving  $\arcsin(u)$  acts as a mathematical “funnel,” enforcing a strict boundary on the domain.

**Problem 7.24: Inverse Trigonometric Identities and Restricted Domains**

Consider the differential equation  $\sqrt{1-x^2} \frac{dy}{dx} + \sqrt{1-y^2} = 0$ . Suppose that  $y = f(x)$  is a solution curve passing through the point  $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ .

- i By separating the variables, integrate to find the general solution in the form  $\arcsin y + \arcsin x = C$ .
- ii By taking the sine of both sides of your general solution, and using the compound angle expansion for  $\sin(A + B)$ , show that the general solution is algebraically equivalent to  $y\sqrt{1-x^2} + x\sqrt{1-y^2} = k$ , where  $k$  is a constant.
- iii Find the explicit values of  $C$  and  $k$  for the given initial condition to establish the particular solution.
- iv By rearranging your inverse trigonometric equation from part i, make  $y$  the explicit subject of the equation. State the maximal valid domain for  $x$  and describe the geometric shape of this specific solution curve.

**Hint:**

- **For i:** Move the  $x$  and  $y$  terms to opposite sides of the equation. Recall the standard integral  $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x$ .
- **For ii:** Let  $A = \arcsin y$  and  $B = \arcsin x$ . This means  $\sin A = y$  and  $\sin B = x$ . To use the expansion  $\sin(A + B) = \sin A \cos B + \cos A \sin B$ , you will need to find expressions for  $\cos A$  and  $\cos B$  by drawing right-angled triangles or using the identity  $\sin^2 \theta + \cos^2 \theta = 1$ .
- **For iii:** Substitute  $x = \frac{1}{2}$  and  $y = \frac{\sqrt{3}}{2}$  into your equations. Note that  $\arcsin(\frac{1}{2}) = \frac{\pi}{6}$  and  $\arcsin(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$ .
- **For iv:** Make  $\arcsin y$  the subject, then take the sine of both sides. You will need to use the complementary angle property  $\sin(\frac{\pi}{2} - \theta) = \cos \theta$ . Critically, think about the range of  $\arcsin y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$  to determine the valid domain for  $x$ .

**Solution 7.24**

i Given  $\sqrt{1-x^2} \frac{dy}{dx} + \sqrt{1-y^2} = 0$ .  
Separate the variables:

$$\frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\int \frac{1}{\sqrt{1-y^2}} dy = -\int \frac{1}{\sqrt{1-x^2}} dx$$

Evaluating the integrals yields:

$$\arcsin y = -\arcsin x + C$$

$$\arcsin y + \arcsin x = C \quad \square$$

ii Take the sine of both sides of the general solution:

$$\sin(\arcsin y + \arcsin x) = \sin C$$

Let  $k = \sin C$ . Apply the compound angle formula  $\sin(A + B) = \sin A \cos B + \cos A \sin B$ :

$$\sin(\arcsin y) \cos(\arcsin x) + \cos(\arcsin y) \sin(\arcsin x) = k$$

We know that  $\sin(\arcsin y) = y$  and  $\sin(\arcsin x) = x$ .

To find  $\cos(\arcsin x)$ , let  $\theta = \arcsin x$ . Then  $\sin \theta = x$ .

Since the range of  $\arcsin$  is  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ ,  $\cos \theta \geq 0$ , so  $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - x^2}$ .

Similarly,  $\cos(\arcsin y) = \sqrt{1 - y^2}$ .

Substituting these back into the expansion gives:

$$y\sqrt{1-x^2} + x\sqrt{1-y^2} = k \quad \square$$

iii Apply the initial condition  $y = \frac{\sqrt{3}}{2}$  when  $x = \frac{1}{2}$  to the equation from part i:

$$C = \arcsin\left(\frac{\sqrt{3}}{2}\right) + \arcsin\left(\frac{1}{2}\right)$$

$$C = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$$

Therefore,  $k = \sin\left(\frac{\pi}{2}\right) = 1$ .

The particular solution is  $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$ .

iv Start with the particular solution from part i:

$$\arcsin y + \arcsin x = \frac{\pi}{2}$$

$$\arcsin y = \frac{\pi}{2} - \arcsin x$$

Take the sine of both sides to make  $y$  the explicit subject:

$$y = \sin\left(\frac{\pi}{2} - \arcsin x\right)$$

Using the complementary angle identity  $\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta$ :

$$y = \cos(\arcsin x)$$

Using the result proven in part ii,  $\cos(\arcsin x) = \sqrt{1-x^2}$ .

Therefore,  $y = \sqrt{1-x^2}$ .

**Domain Restriction:**

The principal range of  $\arcsin y$  is strictly  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ .

Therefore, we must have  $-\frac{\pi}{2} \leq \frac{\pi}{2} - \arcsin x \leq \frac{\pi}{2}$ .

Solving this inequality for  $\arcsin x$ :

$$-\pi \leq -\arcsin x \leq 0 \implies 0 \leq \arcsin x \leq \pi$$

However, the maximal output of  $\arcsin x$  itself is  $\frac{\pi}{2}$ . Thus, the overlapping valid range is  $0 \leq \arcsin x \leq \frac{\pi}{2}$ , which means  $0 \leq x \leq 1$ .

*Geometric Description:* Although  $y = \sqrt{1-x^2}$  generally represents the upper half of a unit circle, the domain restriction forces  $x \in [0, 1]$ . Therefore, the true solution curve is specifically the **quarter of the unit circle located in the first quadrant**.

**Takeaways 7.24**

- **Trigonometry in Calculus:** Differential equations are rarely purely algebraic. Memorizing how to simplify expressions like  $\cos(\arcsin x)$  using right-angled triangles or Pythagorean identities is essential for Extension 2.
- **The Danger of Algebraic Forms:** Notice how the explicit answer  $y = \sqrt{1 - x^2}$  looks like it should be valid for  $-1 \leq x \leq 1$ . However, tracking the inverse trigonometric functions reveals a hidden domain restriction ( $0 \leq x \leq 1$ ). Always check the range of the original functions before simplifying!

**Problem 7.25: Picard Iterations and Series Convergence**

The existence and uniqueness of solutions to first-order differential equations  $y' = f(x, y)$  can be proven constructively using Picard iterations. The sequence of approximation functions is defined by:

$$y_{n+1}(x) = y_0 + \int_{x_0}^x f(t, y_n(t)) dt$$

where  $y(x_0) = y_0$  is the initial condition, and  $y_0(x) = y_0$  is the constant initial guess. Consider the non-separable linear differential equation  $y' = x + y$  with the initial condition  $y(0) = 1$ .

- i** By rearranging the equation and using the method of integrating factors, find the exact analytical solution  $y(x)$  to this Initial Value Problem.
- ii** Compute the first four Picard iterates:  $y_1(x)$ ,  $y_2(x)$ ,  $y_3(x)$ , and  $y_4(x)$ .
- iii** Write down the Maclaurin series expansion for your exact solution from part **i** up to the term involving  $x^4$ . Compare this expansion term-by-term with  $y_4(x)$ . What do you observe about the behaviour of the Picard sequence as  $n$  increases?
- iv** The exact solution evaluated at  $x = 1$  allows us to express Euler's number  $e$  in terms of  $y(1)$ . Rearrange your exact solution to make  $e$  the subject. Then, use  $y_4(1)$  as an approximation for  $y(1)$  to estimate the value of  $e$ . Without using a calculator, explain whether your approximation is an over-estimate or an under-estimate.

**Hint:**

- **For i:** Rearrange to  $\frac{dy}{dx} - y = x$ . The integrating factor is  $e^{-x}$ . Multiply both sides, use the reverse product rule on the left, and use integration by parts for the right-hand side ( $\int x e^{-x} dx$ ).
- **For ii:** For this DE,  $f(t, y) = t + y$ . Start with  $y_0(x) = 1$ . Then  $y_1(t) = 1 + \int_x^0 (t+1) dt$ . To find  $y_2(x)$ , substitute your result for  $y_1(t)$  back into the integral:  $y_2(x) = 1 + \int_x^0 (t + y_1(t)) dt$ .
- **For iii:** Use the standard series  $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$  and substitute it into your exact solution  $y(x)$ . Expand and group the like terms.
- **For iv:** Calculate  $y(1)$  using your exact formula and solve for  $e$ . Then plug  $x = 1$  into your  $y_4(x)$  polynomial. Compare the terms of your polynomial to the infinite Maclaurin series to deduce if pieces are missing.

**Solution 7.25**

i Using integrating factor  $e^{-x}$ ,

$$\frac{d}{dx}(ye^{-x}) = xe^{-x}.$$

Since  $\int x e^{-x} dx = -x e^{-x} - e^{-x} + C$ , we obtain

$$y = -x - 1 + C e^x.$$

From  $y(0) = 1$ ,  $C = 2$ , so  $y = 2e^x - x - 1$ .

ii Starting with  $y_0 = 1$ ,

$$y_1 = 1 + x + \frac{x^2}{2}, \quad y_2 = 1 + x + x^2 + \frac{x^3}{6},$$

$$y_3 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}, \quad y_4 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}.$$

iii From  $y = 2e^x - x - 1$ ,

$$y = 2 \left( 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \right) - x - 1 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \dots$$

so  $y_4$  agrees with the exact series through the  $x^4$  term.

iv Since  $y(1) = 2e - 2$ , we have  $e = \frac{y(1) + 2}{2}$ . Also

$$y_4(1) = 1 + 1 + 1 + \frac{1}{3} + \frac{1}{12} + \frac{1}{120} = \frac{137}{40}.$$

Hence

$$e \approx \frac{\frac{137}{40} + 2}{2} = \frac{217}{80} = 2.7125.$$

This is an under-estimate because the omitted terms in the exponential series are positive.

**Takeaways 7.25**

- **The Magic of Picard Iterations:** This method doesn't just give you a "close" curve; it mechanically derives the exact Taylor Polynomial of the solution without ever requiring you to know the actual function!
- **Self-Correction:** Notice in part **ii** how the  $x^3$  term changed from  $x^3/6$  in  $y_2$  to  $x^3/3$  in  $y_3$ , and then stayed  $x^3/3$  in  $y_4$ . Picard iterations continuously refine the outer terms until they settle into their permanent, correct Taylor coefficients.

**Problem 7.26: Higher-Order Differential Equations and Real Roots**

1. For the following differential equations, use substitution to find all real values of  $\lambda$  that make the function  $y = ke^{\lambda x}$  a solution for all values of the constant  $k$ .  
 (Note:  $y^{(4)}$  denotes the fourth derivative of  $y$  with respect to  $x$ ).

**i**  $y''' - 4y'' - 7y' + 10y = 0$

**ii**  $y^{(4)} - 13y'' + 36y = 0$

**iii**  $y''' - 3y'' + 4y = 0$

2. Consider the differential equation  $y'' + 2cy' + 9y = 0$ . Find the possible values of the real constant  $c$  such that substituting  $y = ke^{\lambda x}$  yields exactly one repeated real root for  $\lambda$ .

- Hint:**
- **For (i):** The third derivative is  $y''' = k\lambda^3 e^{\lambda x}$ . After factoring out the exponential, use the Factor Theorem (testing factors of 10) to find the first root, then use polynomial long division.
  - **For (ii):** You will get a degree-4 polynomial, but notice there are no odd powers of  $\lambda$ . Treat it as a quadratic "in disguise" by letting  $u = \lambda^2$ .
  - **For (iii):** Be careful with missing terms! The characteristic equation is a cubic with no  $\lambda$  term. Once you find the first root, the remaining quadratic will yield a repeated root.
  - **For (iv):** Find the characteristic equation in terms of  $\lambda$  and  $c$ . What must the discriminant ( $\Delta = b^2 - 4ac$ ) of a quadratic equal for it to have exactly one repeated root?

**Solution 7.26**

**i**  $y''' - 4y'' - 7y' + 10y = 0$

Let  $y = ke^{\lambda x}$ . The derivatives are  $y' = k\lambda e^{\lambda x}$ ,  $y'' = k\lambda^2 e^{\lambda x}$ , and  $y''' = k\lambda^3 e^{\lambda x}$ .

Substitute into the DE:

$$k\lambda^3 e^{\lambda x} - 4(k\lambda^2 e^{\lambda x}) - 7(k\lambda e^{\lambda x}) + 10(ke^{\lambda x}) = 0$$

$$ke^{\lambda x}(\lambda^3 - 4\lambda^2 - 7\lambda + 10) = 0$$

Since  $ke^{\lambda x} \neq 0$  (for  $k \neq 0$ ), we solve:

$$P(\lambda) = \lambda^3 - 4\lambda^2 - 7\lambda + 10 = 0$$

By inspection,  $P(1) = 1 - 4 - 7 + 10 = 0$ , so  $(\lambda - 1)$  is a factor.

Using polynomial division or grouping:

$$(\lambda - 1)(\lambda^2 - 3\lambda - 10) = 0$$

$$(\lambda - 1)(\lambda - 5)(\lambda + 2) = 0$$

$$\lambda = 1, \lambda = 5, \text{ or } \lambda = -2$$

**ii**  $y^{(4)} - 13y'' + 36y = 0$

Substitute  $y = ke^{\lambda x}$  and  $y^{(4)} = k\lambda^4 e^{\lambda x}$ :

$$ke^{\lambda x}(\lambda^4 - 13\lambda^2 + 36) = 0$$

$$\lambda^4 - 13\lambda^2 + 36 = 0$$

Let  $u = \lambda^2$ . Then  $u^2 - 13u + 36 = 0$ .

$$(u - 9)(u - 4) = 0 \implies (\lambda^2 - 9)(\lambda^2 - 4) = 0$$

$$(\lambda - 3)(\lambda + 3)(\lambda - 2)(\lambda + 2) = 0$$

$$\lambda = 3, \lambda = -3, \lambda = 2, \text{ or } \lambda = -2$$

**iii**  $y''' - 3y'' + 4y = 0$

Substitute the derivatives:

$$ke^{\lambda x}(\lambda^3 - 3\lambda^2 + 4) = 0$$

$$P(\lambda) = \lambda^3 - 3\lambda^2 + 4 = 0$$

By inspection,  $P(-1) = -1 - 3 + 4 = 0$ , so  $(\lambda + 1)$  is a factor.

By polynomial division:

$$(\lambda + 1)(\lambda^2 - 4\lambda + 4) = 0$$

$$(\lambda + 1)(\lambda - 2)^2 = 0$$

$$\lambda = -1 \text{ or } \lambda = 2$$

**iv**  $y'' + 2cy' + 9y = 0$

Substitute  $y, y', y''$ :

$$ke^{\lambda x}(\lambda^2 + 2c\lambda + 9) = 0$$

The characteristic equation is  $\lambda^2 + 2c\lambda + 9 = 0$ .

For a quadratic to have exactly one repeated real root, its discriminant must be zero ( $\Delta = 0$ ):

$$\Delta = b^2 - 4ac = 0$$

$$(2c)^2 - 4(1)(9) = 0$$

$$4c^2 - 36 = 0$$

$$4c^2 = 36 \implies c^2 = 9$$

$$c = 3 \text{ or } c = -3$$

(Note: If  $c = 3$ , the root is  $\lambda = -3$ . If  $c = -3$ , the root is  $\lambda = 3$ .)

**Takeaways 7.26**

- **Polynomial Masterclass:** Higher-order differential equations are the perfect playground for the polynomials topic. You must confidently switch between the Factor Theorem, polynomial division, and recognizing quadratic forms hidden in degree-4 equations.
- **The Power of Notation:** Notice how elegantly the  $ke^{\lambda x}$  term factors out of every single derivative, leaving behind a clean polynomial? This property is unique to  $e^x$ , making it the most important function in all of calculus.
- **Repeated Roots (Part iii):** When you get a repeated root like  $\lambda = 2$ , it means  $y = ke^{2x}$  is a valid solution, but there's a missing piece to the puzzle. In advanced differential equations, you will learn that the "missing" second solution is actually  $y = kxe^{2x}$ !

**Problem 7.27: Advanced Initial Value Problems**

Solve these initial value problems. In each case, use integration to find the general solution, then use the initial condition to evaluate the constant. Where possible, express  $y$  explicitly as a function of  $x$ .

