Multivariable Calculus: A Summary

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Updated June 11, 2022

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Chapter 0

Background & Review

0.1 Algebra and Pre-Calculus

0.1.1 Sets

Definition. A set A is a collection of distinct elements. Those elements can be anything, like numbers, functions, and even other sets.

We can define a set by giving its elements, like $A = \{-2, 5, 3\}$ or by describing its properties, like $A = \{x \mid x > 0\}$ where the vertical bar means "such that". If an object x is a member of the set A, we write $x \in A$.

A set A is called a subset of a set B if every element of A is also an element of B. We can write this as $A \subseteq B$. For example, $\{7, 10, 16\} \subseteq \{5, 6, 7, 9, 10, 11, 16\}$. Note that this relation can be strict if there exists at least one element in B that is not also an element of A. Some common sets and their informal definitions are given below:

Set Name	Symbol	Informal Definition
Natural numbers	$\mathbb{N}$	$\{1, 2, 3, \dots\}$
Integers	$\mathbb{Z}$	$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
Rational numbers	$\mathbb{Q}$	$\{\frac{m}{n} \mid m, n \in \mathbb{Z}$ and $n \neq 0\}$
Real numbers	$\mathbb{R}$	Any number on the number line[1]

This means that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

There are several common operations that can be performed on sets. The union $A \cup B$ of two sets A and B is the set of all elements that are elements of A or of B. Similarly, the intersection $A \cap B$ of two sets A and B is the set of all elements that are also elements of both A and B.

Example. If $A = \{\sqrt{2}, 2, 5, 8\}$ and $B = \{-9, 8, 2.3\}$, what are $A \cup B$ and $A \cap B$?

To find the union, we combine the sets, making sure to include any repeated element only once:

$$A \cup B = \{-9, \sqrt{2}, 2, 2.3, 5, 8\}.$$

Then, since the only element both sets share is 8, we also have

$$A \cap B = \{8\}.$$

0.1.2 Intervals

Definition. We call a subset I of $\mathbb{R}$ an interval if, for any $a, b \in I$ and $x \in \mathbb{R}$ such that $a \leq x \leq b$, then $x \in I$.

We can write an interval more simply using the notation $[a, b]$, which is equivalent to $\{x \in \mathbb{R} \mid a \leq x \leq b\}$. This is called a closed interval, and to make the inequalities strict, we can also define an open interval by using parantheses instead of square brackets.

In addition, we can mix the two to create half-open intervals, where one inequality is strict and the other isn't. For instance, $(2, 5]$ refers to the set $\{x \in \mathbb{R} \mid x < 2 \leq 5\}$ Finally, if the interval is unbounded in either direction, we use the notations $-\infty$ and ∞ to indicate that there is no minimum or maximum, respectively.

Example. Is $8 \in (-\infty, 4) \cup [8, 100)$?

Since $8 \leq 8 < 100$ is a true statement, $8 \in [8, 100)$. Since we are taking the union with another set, all of the members of the right interval will also be members of the union of intervals. Therefore, the statement is true.

0.1.3 Functions

Definition. A function f is a rule between a pair of sets, denoted $f : D \to C$, that assigns values from the first set, the domain D, to the second set, the codomain C.

We call the subset of the codomain C that constitutes all values f can actually attain the range $R \subseteq C$. Note that when we draw a graph of a function, all we are doing is drawing all ordered pairs $\{(x, f(x)) \mid x \in D\}$.

Example. Find the domain of the following function:

$$f(x) = \frac{1}{(1-x)\sqrt{5-x^2}}$$

We know that $\frac{n}{0}$ is undefined for all $n \in \mathbb{R}$ and $\sqrt{x}$ is only defined for $x \geq 0$. The first condition applies to the first term in the denominator and both conditions apply to the second, giving us

$$(1-x) \neq 0 \text{ and } 5 - x^2 > 0$$

The first condition implies $x \neq 1$ while the second implies $|x| < \sqrt{5}$. Putting these together, we find that the domain is

$$\{x \mid x \neq 1, |x| < \sqrt{5}\} \text{ or } (-\sqrt{5}, 1) \cup (1, \sqrt{5})$$

We can also compose two functions, such that the ouput of one function is the input of another:

$$(f \circ g)(x) = f(g(x)).$$

Definition. A function g is called an inverse function of f if $f(g(x)) = x$ for all x in the domain of g and $g(f(x))$ for all x in the domain of f. We write this as $g = f^{-1}$.

One common algorithm for finding an inverse function is to set $y = f(x)$, substitute all x's for y's, and then solve for y.