- i  $y' = y^2 \cos x$ , with  $y(\pi) = 1$
- ii  $y' = xe^{x-y}$ , with  $y(0) = 0$
- iii  $y' = \frac{x}{y\sqrt{1-x^2}}$ , with  $y(0) = -2$
- iv  $(1 + x^2)y' = xy(1 + y)$ , with  $y(0) = 1$

**Hint:**

- **For i:** Separate the variables to get  $\frac{1}{y^2} dy = \cos x dx$ . Be careful with negative signs when integrating  $y^{-2}$ .
- **For ii:** Use exponent laws to split  $e^{x-y}$  into  $e^x e^{-y}$  before separating variables. You will need to use **Integration by Parts** to integrate the right-hand side ( $\int x e^x dx$ ).
- **For iii:** Integrate both sides. When you rearrange to make  $y$  the explicit subject, you will need to take a square root, which yields a  $\pm$  case. Check your initial condition  $y(0) = -2$  to determine which branch is correct.
- **For iv:** Separating variables yields  $\frac{1}{1+y} dy = \frac{x}{1+x^2} dx$ . You must use **Partial Fractions** to integrate the left-hand side. Use logarithm laws to combine the resulting terms before applying the exponential function to isolate  $y$ .

**Solution 7.27**

i  $\int y^{-2} dy = \int \cos x dx$  gives  $-\frac{1}{y} = \sin x + C$ . Using  $y(\pi) = 1$  gives  $C = -1$ , so  $y = \frac{1}{1 - \sin x}$ .

ii Since  $y' = xe^{x-y} = xe^x e^{-y}$ ,

$$\int e^y dy = \int xe^x dx = xe^x - e^x + C.$$

From  $y(0) = 0$ ,  $1 = -1 + C$ , so  $C = 2$ . Hence

$$e^y = e^x(x - 1) + 2 \Rightarrow y = \ln(e^x(x - 1) + 2).$$

iii  $\int y dy = \int \frac{x}{\sqrt{1-x^2}} dx$  gives

$$\frac{1}{2}y^2 = -\sqrt{1-x^2} + C.$$

Using  $y(0) = -2$  gives  $C = 3$ , so

$$y^2 = 6 - 2\sqrt{1-x^2}.$$

The initial condition selects the negative branch:

$$y = -\sqrt{6 - 2\sqrt{1-x^2}}.$$

iv Since  $\frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$ ,

$$\ln \left| \frac{y}{y+1} \right| = \frac{1}{2} \ln(1+x^2) + C.$$

Using  $y(0) = 1$  gives  $C = \ln \frac{1}{2}$ , hence

$$\frac{y}{y+1} = \frac{\sqrt{1+x^2}}{2}.$$

Solving for  $y$ ,

$$y = \frac{\sqrt{1+x^2}}{2 - \sqrt{1+x^2}}.$$

**Takeaways 7.27**

- **Calculus is the easy part, Algebra is the hard part:** Notice in part iv that the actual integration takes two lines, but cleanly isolating  $y$  requires strong algebraic manipulation and logarithm law fluency. This is a hallmark of Extension 2.
- **The Importance of Branch Selection:** In part iii, lazily writing  $y = \sqrt{\dots}$  without acknowledging the  $\pm$  case will result in a solution curve that physically violates the given initial condition. Always look back at the starting coordinates when taking an even root.
- **Logarithmic Constants:** When an equation is dominated by logarithms (like part iv), leaving the constant as  $C$  makes isolation difficult. Finding  $C$  as a specific logarithmic value (e.g.,  $C = \ln(0.5)$ ) allows you to merge it cleanly using  $\log A + \log B = \log(AB)$ .

**Problem 7.28: Non-Linear Autonomous Initial Value Problems**

Find the explicit solutions  $y = f(x)$  for the following autonomous initial value problems.

- i  $y' = -2y^2$ , with  $y(0) = 1$
- ii  $y' = y^2 + 1$ , with  $y(0) = 1$
- iii  $y' = 4 - y^2$ , with  $y(0) = 0$
- iv  $y' = y \ln y$ , with  $y(0) = e$

- **For i:** Separate the variables to get  $\int \frac{1}{y^2} dy = \int -2 dx$ . To integrate the left side, use the substitution  $u = \ln y$ .
- **For ii:** When you separate the variables, you should recognize a standard inverse trigonometric integral on the left-hand side.
- **For iii:** Separating the variables yields  $\int \frac{1}{4-y^2} dy$ . You must use **Partial Fractions** (difference of two squares) to integrate this. Combine the resulting logarithms before using the exponential function to isolate  $y$ .
- **For iv:** Separate the variables to get  $\int \frac{1}{y \ln y} dy = \int 1 dx$ . To integrate the left side, use the substitution  $u = \ln y$ .

**Hint:**

**Solution 7.28**

i  $y' = -2y^2$ . Then

$$\int y^{-2} dy = \int -2 dx \implies -\frac{1}{y} = -2x + C.$$

Using  $y(0) = 1$  gives  $C = -1$ , so

$$y = \frac{1}{2x + 1}.$$

ii  $y' = y^2 + 1$ . Hence

$$\int \frac{dy}{1 + y^2} = \int dx \implies \arctan y = x + C.$$

Since  $y(0) = 1$ , we get  $C = \frac{\pi}{4}$ , so

$$y = \tan\left(x + \frac{\pi}{4}\right).$$

iii  $y' = 4 - y^2$ . Separating variables,

$$\int \frac{dy}{4 - y^2} = x + C.$$

Using

$$\frac{1}{4 - y^2} = \frac{1}{4} \left( \frac{1}{2 + y} + \frac{1}{2 - y} \right),$$

we obtain

$$\ln \left| \frac{2 + y}{2 - y} \right| = 4x + K.$$

Now  $y(0) = 0$  gives  $K = 0$ , so

$$\frac{2 + y}{2 - y} = e^{4x}.$$

Solving for  $y$ ,

$$y = \frac{2(e^{4x} - 1)}{e^{4x} + 1} = 2 \tanh(2x).$$

iv  $y' = y \ln y$ . Then

$$\int \frac{dy}{y \ln y} = \int dx.$$

Let  $u = \ln y$ , so  $du = \frac{1}{y} dy$ . This gives

$$\ln |\ln y| = x + C.$$

Since  $y(0) = e$ , we have  $C = 0$ , hence

$$\ln y = e^x \implies y = e^{e^x}.$$

**Takeaways 7.28**

- **The Geometry of Autonomous DEs:** An autonomous differential equation lacks an independent variable ( $x$ ) on the right-hand side. Geometrically, this means that the slope of the curve depends *only* on its vertical height ( $y$ ). If you draw a horizontal line across the slope field, every single vector on that line will point at the exact same angle!
- **Integration Toolkit:** When separating variables for  $y' = g(y)$ , the integral  $\int \frac{1}{g(y)} dy$  dictates the shape. A quadratic  $g(y)$  usually leads to inverse trigonometric functions (if roots are complex) or logarithms/hyperbolic functions via partial fractions (if roots are real).

**Problem 7.29: Innovation Diffusion and Logistic Models**

The NSW Education Standards Authority (NESA) is rolling out a new “AI Pedagogical Twin” system to assist high school mathematics teachers across the state. The maximum capacity (total number of target teachers) is  $M = 12\,000$ .

Let  $N(t)$  be the number of teachers who have adopted the system at time  $t$  months after its initial release. The adoption rate is proportional to both the number of current users and the number of teachers who have yet to adopt it. This is modelled by the differential equation:

$$\frac{dN}{dt} = kN(M - N)$$

where  $k$  is a positive constant.

- i By using partial fractions to integrate  $\frac{M}{N(M-N)}$ , show that the explicit solution for  $N(t)$  can be written in the form  $N(t) = \frac{M}{1 + Ae^{-Mkt}}$ , where  $A$  is a constant.
- ii At the initial launch ( $t = 0$ ), exactly 100 early adopters are using the system. After 6 months, the number of users has grown to 1 000. Find the exact value of  $A$ , and show that  $Mk = \frac{1}{6} \ln\left(\frac{119}{11}\right)$ .
- iii The “tipping point” of a technology rollout occurs when the adoption rate  $\frac{dN}{dt}$  is at its absolute maximum. Determine the exact time  $t$  (in months) when this tipping point is reached for the AI Pedagogical Twin.
- iv Suppose NESA realizes that teachers are also retiring or leaving the profession at a constant rate of  $R$  teachers per month, altering the model to  $\frac{dN}{dt} = kN(M - N) - R$ . Show mathematically that if the departure rate  $R$  exceeds  $\frac{kM^2}{4}$ , the rollout will inevitably fail (i.e., the number of users will always decrease regardless of the current adoption numbers).

**Hint:**

- **For i:** Start by showing that  $\frac{N}{M} \frac{d(N-M)}{N} = \frac{N}{1} + \frac{M-N}{1}$ . Separate the variables in the original DE, multiply both sides by  $M$ , and integrate. Use logarithm laws to combine terms before exponentiating.
- **For ii:** Substitute  $t = 0$ ,  $N = 100$ , and  $M = 12000$  to solve for  $A$ . Then substitute  $t = 6$  and  $N = 1000$  to isolate the expression  $e^{-6Mk}$ , which will allow you to solve for  $Mk$ .
- **For iii:** Do not differentiate the ugly  $N(t)$  fraction! Instead, look at the original differential equation. The rate is a downward-opening quadratic function in terms of  $N$ . Find the value of  $N$  that maximizes this quadratic, and then substitute that  $N$  back into your part i equation to find  $t$ .
- **For iv:** Expand the modified differential equation into a quadratic in terms of  $N$ :  $\frac{dN}{dt} = -kN^2 + kMN - R$ . For the rate to be strictly negative for all possible values of  $N$ , what must be true about the discriminant ( $\Delta = b^2 - 4ac$ ) of this quadratic?

**Solution 7.29**

i Since  $\frac{M}{N(M-N)} = \frac{1}{N} + \frac{1}{M-N}$ ,

$$\int \left( \frac{1}{N} + \frac{1}{M-N} \right) dN = \int Mk dt$$

gives

$$\ln \frac{N}{M-N} = Mkt + C.$$

Solving for  $N$ ,

$$N(t) = \frac{M}{1 + Ae^{-Mkt}}.$$

ii With  $M = 12000$ ,

$$100 = \frac{12000}{1 + A} \Rightarrow A = 119.$$

Then

$$1000 = \frac{12000}{1 + 119e^{-6Mk}} \Rightarrow e^{-6Mk} = \frac{11}{119},$$

so

$$Mk = \frac{1}{6} \ln \frac{119}{11}.$$

iii The rate  $kN(M-N)$  is maximal at  $N = M/2 = 6000$ . Hence

$$6000 = \frac{12000}{1 + 119e^{-Mkt}} \Rightarrow 119e^{-Mkt} = 1 \Rightarrow Mkt = \ln 119.$$

Therefore

$$t = \frac{6 \ln 119}{\ln(119/11)}.$$

iv The modified rate is  $-kN^2 + kMN - R$ . For it to be negative for all  $N$ , its discriminant must be negative:

$$(kM)^2 - 4(-k)(-R) < 0 \Rightarrow k^2M^2 - 4kR < 0.$$

Since  $k > 0$ ,

$$R > \frac{kM^2}{4}.$$

**Takeaways 7.29**

- **Beyond Biology:** The logistic equation  $y' = ky(P - y)$  isn't just for animals on an island. It is the fundamental mathematics behind viral marketing, technology adoption (like AI tools in schools), and the spread of rumors.
- **The Point of Inflection:** Part iii asks for the maximum *rate*, which translates graphically to the point of inflection on the S-curve. Noticing that you can maximize the quadratic DE itself rather than taking the second derivative of the complex fraction saves massive amounts of exam time.
- **Calculus meets Quadratics:** Part iv is a brilliant demonstration of Extension 2 synthesis. You are using the Discriminant from Year 11 polynomial theory to prove a physical constraint on a complex differential equation model!

**Problem 7.30: Newton’s Law of Cooling in the Blue Mountains**

A freshly baked meat pie is purchased from a famous bakery in Katoomba in the Blue Mountains. It is taken out of the pie warmer at a temperature of 95°C and placed outside, where the ambient temperature on a crisp winter day is a constant 5°C.

According to Newton’s Law of Cooling, the rate of change of the pie’s temperature  $T$  with respect to time  $t$  (in minutes) is proportional to the difference between its temperature and the ambient temperature. This is modelled by the differential equation:

$$\frac{dT}{dt} = -k(T - 5)$$

where  $k$  is an unknown positive constant.

- i** By separating the variables and integrating, show that the temperature of the pie is given by  $T(t) = 5 + Ae^{-kt}$ , and determine the value of the constant  $A$ .
- ii** After 10 minutes, the pie has cooled to 65°C. Show that  $e^{-10k} = \frac{2}{3}$ , and hence write  $T(t)$  in the alternative form  $T(t) = 5 + 90 \left(\frac{2}{3}\right)^{\frac{t}{10}}$ .
- iii** A local bushwalker considers the pie “safe to eat” without burning their mouth when it reaches 50°C. Calculate exactly how many minutes it takes to reach this temperature. Leave your answer in terms of natural logarithms.
- iv** Evaluate the exact rate at which the pie is cooling at the instant it becomes safe to eat. Briefly explain, referring to the differential equation, why this rate is exactly half of its initial cooling rate.

**Hint:**

- **For i:** Separate the variables to get  $\int \frac{dT}{T-5} = \int -k dt$ . After integrating to get a natural logarithm, exponentiate both sides. Remember that the initial temperature at  $t = 0$  is 95°C.
- **For ii:** Substitute  $t = 10$  and  $T = 65$  into your equation from part **i** to solve for the term  $e^{-10k}$ . Then, use index laws to rewrite  $e^{-kt}$  as  $(e^{-10k})^{\frac{t}{10}}$  to perform the substitution.
- **For iii:** Set  $T = 50$  using the equation you derived in part **ii**. You will end up with an equation like  $\frac{2}{3} = \left(\frac{2}{3}\right)^{\frac{t}{10}}$ . Take the natural logarithm (ln) of both sides and use log laws to make  $t$  the subject.
- **For iv:** You do not need to differentiate your complex equation from part **iii**. Simply use the original differential equation  $\frac{dT}{dt} = -k(T - 5)$ . Find the exact value of  $k$  from part **ii** ( $\ln(e^{-10k}) = \ln(2/3)$ ), and substitute  $T = 50$ .

**Solution 7.30**

i Separating variables gives  $\ln(T - 5) = -kt + C$ , so

$$T(t) = 5 + Ae^{-kt}.$$

Using  $T(0) = 95$  gives  $A = 90$ , hence  $T(t) = 5 + 90e^{-kt}$ .

ii From  $T(10) = 65$ ,

$$65 = 5 + 90e^{-10k} \Rightarrow e^{-10k} = \frac{2}{3}.$$

Therefore

$$e^{-kt} = (e^{-10k})^{t/10} = \left(\frac{2}{3}\right)^{t/10},$$

so

$$T(t) = 5 + 90 \left(\frac{2}{3}\right)^{t/10}.$$

iii Set  $T = 50$ :

$$50 = 5 + 90 \left(\frac{2}{3}\right)^{t/10} \Rightarrow \frac{1}{2} = \left(\frac{2}{3}\right)^{t/10}.$$

Taking logs,

$$t = \frac{10 \ln 2}{\ln 3 - \ln 2} \text{ min}.$$

iv Since  $e^{-10k} = \frac{2}{3}$ ,

$$k = \frac{1}{10} \ln \frac{3}{2}.$$

At  $T = 50$ ,

$$\frac{dT}{dt} = -k(T - 5) = -45k = -\frac{9}{2} \ln \frac{3}{2}.$$

So

$$\frac{dT}{dt} = -\frac{9}{2} \ln \frac{3}{2} \text{ } ^\circ\text{C/min}.$$

This is half the initial cooling rate because  $T - 5$  has fallen from 90 to 45.

**Takeaways 7.30**

- **Base Conversions:** Part ii tests a crucial HSC skill. Exponential decay doesn't always have to be written with base  $e$ . Changing the base to a fraction like  $\frac{2}{3}$  often makes the equation much easier to use for future calculations!
- **Logarithm Laws:** In part iii, using the identity  $\ln(a/b) = \ln(a) - \ln(b)$  allows you to present your final exact answer cleanly without negative signs floating in the numerator.
- **Calculus Shortcuts:** Part iv reminds you that you don't always need to differentiate  $T(t)$  to find a rate. The differential equation *is* the rate! Plugging values directly into  $\frac{dT}{dt}$  saves massive amounts of time.

**Problem 7.31: Torricelli’s Law and the Hurstville Plaza Basin**

Hurstville City Council is draining the large hemispherical water basin in the main plaza for routine maintenance. The basin has a radius of 2 metres and is initially completely full.

Water drains from a hole at the bottom of the basin. According to Torricelli’s Law, the volume  $V$  of water in the basin decreases at a rate proportional to the square root of the depth  $h$  of the water. Thus,  $\frac{dV}{dt} = -k\sqrt{h}$  for some positive constant  $k$ .

- i Given that the volume of a spherical cap is  $V = \frac{\pi}{3}h^2(3R - h)$ , use the chain rule to show that the height of the water satisfies the differential equation  $\frac{dh}{dt} = \frac{-c}{\sqrt{h(4-h)}}$ , where  $c$  is a positive constant.
- ii By separating the variables, find an implicit solution relating the time  $t$  and the height  $h$ .
- iii The council engineers observe that it takes exactly  $28\sqrt{2}$  minutes for the basin to drain completely. Find the exact value of  $c$ , and determine exactly how many minutes it takes for the water level to drop to a height of 1 metre.
- iv Explain mathematically why the rate at which the water level falls is not constant, and determine the exact depth  $h$  at which the water level is falling at its *slowest* rate.

**Hint:**

- **For i:** Start by finding the derivative  $\frac{dV}{dh}$  using the given volume formula (remember to substitute  $R = 2$ ). Then, apply the chain rule:  $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$ .
- **For ii:** Isolate the  $h$  terms with  $dh$  and the constant with  $dt$ . You will need to expand  $\sqrt{h(4-h)}$  into fractional powers ( $h^{1/2}$  and  $h^{3/2}$ ) before integrating.
- **For iii:** First, use the initial condition ( $t = 0, h = 2$ ) to find the constant of integration. Then use the empty condition ( $t = 28\sqrt{2}, h = 0$ ) to find  $c$ . Finally, substitute  $h = 1$  to find  $t$ .
- **For iv:** The rate of change of height is  $\frac{dh}{dt}$ . To find when the water falls slowest, you need to minimize the magnitude of this rate. This is equivalent to maximizing the denominator of your  $\frac{dh}{dt}$  expression. Use calculus to find the stationary point of the denominator.

**Solution 7.31**

i With  $R = 2$ ,

$$V = \frac{\pi}{3}h^2(6 - h), \quad \frac{dV}{dh} = 4\pi h - \pi h^2 = \pi h(4 - h).$$

Hence

$$\frac{dh}{dt} = \frac{dh}{dV} \frac{dV}{dt} = \frac{1}{\pi h(4 - h)}(-k\sqrt{h}) = -\frac{k}{\pi\sqrt{h}(4 - h)}.$$

So, with  $c = \frac{k}{\pi}$ ,

$$\boxed{\frac{dh}{dt} = -\frac{c}{\sqrt{h}(4 - h)}}.$$

ii Separating variables,

$$\sqrt{h}(4 - h) dh = -c dt,$$

so

$$\boxed{\frac{8}{3}h^{3/2} - \frac{2}{5}h^{5/2} = -ct + C}.$$

iii Using  $h(0) = 2$ ,

$$C = \frac{8}{3}(2)^{3/2} - \frac{2}{5}(2)^{5/2} = \frac{56\sqrt{2}}{15}.$$

Using  $h = 0$  at  $t = 28\sqrt{2}$ ,

$$0 = -c(28\sqrt{2}) + \frac{56\sqrt{2}}{15} \Rightarrow \boxed{c = \frac{2}{15}}.$$

For  $h = 1$ ,

$$\frac{8}{3} - \frac{2}{5} = -\frac{2}{15}t + \frac{56\sqrt{2}}{15} \Rightarrow 34 = -2t + 56\sqrt{2},$$

so

$$\boxed{t = 28\sqrt{2} - 17 \text{ min}}.$$

iv Since

$$\left| \frac{dh}{dt} \right| = \frac{c}{\sqrt{h}(4 - h)},$$

the fall is slowest when  $D(h) = \sqrt{h}(4 - h)$  is largest. Now

$$D'(h) = \frac{2}{\sqrt{h}} - \frac{3}{2}\sqrt{h}.$$

Setting  $D'(h) = 0$  gives  $4 = 3h$ , so

$$\boxed{h = \frac{4}{3}}.$$

**Takeaways 7.31**

- **Geometry Matters:** In a perfect cylinder, water level drops according to  $h(t) \propto t^2$  because the surface area is constant. In a hemisphere, the changing surface area competes with the dropping pressure head, creating a complex non-linear curve.
- **Physical Intuition:** Notice part iv. The water falls very fast at the beginning (because the pressure is highest) AND very fast at the end (because the cross-sectional area approaches zero). It slows down in the middle where these two competing geometric factors balance out!

**Problem 7.32: The Wollemi Bushfire Spread**

A bushfire breaks out in a contained valley within Wollemi National Park. The total combustible area is 1200 hectares. If  $A(t)$  is the burnt area after  $t$  hours, the spread is modelled by

$$\frac{dA}{dt} = kA(1200 - A),$$

where  $k > 0$ .

- i Show that  $\frac{1200}{A(1200 - A)} = \frac{1}{A} + \frac{1}{1200 - A}$ , and hence prove that the general solution can be written as

$$A(t) = \frac{1200}{1 + Ce^{-1200kt}},$$

where  $C$  is a constant.

- ii When the fire was first detected,  $A(0) = 20$ . Four hours later,  $A(4) = 100$ . Find  $C$ , and show that

$$1200k = \frac{1}{4} \ln \left( \frac{59}{11} \right).$$

- iii Determine the exact time when the fire is spreading at its maximum rate.
- iv An Emergency Warning is triggered if more than 80% of the valley is projected to burn within 12 hours. Based on this model, should the warning be issued?

**Hint:**

- **For i:** Separate variables first, then multiply by 1200 so the given partial fraction identity applies immediately.
- **For ii:** Use  $A(0) = 20$  to find  $C$ , then substitute  $A(4) = 100$  to isolate  $e^{-4800k}$ .
- **For iii:** The rate equation  $kA(1200 - A)$  is a downward-opening quadratic in  $A$ , so its maximum occurs at the vertex.
- **For iv:** Compare  $A(12)$  with  $0.8 \times 1200 = 960$ , and simplify the exponential exactly before approximating.

**Solution 7.32**

i Since

$$\frac{1200}{A(1200 - A)} = \frac{1}{A} + \frac{1}{1200 - A},$$

separating variables gives

$$\int \frac{1200}{A(1200 - A)} dA = \int 1200k dt.$$

Hence

$$\int \left( \frac{1}{A} + \frac{1}{1200 - A} \right) dA = 1200kt + C$$

so

$$\ln \left( \frac{A}{1200 - A} \right) = 1200kt + C.$$

Solving for  $A$ ,

$$A(t) = \frac{1200}{1 + Ce^{-1200kt}}.$$

ii From  $A(0) = 20$ ,

$$20 = \frac{1200}{1 + C} \Rightarrow C = 59.$$

Then  $A(4) = 100$  gives

$$100 = \frac{1200}{1 + 59e^{-4800k}} \Rightarrow e^{-4800k} = \frac{11}{59},$$

so

$$1200k = \frac{1}{4} \ln \left( \frac{59}{11} \right).$$

iii The rate  $kA(1200 - A)$  is maximal when  $A = 600$ . Thus

$$600 = \frac{1200}{1 + 59e^{-1200kt}} \Rightarrow 59e^{-1200kt} = 1 \Rightarrow 1200kt = \ln 59.$$

Using part ii,

$$t = \frac{4 \ln 59}{\ln(59/11)} \text{ hours}.$$

iv The warning threshold is  $0.8(1200) = 960$ . At  $t = 12$ ,

$$e^{-1200k(12)} = e^{-12(1200k)} = e^{-3 \ln(59/11)} = \left( \frac{11}{59} \right)^3.$$

So

$$A(12) = \frac{1200}{1 + 59(11/59)^3} = \frac{1200}{1 + \frac{1331}{3481}} = \frac{1200 \cdot 3481}{4812} \approx 868.1.$$

Since  $868.1 < 960$ , **no Emergency Warning is triggered.**

**Takeaways 7.32**

- **Contextual Flexibility:** The logistic equation isn't just for biology. It perfectly describes any system (like fire or rumors) that spreads quickly through a medium but eventually slows down as it runs out of "fuel" or "uninformed people."
- **The Point of Inflection:** A classic trap is trying to differentiate the complex fraction  $A(t)$  twice to find the point of maximum growth. Recognizing that the original differential equation *is* the rate equation allows you to use simple Year 9 parabolas to find the maximum instantly.
- **Exact Value Discipline:** In part **iv**, dealing with the exponent  $e^{-3\ln(59/11)}$  tests your index and logarithm laws. Relying on early decimal rounding often leads to significant errors in exponential growth models.

**Problem 7.33: The Rooftop Solar Boom in Western Sydney**

The adoption of rooftop solar panels in a specific Western Sydney municipality is modeled using a logistic differential equation. The local network has a maximum grid capacity to support solar installations on 150 thousand homes.

Let  $S(t)$  be the number of homes (in thousands) with rooftop solar, where  $t$  is the number of years after 2010. The rate of new installations is proportional to both the number of homes that already have solar and the remaining capacity of homes that do not. This is governed by:

$$\frac{dS}{dt} = kS(150 - S)$$

for some positive constant  $k$ .

- By using the substitution  $S(t) = \frac{150}{1+u(t)}$ , where  $u(t)$  is an unknown function of time, show that the non-linear differential equation transforms into the simple linear decay equation  $\frac{du}{dt} = -150ku$ .
- In 2010 ( $t = 0$ ), there were exactly 10 thousand homes with solar. By 2015 ( $t = 5$ ), this number had grown to 50 thousand. By first solving the differential equation for  $u(t)$ , show that the explicit equation for  $S(t)$  is  $S(t) = \frac{150}{1+14(7)^{-t/5}}$ .
- The local council defines the "solar boom" as the period during which the *rate* of new installations is strictly increasing. Find the exact value of  $t$  when this boom ends.
- Suppose a new government rebate is introduced that guarantees a constant minimum of  $R$  thousand new installations per year, modifying the model to  $\frac{dS}{dt} = kS(150 - S) + R$ . Without solving the new differential equation, prove mathematically that the new theoretical maximum limit of solar installations will exceed 150 thousand, and find an expression for this new limit in terms of  $k$  and  $R$ .

**Hint:**

- **For i:** Start by differentiating your substitution  $S = 150(1 + n)^{-1}$  with respect to  $t$  using the chain rule to find  $\frac{dS}{dt}$ . Substitute both  $S$  and  $\frac{dS}{dt}$  into the original differential equation and simplify the algebraic fractions until you isolate  $\frac{dn}{dt}$ .
- **For ii:** Solve  $\frac{dn}{dt} = -150kn$  to get  $n(t) = Ae^{-150kt}$ . Substitute  $t = 0, S = 10$  to find  $A$ . Then substitute  $t = 5, S = 50$  to find the exact value of the exponential term  $e^{-750k}$ . Use index laws to rewrite the general equation cleanly with base 7.
- **For iii:** The rate of installations is  $\frac{dI}{dt}$ . For this rate to stop increasing, it must reach its maximum, which occurs at the point of inflection of  $S(t)$ . For a logistic curve, the maximum rate always occurs at exactly half the carrying capacity.
- **For iv:** The theoretical maximum limit of a system occurs when it reaches equilibrium, meaning the rate of change is zero ( $\frac{dS}{dt} = 0$ ). Set the modified differential equation to zero, which forms a quadratic equation in terms of  $S$ . Solve for  $S$  using the quadratic formula and analyze the roots.

**Solution 7.33**

i With  $S = 150(1 + u)^{-1}$ ,

$$\frac{dS}{dt} = -\frac{150}{(1 + u)^2} \frac{du}{dt}.$$

Also,

$$kS(150 - S) = k \frac{150}{1 + u} \left( 150 - \frac{150}{1 + u} \right) = \frac{150^2 k u}{(1 + u)^2}.$$

Hence

$$\boxed{\frac{du}{dt} = -150ku}.$$

ii So  $u = Ae^{-150kt}$  and

$$S(t) = \frac{150}{1 + Ae^{-150kt}}.$$

Using  $S(0) = 10$  gives  $A = 14$ . Using  $S(5) = 50$  gives

$$50 = \frac{150}{1 + 14e^{-750k}} \Rightarrow e^{-750k} = \frac{1}{7}.$$

Therefore  $e^{-150kt} = (e^{-750k})^{t/5} = 7^{-t/5}$ , so

$$\boxed{S(t) = \frac{150}{1 + 14 \cdot 7^{-t/5}}}.$$

iii The rate  $kS(150 - S)$  is maximal at  $S = 75$ . Thus

$$75 = \frac{150}{1 + 14 \cdot 7^{-t/5}} \Rightarrow 14 \cdot 7^{-t/5} = 1 \Rightarrow 7^{t/5} = 14.$$

Hence

$$\boxed{t = 5 \log_7(14) = 5 + 5 \log_7 2}.$$

iv At equilibrium,

$$kS(150 - S) + R = 0 \iff kS^2 - 150kS - R = 0.$$

So

$$S = 75 \pm \sqrt{75^2 + \frac{R}{k}}.$$

Taking the positive equilibrium,

$$\boxed{S_{\max} = 75 + \sqrt{75^2 + \frac{R}{k}}}.$$

Since  $R > 0$ , the square root exceeds 75, hence  $S_{\max} > 150$ .

**Takeaways 7.33**

- **Alternative Methods:** Partial fractions is the standard way to integrate the logistic equation. However, part i demonstrates a powerful Extension 2 substitution technique ( $S \propto 1/(1+u)$ ) that instantly reduces a complex non-linear system into standard Year 11 exponential decay.
- **Exact Value Precision:** Part ii requires careful manipulation of exponential bases. Recognising that  $e^{-150kt}$  is just the fifth root of  $(e^{-750k})^t$  allows you to bypass messy natural logs and create a clean, exact equation using base 7.
- **Equilibrium Analysis:** Part iv tests a deep understanding of what a differential equation represents physically. You don't always need to solve a DE to understand the system's eventual fate; setting the derivative to zero yields the equilibrium states directly.

**Problem 7.34: Modelling Fish Population in a Protected Lake**

In a wildlife reserve, the fish population is  $N = 2000$  when  $x = 0$ . It is modelled by

$$\frac{dN}{dx} = 0.2N \left( \frac{M - N}{M} \right),$$

where  $M$  is the carrying capacity.

**i** Use

$$\frac{M}{N(M - N)} = \frac{1}{N} + \frac{1}{M - N}$$

to show that

$$\frac{N}{M - N} = \left( \frac{2000}{M - 2000} \right) e^{0.2x}.$$

**ii** If  $N(10) = 6000$ , find  $M$  correct to the nearest hundred.

**Hint:**

• Multiply through by  $\frac{M}{N(M - N)}$  before integrating.

• Use the initial condition only after writing the exponential form.

• In part **ii**, substitute  $x = 10$  and  $N = 6000$ , then solve the resulting linear equation in  $M$ .

**Solution 7.34**

(i) Rearranging,

$$\frac{M}{N(M-N)} dN = 0.2 dx.$$

Hence

$$\int \left( \frac{1}{N} + \frac{1}{M-N} \right) dN = \int 0.2 dx$$

which gives

$$\ln \left( \frac{N}{M-N} \right) = 0.2x + C.$$

So

$$\frac{N}{M-N} = Ae^{0.2x}.$$

Using  $N(0) = 2000$ ,

$$A = \frac{2000}{M-2000},$$

and therefore

$$\frac{N}{M-N} = \left( \frac{2000}{M-2000} \right) e^{0.2x}.$$

(ii) Substitute  $x = 10$  and  $N = 6000$ :

$$\frac{6000}{M-6000} = \left( \frac{2000}{M-2000} \right) e^2.$$

So

$$\frac{3}{M-6000} = \frac{e^2}{M-2000}$$

and

$$3(M-2000) = e^2(M-6000).$$

Hence

$$M = \frac{6000(e^2 - 1)}{e^2 - 3} \approx 8734,$$

so the carrying capacity is

$$M \approx 8700.$$

**Takeaways 7.34**

- The logistic equation keeps the same calculus structure even when the variables are renamed or reinterpreted.
- Two measurements can be enough to estimate a model's long-run limit.

**Problem 7.35: Logistic Growth and Maximum Population Increase**

A population of wild horses satisfies

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{C}\right).$$

Initially  $P(0) = 150000$ , the carrying capacity is  $C = 1200000$ , and  $P(20) = 300000$ .

- i Show that the general solution can be written as  $P(t) = \frac{C}{1 + Ae^{-kt}}$ .
- ii Find the exact value of  $k$ .
- iii Find the exact time when the population is growing fastest.

**Hint:**

- Use the standard logistic partial fraction identity.
- The fastest growth occurs at the vertex of the quadratic rate law in  $P$ .