Example. Find the inverse function of

$$f(x) = \frac{5x+2}{4x-3}.$$

We first make the substitutions to set up the algorithm:

$$y = \frac{5x+2}{4x-3} \text{ becomes } x = \frac{5y+2}{4y-3}.$$

After multiplying both sides by the denominator and simplifying, we have

$$\begin{aligned} 4xy - 3x &= 5y + 2 \\ -3x - 2 &= y(-4x + 5) \\ y = f^{-1}(x) &= \frac{3x+2}{4x-5}. \end{aligned}$$

We say that a function f is even if it satisfies $f(-x) = f(x)$ for all $x \in D$. Likewise, we say that a function f is odd if it satisfies $f(-x) = -f(x)$ for all $x \in D$. Geometrically, we can see that the graph of an even function is symmetric with respect to the y-axis, while the graph of an odd function is symmetric with respect to the origin.

Example. Is $f(x) = 2x - x^2$ even, odd, or neither?

$$f(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, the function is neither even nor odd.

0.1.4 Complex Numbers

Definition. i is called the imaginary unit. It's defined by $i^2 = -1$.

The set of complex numbers ($\mathbb{C}$) is an extension of the real numbers. Complex numbers have the form $z = \alpha + \beta i$, where α and β are real numbers. The α part of z is called the real part, so $\Re(z) = \alpha$. The β part of z is called the imaginary part, so $\Im(z) = \beta i$.

Often, complex numbers are visualized as points or vectors in a 2D plane, called the complex plane, where α is the x-component, and β is the y-component. Thinking of complex numbers like points helps us define the magnitude of complex numbers and compare them. Since a point (x, y) has a distance $\sqrt{x^2+y^2}$ from the origin, we can say the magnitude of z, $|z|$ is $\sqrt{\alpha^2+\beta^2}$. Thinking of complex numbers like vectors helps us understand adding two complex numbers, since you just add the components like vectors.

A common operation on complex numbers is the complex conjugate. The complex conjugate of $z = \alpha + \beta i$ is $\overline{z} = \alpha - \beta i$. z and $\overline{z}$ are called a conjugate pair.

Conjugate pairs have the following properties. Let $z, w \in \mathbb{C}$.

$$\begin{aligned}
\overline{z \pm w} &= \overline{z} \pm \overline{w} \\
\overline{zw} &= \overline{z}\overline{w} \\
\overline{z} &= z \Leftrightarrow z \in \mathbb{R} \\
z\overline{z} &= |z|^2 = |\overline{z}|^2 \\
\overline{\overline{z}} &= z \\
\overline{z}^n &= \overline{z^n} \\
z^{-1} &= \frac{\overline{z}}{|z|^2}
\end{aligned}$$

0.1.5 Factoring Polynomials

We want to break up a polynomial like $f(x) = a_0 + a_1x^1 + \ldots a_nx^n$ into linear factors so that $f(x) = c(x - b_1) \cdot \ldots \cdot (x - b_n)$. This form makes it simple to see that the roots of f, solutions to $f(x) = 0$, are $x = b_1 \ldots b_n$.

For quadratics, $f(x) = ax^2 + bx + c$, there exists a simple formula that will give us both roots, the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We can see that when $b^2 - 4ac < 0$, like for $f(x) = x^2 + 5x + 10$, we will get complex roots $\alpha \pm \beta i$. For any polynomial, these roots come in pairs, so if $\alpha + \beta i$ is a root, then so is $\alpha - \beta i$. This means that every conjugate pair $\alpha \pm \beta i$ has a quadratic equation with those roots. Sometimes we will not factor quadratics with complex roots into linear terms.

Although there do exist explicit formulas for finding roots for cubic (degree 3) and quartic (degree 4) equations, they are too long and not useful enough to memorize. When working by hand, we instead use other tricks to find roots.

There are a few useful tricks that can help. If the polynomial doesn't have a constant term, then 0 is a root. If all the coefficients sum to 0, then 1 is a root. For certain polynomials with an even number of terms, like all cubics of the form $ax^3 + bx^2 + cax + cb$ we can factor out a term from the first two and last two terms to get $x^2(ax + b) + c(ax + b) = (ax + b)(x^2 + c)$. For other polynomials, we might just try guessing and checking values. However, we need a more efficient way that works in general.

Since we are looking to find linear factors $f(x) = (x - b_1) \cdot \ldots \cdot (x - b_n)$, we can see that the constant term in the polynomial is the product of the roots $b_1 \ldots b_n$. In fact, since the coefficients of polynomials are completely determined by the roots and the leading coefficient, all the coefficients are sums and products of roots. You might remember when factoring quadratics that the coefficient of x term is the sum of the two roots. These rules are called Vieta's formulas.

So, if we have the constant term, we can check all of its integer factors to see if any are roots. For each root, we can divide, using a technique like synthetic division, to continue finding the rest of the roots. This method is especially useful on tests because the roots tend to be integers.

Example. Factor the polynomial $x^5 + x^4 - 2x^3 + 4x^2 - 24x$.