**Solution 7.35**

(i) Separating variables gives

$$\int \frac{C}{P(C - P)} dP = \int k dt,$$

so

$$\ln \left( \frac{P}{C - P} \right) = kt + C_1.$$

Rearranging,

$$P(t) = \frac{C}{1 + Ae^{-kt}}.$$

(ii) From  $P(0) = 150000$  and  $C = 1200000$ ,

$$150000 = \frac{1200000}{1 + A} \Rightarrow A = 7.$$

Using  $P(20) = 300000$ ,

$$300000 = \frac{1200000}{1 + 7e^{-20k}} \Rightarrow e^{-20k} = \frac{3}{7}.$$

Hence

$$k = \frac{1}{20} \ln \frac{7}{3}.$$

(iii) The rate  $kP(1 - P/C)$  is maximal at  $P = C/2 = 600000$ . So

$$600000 = \frac{1200000}{1 + 7e^{-kt}} \Rightarrow e^{-kt} = \frac{1}{7}.$$

Therefore

$$t = \frac{\ln 7}{k} = \frac{20 \ln 7}{\ln(7/3)}.$$

**Takeaways 7.35**

- Logistic models often become computationally easy once the standard form is recognised.
- The maximum growth rate occurs at half the carrying capacity.

**Problem 7.36: Autonomous DE with Equilibria and Concavity**

Consider

$$\frac{dy}{dx} = 1 + \sin y.$$

- i Find the equilibrium solutions.
- ii Determine where solution curves are concave up and concave down on  $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ .
- iii If  $f(0) = 0$ , find  $\lim_{x \rightarrow \infty} f(x)$  and  $\lim_{x \rightarrow -\infty} f(x)$ .

**Hint:**

- Set  $1 + \sin y = 0$  for equilibria.
- Differentiate implicitly and factor the result.
- Use the fact that solutions cannot cross equilibrium lines.

**Solution 7.36**

(i) Equilibria satisfy  $\sin y = -1$ , so

$$y = -\frac{\pi}{2} + 2n\pi.$$

(ii) Differentiate implicitly:

$$y'' = \cos y y' = \cos y(1 + \sin y).$$

Since  $1 + \sin y \geq 0$ , the sign of  $y''$  matches  $\cos y$ . Hence the curves are concave up on

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

and concave down on

$$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

(iii) The solution through  $y = 0$  is increasing and trapped between the nearest equilibria below and above it. Therefore

$$\lim_{x \rightarrow \infty} f(x) = \frac{3\pi}{2}, \quad \lim_{x \rightarrow -\infty} f(x) = -\frac{\pi}{2}.$$

**Takeaways 7.36**

- Autonomous equations can often be analysed deeply without solving them explicitly.
- Equilibrium solutions act like barriers for nearby trajectories.

**Problem 7.37: A Separable IVP with Logarithmic Domain Issues**

Consider

$$(x^2 - 4) \frac{dy}{dx} = xy^2, \quad y(0) = 1.$$

- i Find the particular solution.
- ii Determine its maximal domain of validity.
- iii Describe the behaviour near the domain endpoints.

- Separate variables and use  $u = x^2 - 4$ .
- The logarithm forces an interval split at  $x = \pm 2$ .

**Hint:**

**Solution 7.37**

Separating variables,

$$y^{-2} dy = \frac{x}{x^2 - 4} dx.$$

Integrating,

$$-\frac{1}{y} = \frac{1}{2} \ln |x^2 - 4| + C.$$

Using  $y(0) = 1$  gives

$$-1 = \frac{1}{2} \ln 4 + C \Rightarrow C = -1 - \ln 2.$$

So

$$\frac{1}{y} = 1 + \frac{1}{2} \ln \left( \frac{4}{|x^2 - 4|} \right),$$

and

$$y = \frac{2}{2 + \ln \left( \frac{4}{|x^2 - 4|} \right)}.$$

Since the initial point is at  $x = 0$ , the maximal interval avoiding the singular lines  $x = \pm 2$  is

$$(-2, 2).$$

As  $x \rightarrow \pm 2$  from inside this interval, the logarithm tends to  $-\infty$  and the denominator tends to  $-\infty$ , so  $y \rightarrow 0^-$ . The solution stays finite on  $(-2, 2)$  but cannot be continued through  $x = \pm 2$ .

**Takeaways 7.37**

- The DE itself can force a maximal interval even when the explicit formula looks harmless.
- Absolute values inside logarithms are essential when integrating rational expressions of this type.

**Problem 7.38: Linear DE, Isoclines, and Local Minima**

Consider

$$\frac{dy}{dx} = x + y.$$

- i Find the particular solution through  $(-1, 1)$  and its  $y$ -intercept.
- ii Express  $y''$  in terms of  $x$  and  $y$ .
- iii Show that any solution crossing the zero-slope isocline has a local minimum there.

- The zero-slope isocline is where  $x + y = 0$ .
- Differentiate  $y' = x + y$  directly.
- Solve the linear equation with an integrating factor.

**Hint:**

**Solution 7.38**

(i) Writing  $y' - y = x$ , the integrating factor is  $e^{-x}$ . Then

$$\frac{d}{dx}(ye^{-x}) = xe^{-x}.$$

Integrating gives

$$y = -x - 1 + Ce^x.$$

Using  $(-1, 1)$  gives  $C = e$ , so

$$f(x) = -x - 1 + e^{x+1}.$$

Hence

$$f(0) = e - 1.$$

(ii) Differentiate the DE:

$$y'' = 1 + y' = 1 + x + y.$$

(iii) On the zero-slope isocline  $x + y = 0$ , we have  $y' = 0$  and

$$y'' = 1 + (x + y) = 1 > 0.$$

Therefore every solution that crosses this isocline has a stationary point with positive second derivative, so it is a local minimum.

**Takeaways 7.38**

- The DE itself can reveal universal geometric facts about all solutions.
- Isoclines are a useful bridge between algebra and graph shape.

**Problem 7.39: Orthogonal Trajectories of an Elliptic Family**

Consider

$$\frac{dy}{dx} = -\frac{x}{4y}.$$

- i Solve the DE and identify the family of curves.
- ii Find the DE for the orthogonal trajectories.
- iii Solve for the orthogonal family.

- Orthogonal slopes are negative reciprocals.
- Separate variables for both families.

**Hint:****Solution 7.39**

(i) Separating variables,

$$4y \, dy = -x \, dx$$

gives

$$x^2 + 4y^2 = C,$$

so the original family is a set of ellipses centred at the origin.

(ii) The orthogonal slope is

$$m_2 = -\frac{1}{-x/(4y)} = \frac{4y}{x},$$

so the new DE is

$$\frac{dy}{dx} = \frac{4y}{x}.$$

(iii) Separating variables,

$$\frac{1}{y} \, dy = \frac{4}{x} \, dx$$

gives

$$\ln |y| = 4 \ln |x| + C$$

and hence

$$y = Ax^4.$$

These are quartic curves through the origin.

**Takeaways 7.39**

- Orthogonal trajectories convert a geometric condition into a new differential equation.
- Negative reciprocal slopes are the key link.

**Problem 7.40: A Tank-Filling Model with Evaporation**

A paraboloid tank is formed by rotating  $y = x^2$  about the  $y$ -axis. Water is poured in at rate  $2\pi$  and evaporates at rate  $k\pi h$ , where  $h$  is the water depth.

- i Show that  $V = \frac{\pi}{2}h^2$ .
- ii Show that  $\frac{dh}{dt} = \frac{2 - kh}{h}$ .
- iii If  $k = 1$ , find the exact time to reach depth 1.

**Hint:**

- Use the disc method for volume.
- Then apply  $\frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt}$ .

**Solution 7.40**

(i) Since  $x^2 = y$ , the volume is

$$V = \int_0^h \pi x^2 dy = \int_0^h \pi y dy = \boxed{\frac{\pi}{2}h^2}.$$

(ii) Net inflow is

$$\frac{dV}{dt} = 2\pi - k\pi h = \pi(2 - kh).$$

Also

$$\frac{dV}{dh} = \pi h.$$

Hence

$$\boxed{\frac{dh}{dt} = \frac{dV/dt}{dV/dh} = \frac{2 - kh}{h}}.$$

(iii) With  $k = 1$ ,

$$\frac{dt}{dh} = \frac{h}{2 - h} = -1 + \frac{2}{2 - h}.$$

Integrating,

$$t = -h - 2 \ln |2 - h| + C.$$

Using  $t = 0$  at  $h = 0$  gives  $C = 2 \ln 2$ . So at  $h = 1$ ,

$$\boxed{t = 2 \ln 2 - 1}.$$

**Takeaways 7.40**

- Modelling often combines geometry, related rates, and DE solving in one chain.
- Rewriting a rational integrand before integrating can save a lot of work.

**Problem 7.41: Qualitative Analysis of an Allee-Type Population Model**

Consider

$$\frac{dP}{dt} = \frac{1}{100}(P - 20)(80 - P),$$

where  $P$  is measured in thousands.

- i Identify and classify the equilibrium solutions.
- ii Find  $P''$  in terms of  $P$  and  $P'$ .
- iii If  $P(0) = 30$ , find the population where growth is maximal and decide the concavity at  $P = 60$ .

- Differentiate the right-hand side after expanding.
- Sign analysis around the equilibria determines stability.

**Hint:****Solution 7.41**

(i) Equilibria are  $P = 20$  and  $P = 80$ . For  $P < 20$ ,  $P' < 0$ ; for  $20 < P < 80$ ,  $P' > 0$ ; and for  $P > 80$ ,  $P' < 0$ . So  $P = 20$  is unstable and  $P = 80$  is stable.

(ii) Expanding first,

$$P' = \frac{1}{100}(-P^2 + 100P - 1600).$$

Therefore

$$P'' = \frac{1}{50}(50 - P)P'.$$

(iii) Maximum growth occurs when  $P'' = 0$  and  $P' > 0$ , so

$$P = 50.$$

At  $P = 60$ , we have  $50 - P < 0$  while  $P' > 0$ , hence  $P'' < 0$ . The graph is concave down there.

**Takeaways 7.41**

- Many population questions can be answered from equilibrium and sign analysis alone.
- The inflection point of a logistic-style curve is where the growth rate peaks.

**Problem 7.42: Inverse-Trigonometric IVP and Validity Interval**

Consider

$$\frac{dy}{dx} = x\sqrt{9 - y^2}, \quad y(0) = \frac{3\sqrt{3}}{2}.$$

- i Find the particular solution.
- ii Determine its maximal domain of validity.
- iii Find its absolute maximum and minimum values on that domain.

- Use the standard arcsin integral.
- The principal range of arcsin restricts the valid interval.

**Hint:****Solution 7.42**

Separating variables,

$$\int \frac{dy}{\sqrt{9-y^2}} = \int x dx$$

gives

$$\arcsin\left(\frac{y}{3}\right) = \frac{x^2}{2} + C.$$

Using the initial condition,

$$C = \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}.$$

So

$$y = 3 \sin\left(\frac{x^2}{2} + \frac{\pi}{3}\right).$$

For this arcsin form to remain valid,

$$-\frac{\pi}{2} \leq \frac{x^2}{2} + \frac{\pi}{3} \leq \frac{\pi}{2}.$$

Only the upper bound matters, giving

$$x^2 \leq \frac{\pi}{3}.$$

Hence the maximal domain is

$$\left[-\sqrt{\frac{\pi}{3}}, \sqrt{\frac{\pi}{3}}\right].$$

On this interval the angle runs from  $\frac{\pi}{3}$  to  $\frac{\pi}{2}$ , so the minimum occurs at  $x = 0$  and the maximum at the endpoints:

$$y_{\min} = \frac{3\sqrt{3}}{2}, \quad y_{\max} = 3.$$

**Takeaways 7.42**

- Inverse-trig solutions come with hidden validity restrictions from principal ranges.
- The explicit sine form is useful, but the arcsin form explains the true interval of validity.

**Problem 7.43: Exponential Blow-Up and a Vertical Asymptote**

Consider

$$\frac{dy}{dx} = e^{2y}$$

and the solution through  $(0, 0)$ .

- i Find the general solution with  $x$  expressed in terms of  $y$ .
- ii Find the particular solution  $y = f(x)$ .
- iii Determine the maximal domain and the vertical asymptote.

• The logarithm form makes the domain obvious.

• Write the left side as  $e^{-2y} dy$ .

**Hint:**

**Solution 7.43**

Separating variables,

$$e^{-2y} dy = dx.$$

Integrating,

$$-\frac{1}{2}e^{-2y} = x + C.$$

Using  $(0, 0)$  gives  $C = -\frac{1}{2}$ , so

$$e^{-2y} = 1 - 2x.$$

Hence

$$f(x) = -\frac{1}{2} \ln(1 - 2x).$$

The logarithm requires

$$1 - 2x > 0 \Rightarrow x < \frac{1}{2},$$

so the maximal domain is

$$\left(-\infty, \frac{1}{2}\right).$$

As  $x \rightarrow \left(\frac{1}{2}\right)^-$ ,  $\ln(1 - 2x) \rightarrow -\infty$ , so  $f(x) \rightarrow \infty$ . Thus the vertical asymptote is

$$x = \frac{1}{2}.$$

**Takeaways 7.43**

- Some DE solutions blow up to infinity in finite  $x$ .
- Writing the answer in logarithmic form makes the asymptote immediate.

### 7.3 Advanced

**Problem 7.44: The Hyperbolic Tangent and Resisted Motion**

Consider the hyperbolic tangent function, defined as  $y = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ .

- i Show that  $y$  can be rewritten in the form  $y = \frac{2}{1+e^{-2x}} - 1$ , and hence state the equations of its horizontal asymptotes.
- ii Prove that  $y = \tanh(x)$  is a solution to the non-linear differential equation  $\frac{dy}{dx} = 1 - y^2$ .
- iii Using the differential equation from part ii (without differentiating the original fraction again), find an expression for  $\frac{d^2y}{dx^2}$  strictly in terms of  $y$ . Hence, find the coordinates of the unique point of inflection.
- iv In a physics model, a falling object subject to quadratic air resistance has a velocity  $v(t)$  satisfying  $\frac{dv}{dt} = g - \frac{g}{V^2}v^2$ , where  $g$  and  $V$  are positive constants. Given the object is dropped from rest ( $v(0) = 0$ ), solve this differential equation by separating the variables to show that  $v(t) = V \tanh\left(\frac{gt}{V}\right)$ .

**Hint:**

- **For i:** To convert between the forms, try multiplying the numerator and denominator of the original definition by  $e^{-x}$ , then perform algebraic manipulation.
- **For ii:** Differentiate the form you derived in part i using the chain rule. Separately, calculate  $1 - y^2$  by substituting the algebraic fraction, and show the two results are identical.
- **For iii:** Use implicit differentiation on the equation  $\frac{dy}{dx} = 1 - y^2$ . Remember that  $\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$ . Set the second derivative to zero, but be sure to verify that the root you find is actually on the curve.
- **For iv:** Factor out  $\frac{g}{V^2}$  to get  $\frac{dv}{dt} = \frac{g}{V^2}(V^2 - v^2)$ . Separate the variables and use partial fractions to integrate  $\frac{1}{V^2 - v^2}$ .

**Solution 7.44**

i Multiply top and bottom by  $e^{-x}$ :

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{2}{1 + e^{-2x}} - 1.$$

Hence  $y \rightarrow 1$  as  $x \rightarrow \infty$  and  $y \rightarrow -1$  as  $x \rightarrow -\infty$ , so the horizontal asymptotes are  $y = 1$  and  $y = -1$ .

ii From the form above,

$$y' = -2(1 + e^{-2x})^{-2}(-2e^{-2x}) = \frac{4e^{-2x}}{(1 + e^{-2x})^2}.$$

Also

$$1 - y^2 = 1 - \left(\frac{1 - e^{-2x}}{1 + e^{-2x}}\right)^2 = \frac{(1 + e^{-2x})^2 - (1 - e^{-2x})^2}{(1 + e^{-2x})^2} = \frac{4e^{-2x}}{(1 + e^{-2x})^2}.$$

Thus  $y' = 1 - y^2$ .

iii Differentiate the DE:

$$y'' = \frac{d}{dx}(1 - y^2) = -2yy' = -2y(1 - y^2) = 2y^3 - 2y.$$

Set  $y'' = 0$ :

$$-2y(1 - y^2) = 0 \implies y = 0, \pm 1.$$

The values  $\pm 1$  are asymptotes, so the actual inflection point occurs at  $y = 0$ . Since  $\tanh x = 0$  gives  $x = 0$ , the unique point of inflection is  $(0, 0)$ .

iv Write

$$\frac{dv}{dt} = \frac{g}{V^2}(V^2 - v^2), \quad \int \frac{dv}{V^2 - v^2} = \int \frac{g}{V^2} dt.$$

Using

$$\frac{1}{V^2 - v^2} = \frac{1}{2V} \left( \frac{1}{V + v} + \frac{1}{V - v} \right),$$

we get

$$\ln \left( \frac{V + v}{V - v} \right) = \frac{2gt}{V} + C.$$

Since  $v(0) = 0$ , we have  $C = 0$ . Therefore

$$\frac{V + v}{V - v} = e^{2gt/V} \implies v = V \frac{e^{2gt/V} - 1}{e^{2gt/V} + 1} = V \tanh \left( \frac{gt}{V} \right).$$

$$v(t) = V \tanh \left( \frac{gt}{V} \right).$$

**Takeaways 7.44**

- **The Power of the DE:** Part iii highlights a crucial Extension 2 strategy. Differentiating an ugly fraction twice is a nightmare. Using the first-order DE to implicitly find the second derivative is incredibly fast and efficient.
- **Mechanics Connections:** Pure mathematics and mechanics often overlap. The generic logistic/hyperbolic curves you study algebraically perfectly describe terminal velocity  $V$  in quadratic drag models. Notice how the mathematical asymptote  $y = 1$  transforms into the physical speed limit  $v = V$ !