We can immediately see that there is no constant term, so $x = 0$ is a root. Now we need to work on factoring $x^4 + x^3 - 2x^2 + 4x - 24$.
The factors of -24 are: -24, -12, -8, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 8, 12, and 24. Starting from roots close to 0 and working outwards, we find that $x = 2$ is a root. So, we synthetic divide like so

$$\begin{array}{r|ccccl} x=2 & 1 & 1 & -2 & 4 & -24 \\ & \downarrow & 2 & 6 & 8 & 24 \\ \hline & 1 & 3 & 4 & 12 & \mid 0 \end{array}$$

to see that now we need to work on factoring $x^3 + 3x^2 + 4x + 12$. $x^3 + 3x^2 + 4x + 12 = x^2(x + 3) + 4(x + 3) = (x + 3)(x^2 + 4)$, so $x = -3$ is a root, and we need to work on factoring $x^2 + 4$. $x^2 + 4$ has two complex roots $\pm 2i$, so we'll leave it as a quadratic.

$$x^5 + x^4 - 2x^3 + 4x^2 - 24x = x(x - 2)(x - 3)(x^2 + 4)$$

0.1.6 Trig Functions & The Unit Circle

Imagine a circle of radius 1 centered at the origin that we'll call the unit circle. The x and y coordinates of a point on the unit circle are completely determined by the angle θ in radians between the x-axis and a line from the origin to the point.

The function $\cos\theta$ tells us x-coordinate of the point, while $\sin\theta$ tells us the y-coordinate of the point. The function $\tan\theta = \frac{\sin\theta}{\cos\theta}$ tells us the slope of the line from the origin to the point. Most of the trig functions have geometric interpretations as shown below. The most used ones are sin, cos, $\tan = \frac{\sin}{\cos}$, $\cot = \frac{\cos}{\sin}$, $\csc = \frac{1}{\sin}$, and $\sec = \frac{1}{\cos}$.

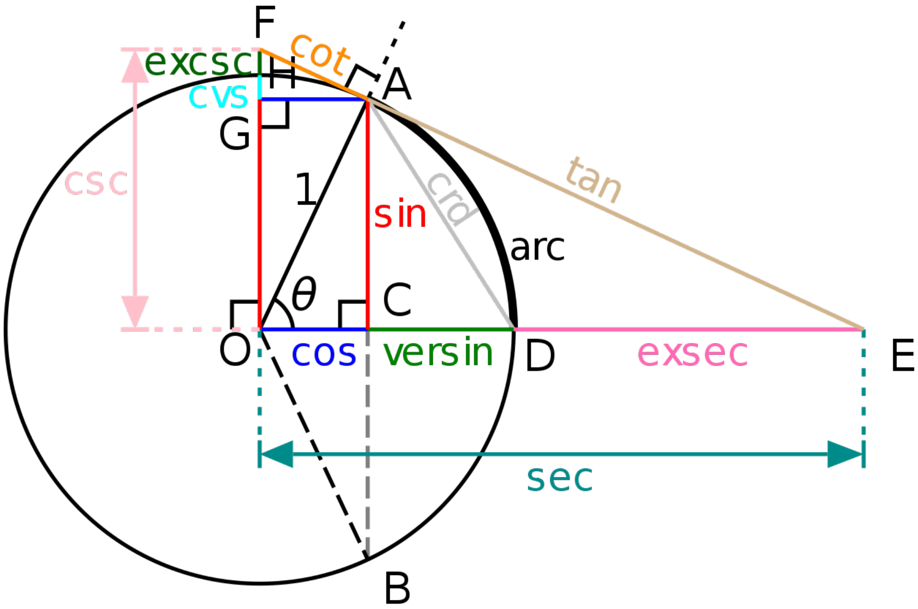


Figure 1: Wikipedia - Unit circle

We can also think about the inverses of these trig functions. These are either notated with a -1 exponent on the function, or the prefix arc in front of the function name. Many of these functions are only defined on a part of the domain $[0, 2\pi]$. Below is a table of the inverse trig functions and their domains.

Function	Domain
arcsin	$[-1, 1]$
arccos	$[-1, 1]$
arctan	$(-\infty, \infty)$
arccot	$(-\infty, \infty)$
arccsc	$(-\infty, -1] \cup [1, \infty)$
arcsec	$(-\infty, -1] \cup [1, \infty)$

0.1.7 Trig Identities

As we could see in Figure 0.1.6, sin and cos form a right triangle with hypotenuse 1. So, using the Pythagorean Theorem,

$$\sin^2\theta + \cos^2\theta = 1.$$

By dividing by $\sin^2$ or $\cos^2$, we can also get

$$1 + \cot^2\theta = \csc^2\theta \text{ and } \tan^2\theta + 1 = \sec^2\theta.$$

Together, these 3 identities are called the Pythagorean Identities.