**Problem 7.45: The Non-Linear Oscillator and Separatrix Curves**

Consider a particle moving along the  $y$ -axis where its acceleration is defined by the non-linear, autonomous differential equation  $\frac{d^2y}{dx^2} = 2y^3 - 2y$ . The particle is subject to the initial conditions  $y(0) = 0$  and  $y'(0) = 1$ . Let  $v = \frac{dy}{dx}$ .

- i By using the substitution  $\frac{d^2y}{dx^2} = v \frac{dv}{dy}$ , show that the phase space trajectory of this specific solution is given by  $v^2 = (1 - y^2)^2$ .
- ii Explain carefully why, for this particular particle, the first-order differential equation simplifies strictly to  $\frac{dy}{dx} = 1 - y^2$ , and not  $y^2 - 1$ .
- iii By separating the variables and using partial fractions, solve this first-order differential equation to find  $x$  as a function of  $y$ .
- iv Hence, show that the explicit solution for  $y$  is given by the hyperbolic tangent function,  $y = \tanh(x)$ , and state the equations of its horizontal asymptotes.

**Hint:**

- **For i:** Use the chain rule  $y'' = v \frac{dv}{dy}$  and integrate both sides with respect to  $y$ . Don't forget to use the given initial conditions ( $x = 0, y = 0, v = 1$ ) to find the exact value of the constant of integration, then factor the resulting polynomial into a perfect square.
- **For ii:** Taking the square root of  $v^2 = (1 - y^2)^2$  yields  $v = \pm(1 - y^2)$ . Test the initial conditions again to determine which sign mathematically must be chosen.
- **For iii:** Rearrange to  $\int \frac{1}{1-y^2} dy = \int 1 dx$ . You can use the algebraic identity  $\frac{1}{1-y^2} = \frac{1}{2} \left( \frac{1}{1-y} + \frac{1}{1+y} \right)$  to integrate. Remember that integrating  $\frac{1}{1-y}$  produces a negative logarithm!
- **For iv:** Once you have  $x = \frac{1}{2} \ln(\dots)$ , multiply by 2, exponentiate both sides to remove the logarithm, and use algebra to group all the  $y$  terms on one side. Remember the definition:  $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$ .

**Solution 7.45**

i With  $v = \frac{dy}{dx}$ , we have  $y'' = v \frac{dv}{dy}$ , so

$$v \frac{dv}{dy} = 2y^3 - 2y.$$

Integrating,

$$\frac{1}{2}v^2 = \frac{1}{2}y^4 - y^2 + C \implies v^2 = y^4 - 2y^2 + C.$$

Using  $y(0) = 0$  and  $v(0) = 1$  gives  $C = 1$ , hence

$$v^2 = y^4 - 2y^2 + 1 = (1 - y^2)^2.$$

ii So  $v = \pm(1 - y^2)$ . At  $x = 0$ ,  $y = 0$  and  $v = 1$ , so the positive branch is forced:

$$\boxed{\frac{dy}{dx} = 1 - y^2}.$$

iii Separate variables:

$$\int \frac{dy}{1 - y^2} = \int dx.$$

Since

$$\frac{1}{1 - y^2} = \frac{1}{2} \left( \frac{1}{1 + y} + \frac{1}{1 - y} \right),$$

we obtain

$$\frac{1}{2} \ln \left| \frac{1 + y}{1 - y} \right| = x + C.$$

Using  $(x, y) = (0, 0)$  gives  $C = 0$ , so

$$\boxed{x = \frac{1}{2} \ln \left( \frac{1 + y}{1 - y} \right)}.$$

iv Exponentiating,

$$e^{2x} = \frac{1 + y}{1 - y} \implies y = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x.$$

Thus

$$\boxed{y = \tanh x},$$

with horizontal asymptotes  $\boxed{y = 1}$  and  $\boxed{y = -1}$ .

**Takeaways 7.45**

- **The Power of  $v \frac{dv}{dy}$ :** This substitution is the master key for solving *any* autonomous second-order differential equation (where  $x$  is missing). It essentially allows you to integrate an acceleration function to find an energy/velocity equation.
- **Beware the Square Root:** In part ii, lazily writing  $v = \pm\sqrt{(y^2 - 1)^2}$  without checking the initial conditions will doom the rest of the problem. Always verify your branch cuts in differential equations!
- **Separatrix Curves:** In physics, the curve  $y = \tanh(x)$  represents a “separatrix” in phase space. It is the exact boundary trajectory between a particle that is trapped oscillating in a potential well, and a particle that has enough energy to escape entirely.

**Problem 7.46: Damped Harmonic Motion**

Consider the function  $y = e^{-x}(A \cos 2x + B \sin 2x)$ , where  $A$  and  $B$  are arbitrary constants.

- i Use the product rule to find the first derivative,  $y'$ , and the second derivative,  $y''$ , simplifying your answers by grouping the sine and cosine terms.
- ii Verify that the given function is a general solution to the differential equation  $y'' + 2y' + 5y = 0$ .
- iii Find the exact values of  $A$  and  $B$  given the initial conditions  $y(0) = 2$  and  $y'(0) = -2$ .
- iv Using the values of  $A$  and  $B$  found in part iii, describe the behavior of the solution curve as  $x \rightarrow \infty$ .

**Hint:**

- **For i:** Let  $u = e^{-x}$  and  $v = A \cos 2x + B \sin 2x$ . When finding the second derivative, be careful with the negative signs from the chain rule.
- **For ii:** Substitute your expressions for  $y, y'$ , and  $y''$  into the left-hand side of the differential equation. Group all the  $e^{-x} \cos 2x$  terms together and all the  $e^{-x} \sin 2x$  terms together to show they sum to zero.
- **For iii:** Substitute  $x = 0$  into your original equation for  $y$  to easily find  $A$ . Then substitute  $x = 0$  and your value for  $A$  into your equation for  $y'$  from part i to solve for  $B$ .
- **For iv:** Consider the limits of the exponential and trigonometric components separately. What happens to  $e^{-x}$  as  $x$  gets infinitely large?

**Solution 7.46**

**i** Given  $y = e^{-x}(A \cos 2x + B \sin 2x)$ .

Using the product rule:

$$y' = -e^{-x}(A \cos 2x + B \sin 2x) + e^{-x}(-2A \sin 2x + 2B \cos 2x)$$

$$y' = e^{-x}[(-A + 2B) \cos 2x + (-2A - B) \sin 2x]$$

Differentiating again using the product rule:

$$y'' = -e^{-x}[(-A + 2B) \cos 2x + (-2A - B) \sin 2x] + e^{-x}[-2(-A + 2B) \sin 2x + 2(-2A - B) \cos 2x]$$

Grouping the cosine terms:  $-(-A + 2B) + 2(-2A - B) = A - 2B - 4A - 2B = -3A - 4B$

Grouping the sine terms:  $-(-2A - B) - 2(-A + 2B) = 2A + B + 2A - 4B = 4A - 3B$

$$y'' = e^{-x}[(-3A - 4B) \cos 2x + (4A - 3B) \sin 2x]$$

**ii** Substitute  $y, y'$ , and  $y''$  into the LHS of  $y'' + 2y' + 5y = 0$ :

$$\text{LHS} = e^{-x}[(-3A - 4B) \cos 2x + (4A - 3B) \sin 2x] + 2e^{-x}[(-A + 2B) \cos 2x + (-2A - B) \sin 2x] + 5e^{-x}[A \cos 2x + B \sin 2x]$$

Factor out  $e^{-x}$  and group by trig function:

$$\cos 2x \text{ coefficient: } (-3A - 4B) + 2(-A + 2B) + 5A = -3A - 4B - 2A + 4B + 5A = 0$$

$$\sin 2x \text{ coefficient: } (4A - 3B) + 2(-2A - B) + 5B = 4A - 3B - 4A - 2B + 5B = 0$$

$$\text{LHS} = e^{-x}[0 \cos 2x + 0 \sin 2x] = 0 = \text{RHS} \quad \square$$

**iii** Apply  $y(0) = 2$ :

$$2 = e^0(A \cos 0 + B \sin 0) \implies 2 = 1(A + 0) \implies \mathbf{A = 2}$$

Apply  $y'(0) = -2$  using the formula from part **i**:

$$-2 = e^0[(-A + 2B) \cos 0 + (-2A - B) \sin 0]$$

$$-2 = 1[-A + 2B]$$

Substitute  $A = 2$ :

$$-2 = -2 + 2B \implies 0 = 2B \implies \mathbf{B = 0}$$

**iv** With  $A = 2$  and  $B = 0$ , the particular solution is  $y = 2e^{-x} \cos 2x$ .

As  $x \rightarrow \infty$ , the trigonometric term  $\cos 2x$  oscillates continuously between  $-1$  and  $1$ . However, the exponential decay term  $e^{-x} \rightarrow 0$ .

Therefore, the amplitude of the oscillation approaches zero, and  $y \rightarrow 0$  as  $x \rightarrow \infty$ .

**Takeaways 7.46**

- A damped oscillation combines periodic behavior with exponential decay.
- Verifying a second-order solution is often faster than solving the differential equation from scratch.
- Initial conditions usually determine the phase constants most cleanly when evaluated at  $x = 0$ .

**Problem 7.47: Fourth-Order Verification**

Consider the function  $y = Ae^x + Be^{-x} + C \cos 2x + D \sin 2x$ , where  $A, B, C$ , and  $D$  are arbitrary constants.

- i** Find the second derivative  $y''$  and the fourth derivative  $y^{(4)}$ .
- ii** Verify that the given function is a general solution to the fourth-order differential equation  $y^{(4)} + 3y'' - 4y = 0$ .
- iii** A specific physical system modeled by this equation requires the solution  $y(x)$  to remain bounded (finite) as  $x \rightarrow \infty$  and also as  $x \rightarrow -\infty$ . Deduce the necessary values of the constants  $A$  and  $B$ .
- iv** Using your findings from part **iii**, and given the initial conditions  $y(0) = 3$  and  $y'(0) = 4$ , find the exact particular solution for the system.

**Hint:**

- **For i:** The derivatives of exponential functions cycle simply, and trigonometric functions cycle every four derivatives.
- **For ii:** Substitute  $y^{(4)}, y'',$  and  $y$  into the left-hand side of the equation. Group the terms by their function type ( $e^x, e^{-x}, \cos 2x, \sin 2x$ ).
- **For iii:** Analyze the limits. What happens to  $Ae^x$  as  $x \rightarrow \infty$ ? What happens to  $Be^{-x}$  as  $x \rightarrow -\infty$ ? If the solution must be bounded, these terms cannot be allowed to grow infinitely.
- **For iv:** Set  $A$  and  $B$  to the values you deduced, leaving a much simpler equation with only  $C$  and  $D$ . Substitute  $x = 0$  into your simplified  $y$  and  $y'$  to find the remaining constants.

**Solution 7.47**

**i** Given  $y = Ae^x + Be^{-x} + C \cos 2x + D \sin 2x$ .

First derivative:

$$y' = Ae^x - Be^{-x} - 2C \sin 2x + 2D \cos 2x$$

Second derivative:

$$y'' = Ae^x + Be^{-x} - 4C \cos 2x - 4D \sin 2x$$

Third derivative:

$$y''' = Ae^x - Be^{-x} + 8C \sin 2x - 8D \cos 2x$$

Fourth derivative:

$$y^{(4)} = Ae^x + Be^{-x} + 16C \cos 2x + 16D \sin 2x$$

**ii** Substitute into the LHS of  $y^{(4)} + 3y'' - 4y = 0$ :

$$\begin{aligned} \text{LHS} &= (Ae^x + Be^{-x} + 16C \cos 2x + 16D \sin 2x) + 3(Ae^x + Be^{-x} - 4C \cos 2x - 4D \sin 2x) \\ &\quad - 4(Ae^x + Be^{-x} + C \cos 2x + D \sin 2x) \end{aligned}$$

Group the terms:

$$e^x \text{ terms: } A + 3A - 4A = 0$$

$$e^{-x} \text{ terms: } B + 3B - 4B = 0$$

$$\cos 2x \text{ terms: } 16C - 12C - 4C = 0$$

$$\sin 2x \text{ terms: } 16D - 12D - 4D = 0$$

$$\text{LHS} = 0 = \text{RHS} \quad \square$$

**iii** Consider the behavior of the terms as  $x \rightarrow \pm\infty$ :

The trigonometric terms  $C \cos 2x$  and  $D \sin 2x$  are always bounded between  $-|C|$  and  $|C|$ , and  $-|D|$  and  $|D|$ .

As  $x \rightarrow \infty$ , the term  $Ae^x \rightarrow \pm\infty$  unless  $A = 0$ . Therefore, to remain bounded as  $x$  gets infinitely large, we must have  $A = 0$ .

As  $x \rightarrow -\infty$ , the term  $Be^{-x} \rightarrow \pm\infty$  unless  $B = 0$ . Therefore, to remain bounded as  $x$  gets infinitely negative, we must have  $B = 0$ .

**iv** Given  $A = 0$  and  $B = 0$ , the general solution simplifies to  $y = C \cos 2x + D \sin 2x$ .

The first derivative is  $y' = -2C \sin 2x + 2D \cos 2x$ .

Apply  $y(0) = 3$ :

$$3 = C \cos 0 + D \sin 0 \implies \mathbf{C = 3}$$

Apply  $y'(0) = 4$ :

$$4 = -2(3) \sin 0 + 2D \cos 0 \implies 4 = 2D \implies \mathbf{D = 2}$$

The exact particular solution is:

$$\mathbf{y = 3 \cos 2x + 2 \sin 2x}$$

**Takeaways 7.47**

- Higher-order verification still follows the same pattern: differentiate, substitute, and group like terms.
- Boundedness conditions can eliminate entire exponential branches immediately.
- Even a fourth-order model can collapse to a short system for the remaining constants once the physical constraints are used.

**Problem 7.48: The Vibrating Cantilever Beam**

In the study of vibrating cantilever beams (such as a diving board or an airplane wing), the vertical deflection  $y(x)$  satisfies the fourth-order differential equation  $\frac{d^4y}{dx^4} = \lambda^4 y$ , where  $\lambda$  is a positive constant related to the beam's material and frequency. The general solution is given by:

$$y(x) = C_1 \cosh(\lambda x) + C_2 \sinh(\lambda x) + C_3 \cos(\lambda x) + C_4 \sin(\lambda x)$$

A cantilever beam of length  $L$  is clamped firmly into a wall at  $x = 0$  and is completely free at  $x = L$ .

- i** At the clamped end ( $x = 0$ ), there is zero deflection ( $y = 0$ ) and zero slope ( $y' = 0$ ). Use these boundary conditions to show that  $C_3 = -C_1$  and  $C_4 = -C_2$ , reducing the solution to  $y(x) = C_1(\cosh \lambda x - \cos \lambda x) + C_2(\sinh \lambda x - \sin \lambda x)$ .
- ii** At the free end ( $x = L$ ), there is no bending moment and no shear force, which mathematically requires  $y''(L) = 0$  and  $y'''(L) = 0$ . Calculate  $y''(x)$  and  $y'''(x)$ , and write down the two equations that  $C_1$  and  $C_2$  must satisfy at  $x = L$ .
- iii** For the beam to actually vibrate, it must have a non-trivial solution (meaning  $C_1$  and  $C_2$  cannot both be zero). By eliminating  $C_1$  and  $C_2$  from your equations in part **ii**, prove that  $\lambda L$  must satisfy the characteristic equation  $\cos(\lambda L) \cosh(\lambda L) = -1$ .
- iv** Let the amplitude constant be  $A = C_1$ . Using your equations from part **ii**, express  $C_2$  in terms of  $A$ ,  $\lambda$ , and  $L$ . Hence, write down the final equation for the mode shape of the vibrating beam,  $y(x)$ , entirely in terms of  $A$ ,  $\lambda$ ,  $L$ , and  $x$ .

**Hint:**

- **For i:** Remember that  $\cosh(0) = 1, \sinh(0) = 0, \cos(0) = 1, \sin(0) = 0$ . When differentiating, recall that  $\frac{d}{dx} \cosh(\lambda x) = \lambda \sinh(\lambda x)$  and  $\frac{d}{dx} \sinh(\lambda x) = \lambda \cosh(\lambda x)$ .
- **For ii:** Differentiate the reduced equation from part **i** twice more. Set  $x = L$  and equate both expressions to 0. You can divide out the common  $\lambda^2$  and  $\lambda^3$  factors since  $\lambda > 0$ .
- **For iii:** You have a system of two linear equations. Solve the first equation for  $C_2$  in terms of  $C_1$ , and substitute it into the second equation. Since  $C_1 \neq 0$ , you can divide it out. Expand the brackets carefully and use the fundamental identities  $\cosh^2 \theta - \sinh^2 \theta = 1$  and  $\cos^2 \theta + \sin^2 \theta = 1$  to simplify the massive resulting fraction down to  $-1$ .
- **For iv:** Rearrange either of the boundary equations from part **ii** to make  $C_2$  the subject, and simply substitute it back into the equation from part **i**.

**Solution 7.48**

**i** Given  $y(x) = C_1 \cosh(\lambda x) + C_2 \sinh(\lambda x) + C_3 \cos(\lambda x) + C_4 \sin(\lambda x)$ .

Apply  $y(0) = 0$ :

$$0 = C_1(1) + C_2(0) + C_3(1) + C_4(0) \implies C_3 = -C_1$$

Differentiate to find  $y'(x)$ :

$$y'(x) = \lambda C_1 \sinh(\lambda x) + \lambda C_2 \cosh(\lambda x) - \lambda C_3 \sin(\lambda x) + \lambda C_4 \cos(\lambda x)$$

Apply  $y'(0) = 0$ :

$$0 = \lambda[C_1(0) + C_2(1) - C_3(0) + C_4(1)] \implies C_4 = -C_2$$

Substitute  $C_3$  and  $C_4$  back into the general solution and factor:

$$\begin{aligned} y(x) &= C_1 \cosh(\lambda x) + C_2 \sinh(\lambda x) - C_1 \cos(\lambda x) - C_2 \sin(\lambda x) \\ y(x) &= C_1(\cosh \lambda x - \cos \lambda x) + C_2(\sinh \lambda x - \sin \lambda x) \quad \square \end{aligned}$$

**ii** Differentiate  $y(x)$  from part **i** to find the second and third derivatives:

$$\begin{aligned} y'(x) &= \lambda[C_1(\sinh \lambda x + \sin \lambda x) + C_2(\cosh \lambda x - \cos \lambda x)] \\ y''(x) &= \lambda^2[C_1(\cosh \lambda x + \cos \lambda x) + C_2(\sinh \lambda x + \sin \lambda x)] \\ y'''(x) &= \lambda^3[C_1(\sinh \lambda x - \sin \lambda x) + C_2(\cosh \lambda x + \cos \lambda x)] \end{aligned}$$

Applying the boundary conditions at  $x = L$ :

$$y''(L) = 0 \implies C_1(\cosh \lambda L + \cos \lambda L) + C_2(\sinh \lambda L + \sin \lambda L) = 0 \quad (\text{Eq. 1})$$

$$y'''(L) = 0 \implies C_1(\sinh \lambda L - \sin \lambda L) + C_2(\cosh \lambda L + \cos \lambda L) = 0 \quad (\text{Eq. 2})$$

**iii** From (Eq. 1), isolate  $C_2$ :

$$C_2 = -C_1 \frac{\cosh \lambda L + \cos \lambda L}{\sinh \lambda L + \sin \lambda L}$$

Substitute this expression for  $C_2$  into (Eq. 2):

$$C_1(\sinh \lambda L - \sin \lambda L) - C_1 \frac{(\cosh \lambda L + \cos \lambda L)^2}{\sinh \lambda L + \sin \lambda L} = 0$$

Since we seek a non-trivial solution,  $C_1 \neq 0$ , so we can divide it out. Multiply the entire equation by  $(\sinh \lambda L + \sin \lambda L)$  to clear the denominator:

$$(\sinh \lambda L - \sin \lambda L)(\sinh \lambda L + \sin \lambda L) - (\cosh \lambda L + \cos \lambda L)^2 = 0$$

Expand using difference of two squares and perfect squares:

$$(\sinh^2 \lambda L - \sin^2 \lambda L) - (\cosh^2 \lambda L + 2 \cosh \lambda L \cos \lambda L + \cos^2 \lambda L) = 0$$

Group the hyperbolic and trigonometric terms together:

$$(\sinh^2 \lambda L - \cosh^2 \lambda L) - (\sin^2 \lambda L + \cos^2 \lambda L) - 2 \cosh \lambda L \cos \lambda L = 0$$

Using the identities  $\cosh^2 \theta - \sinh^2 \theta = 1 \implies \sinh^2 \theta - \cosh^2 \theta = -1$ , and  $\sin^2 \theta + \cos^2 \theta = 1$ :

$$\begin{aligned} -1 - 1 - 2 \cosh \lambda L \cos \lambda L &= 0 \\ -2 &= 2 \cosh \lambda L \cos \lambda L \\ \cos(\lambda L) \cosh(\lambda L) &= -1 \quad \square \end{aligned}$$

**iv** Using the substitution from part **iii** and letting  $C_1 = A$ :

$$C_2 = -A \frac{\cosh \lambda L + \cos \lambda L}{\sinh \lambda L + \sin \lambda L}$$

Substitute  $C_1$  and  $C_2$  into the equation from part **i**:

$$y(x) = A \left[ (\cosh \lambda x - \cos \lambda x) - \frac{\cosh \lambda L + \cos \lambda L}{\sinh \lambda L + \sin \lambda L} (\sinh \lambda x - \sin \lambda x) \right]$$

**Takeaways 7.48**

- **Eigenvalue Problems:** The equation  $\cos(\lambda L) \cosh(\lambda L) = -1$  cannot be solved algebraically. It has infinitely many discrete roots (eigenvalues), which correspond to the fundamental frequency and the higher harmonics (overtones) of the vibrating beam.
- **Linear Algebra meets Calculus:** In part **iii**, finding a “non-trivial solution” to a system of two equations equal to zero is equivalent to saying the determinant of their coefficients must be zero. Setting the determinant of the coefficient matrix to zero yields the exact same characteristic equation much faster!
- Using complex exponentials  $e^{i\theta} = \cos \theta + i \sin \theta$  and  $e^{\lambda x} = \cosh \lambda x + \sinh \lambda x$  can often simplify the algebra in problems like this, but note that it is out of syllabus for MX1 and MX2.

**Problem 7.49: Pharmacokinetics and Two-Compartment Diffusion**

According to Fick’s law, diffusion across a semi-permeable membrane is driven by the concentration gradient. The classic model assumes the surrounding concentration is constant. However, in closed medical systems, this is not true.

A drug is injected into a patient’s bloodstream. The blood has a volume of  $V_B = 4$  Litres, and the drug diffuses into a target organ with a volume of  $V_O = 1$  Litre. Let  $B(t)$  and  $C(t)$  be the concentration of the drug (in mg/L) in the blood and the organ respectively, at time  $t$  hours.

A total mass of 100 mg of the drug is injected into the bloodstream at  $t = 0$ , meaning  $B(0) = 25$  mg/L and  $C(0) = 0$  mg/L. The total mass of the drug in the system is conserved such that  $4B(t) + 1C(t) = 100$  for all  $t \geq 0$ .

The rate at which the drug diffuses into the organ is governed by Fick’s Law:

$$\frac{dC}{dt} = k(B - C)$$

where  $k$  is a positive constant of permeability.

- Use the conservation of mass equation to express  $B$  in terms of  $C$ , and hence show that the differential equation can be written purely in terms of  $C$  as  $\frac{dC}{dt} = \frac{5k}{4}(20 - C)$ .
- By separating the variables, solve this initial value problem to find an explicit formula for the concentration of the drug in the organ,  $C(t)$ .
- Determine the limiting values of both  $C(t)$  and  $B(t)$  as  $t \rightarrow \infty$ . Explain the physical significance of this result in the context of diffusion.
- Suppose medical imaging shows that it takes exactly 2 hours for the concentration in the organ to reach 10 mg/L. Find the exact value of  $k$ . Hence, determine the exact rate of diffusion ( $\frac{dC}{dt}$ ) at this instant.

**Hint:**

- **For i:** Rearrange the linear equation  $4B + C = 100$  to make  $B$  the subject. Substitute this expression for  $B$  into the original differential equation and factorise the right-hand side.
- **For ii:** Separate the variables to get  $\int \frac{dC}{20-C} = \int \frac{5k}{4} dt$ . Be very careful with the negative sign when integrating the left side! Apply the initial condition  $C(0) = 0$  to find your constant.
- **For iii:** Consider the behaviour of the exponential term  $e^{-rt}$  as time goes to infinity. Once you find the limit of  $C(t)$ , use the conservation equation to find the corresponding limit of  $B(t)$ .
- **For iv:** Substitute  $t = 2$  and  $C = 10$  into your explicit function from part ii and use logarithms to solve for  $k$ . To find the rate, you do not need to differentiate your function—simply substitute your values for  $k$  and  $C$  back into the differential equation from part i.

**Solution 7.49**

i From  $4B + C = 100$ ,

$$B = 25 - \frac{C}{4}.$$

Substituting into  $\frac{dC}{dt} = k(B - C)$  gives

$$\frac{dC}{dt} = k \left( 25 - \frac{5}{4}C \right) = \frac{5k}{4} (20 - C).$$

ii Separate variables:

$$\int \frac{dC}{20 - C} = \int \frac{5k}{4} dt.$$

Hence

$$-\ln(20 - C) = \frac{5k}{4}t + A$$

and, using  $C(0) = 0$ ,

$$20 - C = 20e^{-5kt/4}.$$

Therefore

$$C(t) = 20 \left( 1 - e^{-5kt/4} \right).$$

iii As  $t \rightarrow \infty$ , the exponential term vanishes, so

$$\lim_{t \rightarrow \infty} C(t) = 20.$$

Then  $4B + 20 = 100$ , giving

$$\lim_{t \rightarrow \infty} B(t) = 20.$$

So both compartments approach the same concentration, and diffusion stops at equilibrium.

iv Since  $C(2) = 10$ ,

$$10 = 20 \left( 1 - e^{-5k/2} \right) \implies e^{-5k/2} = \frac{1}{2} \implies k = \frac{2}{5} \ln 2.$$

At that instant,

$$\frac{dC}{dt} = \frac{5k}{4} (20 - C) = \frac{5}{4} \cdot \frac{2 \ln 2}{5} \cdot 10 = 5 \ln 2 \text{ mg/L/hr}.$$

**Takeaways 7.49**

- **Coupled Systems:** In real life, biological compartments aren't infinite. As the organ absorbs the drug, the blood loses it. This problem shows how to use a secondary relationship (conservation of mass) to collapse a two-variable system into a solvable, single-variable differential equation.
- **Equilibrium isn't just zero:** In exponential decay problems, things usually drop to zero. In coupled diffusion, the system shifts until the *gradient* is zero. Calculating the limits proves that diffusion stops exactly when the concentrations match.
- **Work Smarter:** In part **iv**, calculating the rate directly from the  $\frac{dC}{dt}$  equation is much faster and less prone to algebraic errors than differentiating the explicit solution  $C(t)$ . Always look back to the original DE!

**Problem 7.50: Radiosonde Telemetry in the Snowy Mountains**

A Bureau of Meteorology (BOM) radiosonde balloon is launched from a station near the Snowy Mountains. The atmospheric pressure  $P$  (in kPa) at altitude  $h$  (in meters) decreases at a rate proportional to the current pressure, governed by the differential equation  $\frac{dP}{dh} = -kP$ , for some positive constant  $k$ .

- By separating the variables, find the general solution for  $P$  in terms of  $h$ , initial pressure  $P_0$ , and  $k$ .
- Tracking data shows that the pressure at an altitude of 2000 m is 80 kPa, and at 6000 m it is 40 kPa. Find the exact value of the decay constant  $k$ , and show that the ground-level pressure at the launch station ( $h = 0$ ) is exactly  $80\sqrt{2}$  kPa.
- The balloon ascends at a constant vertical velocity of 5 m/s. Let  $t$  be the time in seconds after launch. Use the chain rule to formulate a differential equation for  $\frac{dP}{dt}$ , and hence express  $P$  explicitly as a function of time  $t$ .
- The radiosonde's pressure sensor requires the magnitude of the depressurization rate ( $\left|\frac{dP}{dt}\right|$ ) to remain above 0.05 kPa/s to calibrate its internal altimeter accurately. Calculate the exact altitude  $h$  at which the sensor loses accurate calibration.

- Hint:**
- **For i:** Move  $P$  to the left and  $dh$  to the right. Integrate both sides to get a natural logarithm, then exponentiate. Apply the condition that at  $h = 0$ ,  $P = P_0$ .
  - **For ii:** Set up two equations using the two data points. Divide the 6 000 m equation by the 2 000 m equation to eliminate  $P_0$  and solve for  $k$ . To find  $P_0$ , substitute  $k$  back into the first equation and use index laws like  $e^{A \ln B} = B^A$ .
  - **For iii:** The vertical velocity is  $\frac{dh}{dt} = 5$ . Apply the chain rule:  $\frac{dP}{dt} = \frac{dP}{dh} \cdot \frac{dh}{dt}$ . Solve this new differential equation with respect to  $t$ .
  - **For iv:** The magnitude of depressurization is simply the positive version of your rate from part iii. Set  $5kP = 0.05$  to find the exact pressure  $P$  at which failure occurs. Then, equate this  $P$  to your altitude equation  $P(h)$  to find  $h$ . You will need to use logarithm laws carefully, resulting in a nested logarithm  $\ln(\ln 2)$ .

**Solution 7.50**

i Separating variables,

$$\int \frac{1}{P} dP = \int -k dh \implies \ln P = -kh + C.$$

If  $P(0) = P_0$ , then  $e^C = P_0$ , so

$$P(h) = P_0 e^{-kh}.$$

ii Using  $P(2000) = 80$  and  $P(6000) = 40$ ,

$$80 = P_0 e^{-2000k}, \quad 40 = P_0 e^{-6000k}.$$

Dividing gives

$$\frac{1}{2} = e^{-4000k} \implies k = \frac{\ln 2}{4000}.$$

Then

$$80 = P_0 e^{-(1/2)\ln 2} = P_0 \cdot \frac{1}{\sqrt{2}},$$

so

$$P_0 = 80\sqrt{2} \text{ kPa}.$$

iii Since  $\frac{dh}{dt} = 5$ ,

$$\frac{dP}{dt} = \frac{dP}{dh} \frac{dh}{dt} = -kP \cdot 5 = -5kP = -\frac{\ln 2}{800} P.$$

Hence

$$P(t) = P_0 e^{-(\ln 2)t/800} = 80\sqrt{2} \cdot 2^{-t/800}.$$

iv The calibration fails when

$$\left| \frac{dP}{dt} \right| = 5kP = 0.05 \implies P = \frac{0.01}{k} = \frac{40}{\ln 2}.$$

Now solve

$$80\sqrt{2} e^{-kh} = \frac{40}{\ln 2} \implies e^{-kh} = \frac{1}{2\sqrt{2} \ln 2}.$$

Taking logs,

$$-kh = -\frac{3}{2} \ln 2 - \ln(\ln 2),$$

so

$$h = \frac{\frac{3}{2} \ln 2 + \ln(\ln 2)}{k} = 6000 + 4000 \frac{\ln(\ln 2)}{\ln 2} \text{ m}.$$

**Takeaways 7.50**

- **Linking Variables via Chain Rule:** Differential equations in textbooks often stick to just two variables (like  $y$  and  $x$ ). In reality, systems move through time *and* space. Part **iii** demonstrates how the chain rule allows you to transform a spatial differential equation ( $\frac{dP}{dh}$ ) into a temporal one ( $\frac{dP}{dt}$ ).
- **Exact Value Discipline:** Decimals will ruin your accuracy in exponential models. Learning to manipulate terms like  $e^{-0.5 \ln 2}$  into  $\frac{1}{\sqrt{2}}$  is a crucial algebraic skill for high-level HSC Mathematics.
- **Nested Logarithms:** Don't panic if you see  $\ln(\ln x)$ . It behaves exactly like any other term algebraically. Just trust the logarithm laws!

**Problem 7.51: Non-Linear Differential Equations and Trigonometric Families**

The non-linear differential equation

$$(y')^2 = 4y(1 - y)$$

has non-constant solution families

$$y_1(x) = \sin^2(x + \alpha), \quad y_2(x) = \cos^2(x + \beta).$$

- i** Find the constant solutions and explain their geometric significance.
- ii** Verify that both  $y_1$  and  $y_2$  satisfy the differential equation.
- iii** Show that the two families are the same up to a phase shift, and find a relation between  $\alpha$  and  $\beta$ .
- iv** If  $y(0) = \frac{3}{4}$  and  $y'(0) < 0$ , determine  $\alpha$  and  $\beta$  in the interval  $0 \leq \alpha, \beta < \pi$ .

**Hint:**

- Set  $y' = 0$  first to find constant solutions.
- Differentiate the trial solutions carefully and square only at the end.
- Use  $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$  and  $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ .
- The sign condition on  $y'(0)$  selects the correct branch.

**Solution 7.51**

i If  $y = c$  is constant, then  $y' = 0$ , so

$$0 = 4c(1 - c).$$

Hence the constant solutions are

$$\boxed{y = 0 \quad \text{and} \quad y = 1.}$$

They act as horizontal envelope curves because the non-constant trigonometric solutions stay between 0 and 1.

ii For  $y_1(x) = \sin^2(x + \alpha)$ ,

$$y_1' = 2 \sin(x + \alpha) \cos(x + \alpha) = \sin(2x + 2\alpha),$$

so

$$(y_1')^2 = \sin^2(2x + 2\alpha) = 4 \sin^2(x + \alpha) \cos^2(x + \alpha) = 4y_1(1 - y_1).$$

Similarly, for  $y_2(x) = \cos^2(x + \beta)$ ,

$$y_2' = -\sin(2x + 2\beta), \quad (y_2')^2 = 4y_2(1 - y_2).$$

iii Rewrite both families:

$$y_1(x) = \frac{1}{2}(1 - \cos(2x + 2\alpha)), \quad y_2(x) = \frac{1}{2}(1 + \cos(2x + 2\beta)).$$

For the families to describe the same curves we need

$$-\cos(2x + 2\alpha) = \cos(2x + 2\beta).$$

Using  $-\cos \theta = \cos(\theta + \pi)$  gives

$$2\beta = 2\alpha \pm \pi \quad \implies \quad \boxed{\beta = \alpha \pm \frac{\pi}{2}.}$$

iv From  $y_1(0) = \sin^2 \alpha = \frac{3}{4}$ , we get  $\alpha = \frac{\pi}{3}$  or  $\alpha = \frac{2\pi}{3}$ . Also

$$y_1'(0) = \sin(2\alpha).$$

The condition  $y'(0) < 0$  forces  $\sin(2\alpha) < 0$ , so

$$\boxed{\alpha = \frac{2\pi}{3}.}$$

Then

$$\beta = \alpha - \frac{\pi}{2} = \frac{2\pi}{3} - \frac{\pi}{2} = \boxed{\frac{\pi}{6}}.$$

**Takeaways 7.51**

- Non-linear equations can admit envelope solutions such as  $y = 0$  and  $y = 1$ .
- Squaring a derivative hides sign information, so initial conditions matter.
- Two different-looking trigonometric answers may represent the same family up to phase shift.

**Problem 7.52: Non-Dimensionalisation and the Richards Growth Model**

Let  $A(t)$  satisfy

$$\frac{dA}{dt} = kA(M^2 - A^2), \quad A(0) = A_0,$$

where  $k > 0$  and  $M > 0$  are constants.

**i** Use the substitution  $A = M\sqrt{y}$  to show that  $y$  satisfies

$$\frac{dy}{dt} = 2kM^2y(1 - y).$$

**ii** Define  $\tau = 2kM^2t$  and show that the equation reduces to

$$\frac{dy}{d\tau} = y(1 - y).$$

**iii** Use  $v = \frac{1}{y}$  to transform this into a linear equation for  $v$ , and solve it.

**iv** Hence find an explicit formula for  $A(t)$ .

**Hint:**

- Differentiate  $A = My^{1/2}$  using the chain rule.
- The substitution  $\tau = 2kM^2t$  removes all parameters from the equation.
- If  $v = y^{-1}$ , then  $\frac{dv}{dy} = -y^{-2} \frac{dy}{dy}$ .
- Apply  $A(0) = A_0$  only after converting back from  $v$  to  $y$  and then to  $A$ .

**Solution 7.52**

i Let  $A = M\sqrt{y}$ . Then

$$\frac{dA}{dt} = \frac{1}{2}My^{-1/2}\frac{dy}{dt}.$$

Substituting into the original equation gives

$$\frac{1}{2}My^{-1/2}\frac{dy}{dt} = k(My^{1/2})(M^2 - M^2y).$$

After simplifying,

$$\frac{dy}{dt} = 2kM^2y(1 - y).$$

ii If  $\tau = 2kM^2t$ , then

$$\frac{dy}{dt} = \frac{dy}{d\tau} \frac{d\tau}{dt} = 2kM^2 \frac{dy}{d\tau},$$

so

$$\frac{dy}{d\tau} = y(1 - y).$$

iii Set  $v = \frac{1}{y}$ . Then

$$\frac{dv}{d\tau} = -y^{-2}\frac{dy}{d\tau} = -y^{-2}(y - y^2) = -\frac{1}{y} + 1 = -v + 1.$$

Hence

$$\frac{dv}{d\tau} + v = 1.$$

Using the integrating factor  $e^\tau$ ,

$$\frac{d}{d\tau}(ve^\tau) = e^\tau,$$

so after integrating,

$$v(\tau) = 1 + Ce^{-\tau}.$$

iv Since  $y = \frac{1}{v}$ ,

$$y(\tau) = \frac{1}{1 + Ce^{-\tau}}.$$

Therefore

$$A(t) = M\sqrt{y} = \frac{M}{\sqrt{1 + Ce^{-2kM^2t}}}.$$

Use  $A(0) = A_0$ :

$$A_0 = \frac{M}{\sqrt{1 + C}} \implies C = \frac{M^2 - A_0^2}{A_0^2}.$$

Hence

$$A(t) = \frac{MA_0}{\sqrt{A_0^2 + (M^2 - A_0^2)e^{-2kM^2t}}}.$$

**Takeaways 7.52**

- Non-dimensionalisation strips away parameters and reveals the universal structure of a model.
- A substitution can reduce a non-linear growth law to a standard logistic form.
- Reciprocal substitutions such as  $v = \frac{1}{y}$  are a powerful linearisation tool.

**Problem 7.53: Verification of a Third-Order Solution**

Consider

$$y''' - y = 0.$$

Verify that

$$y = C_1 e^x + C_2 e^{-x/2} \cos\left(\frac{\sqrt{3}}{2}x\right) + C_3 e^{-x/2} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

is a general solution.

- Hint:**
- Let  $\omega = \frac{\sqrt{3}}{2}$  and keep the common factor  $e^{-x/2}$  outside.
  - Differentiate the oscillatory part in grouped cosine-sine form instead of expanding everything.
  - Because the equation is linear, it is enough to show that the full expression reproduces itself after three derivatives.

**Solution 7.53**

Let

$$\omega = \frac{\sqrt{3}}{2}, \quad u = e^{-x/2}(C_2 \cos \omega x + C_3 \sin \omega x).$$

Then  $y = C_1 e^x + u$ . A tidy regrouping gives

$$u' = e^{-x/2} \left[ \left( -\frac{1}{2}C_2 + \omega C_3 \right) \cos \omega x + \left( -\omega C_2 - \frac{1}{2}C_3 \right) \sin \omega x \right],$$

$$u'' = e^{-x/2} \left[ \left( -\frac{1}{2}C_2 - \omega C_3 \right) \cos \omega x + \left( \omega C_2 - \frac{1}{2}C_3 \right) \sin \omega x \right].$$

Differentiating once more and using  $\omega^2 = \frac{3}{4}$  returns

$$u''' = e^{-x/2}(C_2 \cos \omega x + C_3 \sin \omega x) = u.$$

Also  $(C_1 e^x)''' = C_1 e^x$ , so  $y''' = y$ . Therefore

$$\boxed{y''' - y = 0},$$

which verifies the proposed general solution.

**Takeaways 7.53**

- Complex roots of characteristic equations naturally produce exponentially damped sine-cosine terms.
- Grouping terms before the next derivative keeps higher-order verification manageable.
- Refer to the Appendix for a more info about linear homogeneous differential equations.

**Problem 7.54: The Fourth-Order Oscillator (Enrichment)**

This problem is for enrichment only and is outside the NESAsyllabus.  
Verify that

$$y = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x$$

solves

$$y^{(4)} - y = 0.$$

**Hint:**

- Exponentials and trigonometric functions cycle predictably under repeated differentiation.
- Use linearity and check the four building blocks separately.

**Solution 7.54**

Differentiate each component four times:

$$\begin{aligned} \frac{d^4}{dx^4}(e^x) &= e^x, & \frac{d^4}{dx^4}(e^{-x}) &= e^{-x}, \\ \frac{d^4}{dx^4}(\cos x) &= \cos x, & \frac{d^4}{dx^4}(\sin x) &= \sin x. \end{aligned}$$

Therefore

$$y^{(4)} = C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x = y,$$

so

$$y^{(4)} - y = 0.$$

This verifies the proposed general solution.

**Takeaways 7.54**

- The roots of  $r^4 - 1 = 0$  explain why exponentials and trig functions appear together.
- Fourth-order verification is often shorter than expected because the derivative cycles are simple.

**Problem 7.55: Integrating Factor with Oscillatory Forcing**

For  $x > 0$ , consider

$$x \frac{dy}{dx} - 2y = x^4 \sin x.$$

- i Find the general solution.
- ii Find the particular solution satisfying  $y(\pi) = 0$ .
- iii Find two simple bounding curves for that particular solution.

**Hint:**

- Divide by  $x$  first.
- The integrating factor is  $x^{-2}$ .
- Group the final answer as  $x^2(\sin x - x \cos x - \pi)$ .

**Solution 7.55**

Writing the equation as

$$y' - \frac{2}{x}y = x^3 \sin x,$$

the integrating factor is

$$\mu(x) = e^{\int -2/x dx} = x^{-2}.$$

Hence

$$\frac{d}{dx} \left( \frac{y}{x^2} \right) = x \sin x.$$

Integrating by parts,

$$\frac{y}{x^2} = -x \cos x + \sin x + C,$$

so

$$y = -x^3 \cos x + x^2 \sin x + Cx^2.$$

Using  $y(\pi) = 0$  gives

$$0 = \pi^3 + C\pi^2 \Rightarrow C = -\pi.$$

Thus

$$y = -x^3 \cos x + x^2 \sin x - \pi x^2.$$

Since  $-1 \leq \sin x, \cos x \leq 1$ , we have

$$-x^3 - (\pi + 1)x^2 \leq y \leq x^3 + (1 - \pi)x^2.$$

These give convenient envelope curves.

**Takeaways 7.55**

- With a non-unit coefficient of  $y'$ , putting the equation into standard form is essential.
- Bounding trig terms by  $\pm 1$  is a quick way to control oscillatory solutions.

**Problem 7.56: Isoclines and Optimisation on the Unit Circle**

Consider

$$\frac{dy}{dx} = \frac{x - 2y}{x^2 + y^2}.$$

- i Find the curve on which all slopes are horizontal.
- ii Show that the isocline of slope  $k \neq 0$  is a circle.
- iii Find the maximum and minimum slopes on  $x^2 + y^2 = 1$ .

- On the unit circle, parametrize  $x = \cos \theta$ ,  $y = \sin \theta$ .
- For isoclines, set the whole fraction equal to  $k$  and complete the square.
- Set the numerator to zero for horizontal tangents.

**Hint:**

**Solution 7.56**

(i) Horizontal slopes satisfy  $x - 2y = 0$ , so

$$y = \frac{x}{2},$$

excluding the origin, where the DE is undefined.

(ii) Setting the slope equal to  $k$  gives

$$\frac{x - 2y}{x^2 + y^2} = k \Rightarrow kx^2 - x + ky^2 + 2y = 0.$$

Dividing by  $k$  and completing the square,

$$\left(x - \frac{1}{2k}\right)^2 + \left(y + \frac{1}{k}\right)^2 = \frac{5}{4k^2}.$$

So the isocline is a circle with centre

$$\left(\frac{1}{2k}, -\frac{1}{k}\right)$$

and radius

$$\frac{\sqrt{5}}{2|k|}.$$

(iii) On  $x^2 + y^2 = 1$ , the slope is  $x - 2y$ . Writing  $x = \cos \theta$  and  $y = \sin \theta$ ,

$$x - 2y = \cos \theta - 2 \sin \theta = \sqrt{5} \cos(\theta + \alpha).$$

Hence the extreme values are

$$\sqrt{5} \text{ and } -\sqrt{5}.$$

**Takeaways 7.56**

- Isoclines convert a DE question into an algebraic locus question.
- Parametrising a circle is often the cleanest route to constrained extrema.

**Problem 7.57: Logistic Growth and the Uniqueness Barrier**

Consider

$$\frac{dP}{dt} = \frac{1}{500}P(2000 - P), \quad P(0) = 500.$$

- i Explain why the solution can never reach  $P = 2500$ .
- ii Find the exact point  $(t, P)$  where the population grows fastest.

- The fastest growth occurs at the inflection point  $P = 1000$ .
- Use the equilibrium solutions and uniqueness.

**Hint:****Solution 7.57**

(i) The equilibrium solutions are  $P = 0$  and  $P = 2000$ . Since  $P = 2000$  is itself a solution, uniqueness implies that a solution starting below it cannot cross it. Because  $P(0) = 500 < 2000$ , the population can never reach 2500.

(ii) Differentiate implicitly:

$$P'' = \frac{1}{500}(2000 - 2P)P'$$

Maximum growth occurs when  $P'' = 0$  and  $P' > 0$ , so

$$P = 1000.$$

Now solve the logistic DE:

$$\int \frac{500}{P(2000 - P)} dP = \int dt.$$

This gives

$$\frac{1}{4} \ln \left( \frac{P}{2000 - P} \right) = t + C.$$

Using  $P(0) = 500$  gives

$$C = -\frac{1}{4} \ln 3.$$

At  $P = 1000$ ,

$$t = \frac{1}{4} \ln 1 + \frac{1}{4} \ln 3 = \frac{1}{4} \ln 3.$$

So the point of fastest growth is

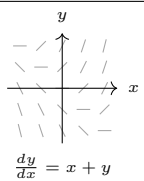
$$\left( \frac{\ln 3}{4}, 1000 \right).$$

**Takeaways 7.57**

- Equilibrium solutions can act as impenetrable barriers when uniqueness applies.
- In logistic models, the inflection point combines qualitative analysis with exact integration.

## 8 Appendices

### 8.1 Appendix A: Differential Equations Quick Reference

Concept / Model	Key Form / Formula	Key Interpretation
General first-order DE	$\frac{dy}{dx} = F(x, y)$	One unknown function; need one condition to fix the solution.
Initial condition	$y(x_0) = y_0$	Picks out one particular solution from the family.
Separable DE	$A(y) dy = B(x) dx$	Separate variables, then integrate both sides.
Equilibrium solution	$\frac{dy}{dx} = 0 \Rightarrow y = k$	Constant solution; solution curve is a horizontal line.
Slope field		Each segment shows the tangent direction of a solution curve at that point.
Exponential growth/decay	$\frac{dy}{dx} = ky \Rightarrow y = Ae^{kx}$	$k > 0$ : growth; $k < 0$ : decay. $y = 0$ is the only equilibrium.
Logistic growth	$\frac{dy}{dt} = ry \left(1 - \frac{y}{M}\right)$	Equilibria $y = 0$ (unstable) and $y = M$ (stable); $M$ = carrying capacity.
Logistic with harvesting	$\frac{dy}{dt} = ry \left(1 - \frac{y}{M}\right) - h$	High $h$ removes the positive equilibrium; population can collapse.
Newton's law of cooling	$\frac{dT}{dt} = -k(T - T_a) \Rightarrow T = T_a + Ce^{-kt}$	$T \rightarrow T_a$ as $t \rightarrow \infty$ ; $T_a$ is the only stable equilibrium.
Torricelli draining	$\frac{dh}{dt} = -k\sqrt{h} \Rightarrow 2\sqrt{h} = -kt + C$	Draining rate slows as water level $h$ falls; empties in finite time.
Particle on a curve	$\mathbf{r} = x\mathbf{i} + y\mathbf{j}, \quad \mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}, \quad \mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$	Differentiate position to get velocity; differentiate again for acceleration.

### 8.2 Appendix B: Reading and Drawing Slope Fields

Slope fields are useful for two things: **reading behavior quickly** and **sketching solutions when algebra is hard**.

#### Read a slope field (fast workflow)

1. **Mark zero-slope curves:** solve  $f(x, y) = 0$  in  $y' = f(x, y)$ . These are horizontal tangent locations.

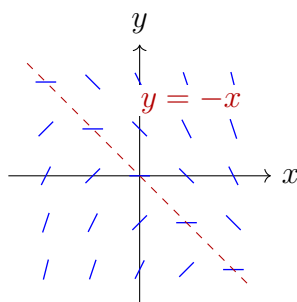
2. **Check sign regions:** where  $f(x, y) > 0$ , solutions rise; where  $f(x, y) < 0$ , solutions fall.
3. **Estimate speed of change:** larger  $|f(x, y)|$  means steeper segments and faster change.
4. **Use the initial condition:** the particular solution must pass through the given point and follow local segment directions.

**Draw a slope field (quick method)**

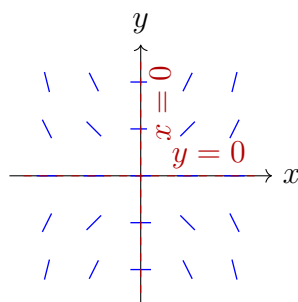
For  $y' = f(x, y)$ :

1. Choose a rectangular grid of points.
2. At each grid point  $(x_i, y_j)$ , compute  $m_{ij} = f(x_i, y_j)$ .
3. Draw a short line segment with slope  $m_{ij}$  (keep segment lengths similar).
4. Add key guide curves first (for example,  $f(x, y) = 0$  and any undefined lines).
5. Sketch solution curves as smooth paths tangent to the local segments.

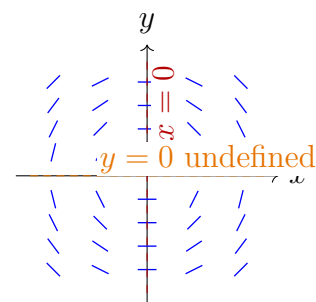
**Three compact visual examples**



**Example 1:**  $y' = -x - y$



**Example 2:**  $y' = xy$



**Example 3:**  $yy' + x = 0$

- **Example 1:**  $y = -x$  is the zero-slope isocline. Above it slopes are negative; below it slopes are positive.
- **Example 2:** the axes are zero-slope guide lines because  $xy = 0$  whenever  $x = 0$  or  $y = 0$ .
- **Example 3:** rewriting as  $y' = -\frac{x}{y}$  shows a zero-slope isocline at  $x = 0$  and an undefined line at  $y = 0$ .

**Common mistakes**

- Drawing curves that cut across local segment directions.
- Treating slope marks as disconnected graph pieces rather than tangent information.
- Giving only a sketch when the question also asks for long-run behavior in words.

### 8.3 Appendix C: Modelling Workflow

Many differential equations begin with language such as "the rate of change is proportional to..." or "the rate depends on the current amount." A clean modelling workflow helps turn those words into mathematics.

1. **Define the variable.** State what  $y$  represents and what the independent variable is. For example,  $y$  could be a population and  $x$  could be time in years.
2. **Translate the rate statement.** If the growth rate is proportional to the amount present, write

$$\frac{dy}{dx} = ky.$$

If the rate depends on both the amount present and remaining capacity, write a logistic-style equation.

3. **Solve the differential equation.** Use separation of variables or a known standard form.
4. **Apply the condition.** Use initial data such as  $y(0) = y_0$  to determine the constant.
5. **Interpret constants.** Explain what  $k$ ,  $r$ , or  $M$  mean in the situation.
6. **Check the answer against the story.** Ask whether the function stays positive when it should, whether units are sensible, and whether the long-run behavior is realistic.

#### Useful sentence patterns

- "Rate proportional to the current amount" usually leads to  $\frac{dy}{dx} = ky$ .
- "Rate proportional to the difference from a limiting value" often leads to  $\frac{dy}{dx} = k(M - y)$ .
- "Growth slows as the population approaches a maximum level" suggests a logistic model.

#### Mechanics substitution choices

When a modelling question moves into Extension 2 mechanics, the key choice is often how to rewrite acceleration.

- Use

$$a = \frac{dv}{dt}$$

when the equation is naturally expressed in terms of time.

- Use

$$a = v \frac{dv}{dx}$$

when the question connects velocity and displacement directly.

- Use

$$a = \frac{d}{dx} \left( \frac{1}{2} v^2 \right)$$

when an energy-style integral is the cleanest route.

These choices appear repeatedly in resistive motion, gravitation, and spring-motion models.

## 8.4 Appendix D: Enrichment — Vector Fields and Trajectories

*Note: This section is out of syllabus for NSW HSC Mathematics Extension 1 and Extension 2. It is included for enrichment and for students who want a first bridge to university mathematics, physics, and engineering.*

Slope fields and vector fields are closely related, but they are not the same. A slope field records only the local direction of a solution curve. A vector field records both direction and magnitude. In that sense, a slope field is a direction-only cousin of a vector field.

### Vector fields

A vector field assigns a vector to each point in the plane. In a system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y),$$

the vector

$$\begin{pmatrix} P(x, y) \\ Q(x, y) \end{pmatrix}$$

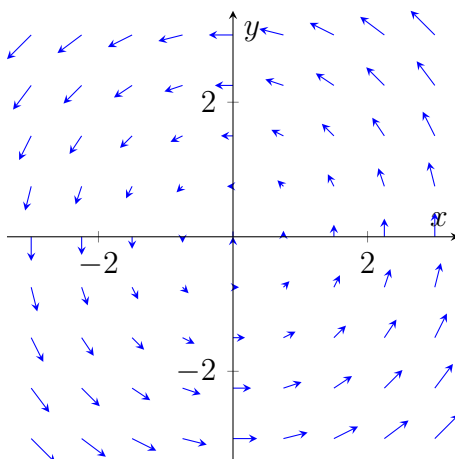
tells us the local direction and speed of motion.

A standard example is

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x.$$

At  $(1, 0)$  the vector points upward, and at  $(0, 1)$  it points left. This produces a rotational field.

Vector Field:  $\vec{v} = (-y, x)$

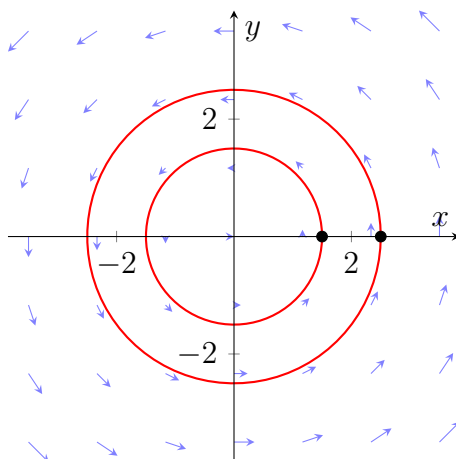


### Trajectories

A **trajectory** is the actual path followed by a particle moving according to a vector field. If the vector field gives the local velocity, the trajectory is the curve obtained by following that velocity over time.

For the rotational field above, trajectories are circles centered at the origin. The arrows tell the particle how to move at each instant; the trajectory shows the complete journey.

## Trajectories in a Rotational Field



## Why this matters

- Slope fields describe first-order scalar equations such as  $\frac{dy}{dx} = f(x, y)$ .
- Vector fields naturally describe systems of equations and particle motion.
- Trajectories turn a local rule into a global picture of motion.

## 8.5 Appendix E: Enrichment — Partial Differentiation and PDEs

*Note: This section is out of syllabus for NSW HSC Mathematics Extension 1 and Extension 2. It is included for enrichment and to give a first glimpse of higher-dimensional calculus.*

## Partial differentiation

In ordinary differentiation, a function depends on one independent variable, such as  $y = f(x)$ . In higher mathematics, functions often depend on several variables at once.

For example, a temperature function might be written as

$$T = T(x, y, z, t),$$

where temperature depends on position and time.

When differentiating with respect to one variable while treating the others as constants, we use a **partial derivative**. The notation is

$$\frac{\partial f}{\partial x}.$$

If

$$f(x, y) = x^3y^2 + 5x,$$

then

$$\frac{\partial f}{\partial x} = 3x^2y^2 + 5, \quad \frac{\partial f}{\partial y} = 2x^3y.$$

## Partial differential equations

A **partial differential equation** (PDE) relates a multivariable function to one or more of its partial derivatives.

For example, the one-dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2},$$

where  $u(x, t)$  is temperature and  $\alpha$  is a constant.

This equation says that the rate of change of temperature over time depends on how curved the temperature profile is in space.

### Why PDEs matter

PDEs are central in heat flow, fluid motion, wave propagation, electromagnetism, and quantum mechanics. They sit well beyond the HSC course, but they grow naturally from the same calculus ideas students already know.

## 8.6 Appendix F: Enrichment — Power Series Methods

*Note: This section is out of syllabus for NSW HSC Mathematics Extension 1 and Extension 2. It is included for enrichment and to show one way mathematicians handle differential equations that resist standard integration.*

### The basic idea

Sometimes an ordinary differential equation has no simple closed-form solution. One strategy is to assume the solution can be written as a power series and then determine its coefficients.

For a Maclaurin series,

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$

the differential equation itself can be used to compute successive derivatives at  $x = 0$ .

#### Example 1: Solve $y' = y$ , $y(0) = 1$

From the differential equation,

$$y'(0) = y(0) = 1.$$

Differentiating again gives  $y'' = y'$ , so  $y''(0) = 1$ , and similarly every higher derivative at 0 is also 1.

Therefore,

$$y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x.$$

#### Example 2: Solve $y' = x + y$ , $y(0) = 1$

We have

$$y'(0) = 0 + y(0) = 1.$$

Differentiate the equation:

$$y'' = 1 + y',$$

so

$$y''(0) = 2.$$

Differentiate again:

$$y''' = y'',$$

so  $y'''(0) = 2$ , and the same pattern continues.

This gives the beginning of the series

$$y(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \dots$$

which matches the exact solution,  $y(x) = 2e^x - x - 1$ , as can be verified by substitution.

### Takeaway

Power-series methods turn a differential equation into a coefficient-finding problem. They are one of the main tools used when exact antiderivatives are not enough.

## 8.7 Appendix G: Enrichment — The Characteristic Equation

*Note: This section is provided for enrichment only. It is out of syllabus for NSW HSC Mathematics Extension 1 and Extension 2, but it is an important method in higher mathematics, engineering, and physics.*

### Definitions

Consider a linear homogeneous differential equation with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0.$$

If we try a solution of the form  $y = e^{\lambda x}$ , substitution gives

$$(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0) e^{\lambda x} = 0.$$

Since  $e^{\lambda x} \neq 0$ , the polynomial factor must vanish.

- The polynomial

$$P(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0$$

is the **characteristic polynomial**.

- The equation

$$P(\lambda) = 0$$

is the **characteristic equation**.

### Example 1: First order

Solve

$$y' - 5y = 0.$$

The characteristic equation is

$$\lambda - 5 = 0,$$

so  $\lambda = 5$  and

$$y = Ce^{5x}.$$

**Example 2: Second order**

Solve

$$y'' + 3y' + 2y = 0.$$

The characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0 = (\lambda + 1)(\lambda + 2),$$

so the roots are  $-1$  and  $-2$ . Hence

$$y = Ae^{-x} + Be^{-2x}.$$

**Example 3: Third order**

Solve

$$y''' = y.$$

The characteristic equation is

$$\lambda^3 - 1 = 0 = (\lambda - 1)(\lambda^2 + \lambda + 1).$$

This gives one real root  $\lambda = 1$  and two complex roots

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

Therefore the general solution is

$$y = C_1e^x + C_2e^{-x/2} \cos\left(\frac{\sqrt{3}}{2}x\right) + C_3e^{-x/2} \sin\left(\frac{\sqrt{3}}{2}x\right).$$

**Key takeaways**

- The method converts a differential-equation problem into an algebra problem about polynomial roots.
- Distinct real roots give exponential terms.
- Complex roots produce oscillatory sine and cosine terms multiplied by exponentials.
- The number and type of roots determine the shape of the general solution.

**8.8 Appendix H: Enrichment — The Complex Harmony of DEs**

*Syllabus note: This section is provided for enrichment only. Complex-valued methods for mechanics are outside the scope of NSW HSC Mathematics Extension 1 and Extension 2, but they give a useful preview of undergraduate mathematics, physics, and engineering.*

**1. The complex exponential  $e^z$** 

In Extension 2, the exponential function is first met for real inputs. If we extend the exponent to a complex number  $z = x + iy$ , Euler's formula gives

$$e^{iy} = \cos y + i \sin y.$$

Hence

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

This is the bridge between oscillation and exponential form: sine-cosine motion can be packaged into one algebraically convenient object.

## 2. Homogeneous differential equations and imaginary roots

Consider the simple harmonic motion equation

$$\frac{d^2y}{dx^2} + n^2y = 0.$$

Its characteristic equation is

$$\lambda^2 + n^2 = 0,$$

so the roots are  $\lambda = \pm ni$ . The complex-form solution is

$$y = C_1 e^{inx} + C_2 e^{-inx}.$$

Applying Euler's formula turns this back into the familiar real form

$$y = A \cos(nx) + B \sin(nx).$$

So the HSC trigonometric answer is really the real-valued version of a compact exponential calculation.

## 3. Streamlining 2D resistive motion

In projectile motion with linear resistance  $R = -mkv$ , the standard Extension 2 approach solves

$$\ddot{x} = -k\dot{x}, \quad \ddot{y} = -g - k\dot{y}$$

as two separate equations. If we instead define the complex displacement

$$z = x + iy,$$

then the complex velocity is  $\dot{z} = \dot{x} + i\dot{y}$  and

$$\ddot{z} = \ddot{x} + i\ddot{y} = -k(\dot{x} + i\dot{y}) - ig.$$

Therefore

$$\ddot{z} = -k\dot{z} - ig.$$

This combines the whole 2D system into one linear equation for the complex velocity.

### Case study: the elegance of the complex plane

**Traditional split:**

$$\dot{x}(t) = u \cos \theta e^{-kt}, \quad \dot{y}(t) = \left( u \sin \theta + \frac{g}{k} \right) e^{-kt} - \frac{g}{k}.$$

**Complex method:** Let  $W = \dot{z}$ . Then

$$\dot{W} + kW = -ig.$$

Using the integrating factor  $e^{kt}$ ,

$$\frac{d}{dt}(W e^{kt}) = -ig e^{kt}$$

so

$$W(t) = C e^{-kt} - \frac{ig}{k}.$$

Since

$$W(0) = u(\cos \theta + i \sin \theta) = ue^{i\theta},$$

we get

$$C = ue^{i\theta} + \frac{ig}{k}.$$

Hence the full complex velocity is

$$W(t) = \left( ue^{i\theta} + \frac{ig}{k} \right) e^{-kt} - \frac{ig}{k}.$$

Taking the real and imaginary parts recovers the horizontal and vertical velocity components at once.

### Why this matters

- Complex numbers are not just abstract algebra; they can compress a two-equation mechanics problem into one line.
- Imaginary roots in differential equations are the reason sine and cosine appear in oscillation models.
- This style of thinking becomes standard in university differential equations, control, waves, and electrical engineering.

## 9 Conclusion

Differential equations are one of the clearest examples of how calculus becomes a modelling language. The topic asks students to do more than differentiate and integrate: it asks them to connect rate, graph, formula, and interpretation in a single chain of reasoning. That combination is exactly why the topic is so useful later in Extension 2, mechanics, engineering, and university mathematics.

From what I have seen in HSC Math Extension 1 exams in recent years, the most asked questions on this topic are: (i) understanding slope fields, (ii) solving separable equations, and (iii) interpreting logistic growth models with difficulty levels ranging from basic to medium in this booklet. The more advanced problems in this booklet are designed to stretch your understanding and problem-solving skills, and they may help you prepare for Mechanics problems in Extension 2 or university courses.

As you work through this booklet, aim to build two habits together: solve carefully, and explain what the solution means. Strong HSC answers usually do both. Once the problem bank is filled in, this booklet will become a natural place to practice that full cycle repeatedly.

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