We can also relate functions and co-functions.

$$\text{xxx}(\theta) = \text{coxxx}\left(\frac{\pi}{2} - \theta\right).$$

Some of the most useful and used identities are the sum and difference.

$$\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\tan(\alpha \pm \beta) &= \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \\
\sin\alpha \pm \sin\beta &= 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \\
\cos\alpha + \cos\beta &= 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \\
\cos\alpha - \cos\beta &= -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)
\end{aligned}$$

0.1.8 Exponentials & Logarithms

Definition. e is the base of the natural logarithm. It's defined by the limit

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

$\exp x = e^x$ and $\ln x$ are inverse functions of each other such that

$$e^{\ln x} = x \text{ and } \ln e^x = x.$$

Just like other exponentials, the normal rules for adding, subtracting, and multiplying exponents apply:

$$e^x e^y = e^{x+y},\ \frac{e^x}{e^y} = e^{x-y}, \text{ and } (e^x)^k = e^{xk}.$$

Similar rules apply for logarithms:

$$\ln x + \ln y = \ln xy,\ \ln x - \ln y = \ln\left(\frac{x}{y}\right), \text{ and } \ln\left(a^b\right) = b\ln a.$$

We can also write a logarithm of any base using natural logarithms:

$$\log_b a = \frac{\ln a}{\ln b}.$$

e is also unique in that it is the only real number a satisfying the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x,$$

meaning e^x is its own derivative.

0.1.9 Partial Fractions

If we have a function of two polynomials $f(x) = \frac{P(x)}{Q(x)}$, it's often easier to break this quotient into a sum of parts where the denominator is a linear or quadratic factor and the numerator is always a smaller degree than the denominator.

Example.

$$\frac{2x-1}{x^3-6x^2+11x-6} = \frac{1/2}{x-1} + \frac{-3}{x-2} + \frac{5/2}{x-3}.$$

One natural way to find these small denominators comes from the linear factors of the denominator where we keep quadratics with complex roots. This way, when making a common denominator, we get back the original big denominator. However, there are a few special cases we have to take care of.

Linear Factors

This is the the most basic type where the degree of the numerator is less than the degree of the denominator and the denominator factors into all linear factors with no repeated roots. In this case we can write

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \ldots + \frac{A_n}{(x-a_n)}.$$

Multiplying each side by $Q(x)$,

$$P(x) = A_1(x-a_2)\ldots(x-a_n) + \ldots + A_n(x-a_1)\ldots(x-a_{n-1}).$$

We can then find each A_i by evaluating both sides at $x = a_i$, since every term except the ith has an $(x - a_i)$ factor that will go to 0. So,

$$A_i = \frac{P(a_i)}{(x-a_i)\ldots(x-a_{i-1})(x-a_{i+1})\ldots(x-a_n)}.$$

Example. Find the partial fraction decomposition of the following expression:

$$\frac{2x-1}{x^3-6x^2+11x-6}.$$

Factoring,

$$x^3-6x^2+11x-6 = (x-1)(x-2)(x-3).$$

So,

$$\frac{2x-1}{x^3-6x^2+11x-6} = \frac{A_1}{x-1} + \frac{A_2}{x-2} + \frac{A_3}{x-3}.$$

Multiplying each side by the denominator,

$$2x-1 = A_1(x-2)(x-3) + A_2(x-1)(x-3) + A_3(x-1)(x-2).$$

At $x = 1$,

$$1 = A_1(1-2)(1-3) \implies A_1 = \frac{1}{2}.$$

At $x = 2$,

$$3 = A_2(2-1)(2-3) \implies A_2 = -3.$$

At $x = 3$,

$$5 = A_3(3-1)(3-2) \implies A_3 = \frac{5}{2}.$$

So,

$$\frac{2x-1}{x^3-6x^2+11x-6} = \frac{1/2}{x-1} + \frac{-3}{x-2} + \frac{5/2}{x-3},$$

just as was shown in the previous example.

Repeated Linear Factors

If $Q(x)$ has repeated roots, it factors into

$$Q(x) = R(x)(x-a)^k,\ k \geq 2 \text{ and } R(a) \neq 0.$$

When making the common denominator for each repeated root of multiplicity k, we do

$$\frac{P(x)}{R(x)(x-a)^k} = (\text{Decomposition of } R(x)) + \frac{A_1}{x-a} + \ldots + \frac{A_k}{(x-a)^k}.$$

You would then multiply each side by the denominator like in the linear factors case and solve for the coefficients. The only additional difficulty is that you might have to use previous results or solve a system of linear equations to get some of the constants.

Example. Find the partial fraction of the following expression:

$$\frac{x^2+5x-6}{x^3-7x^2+16x-12}.$$

Factoring,

$$x^3 - 7x^2 + 16x - 12 = (x-3)(x-2)^2.$$

So,

$$\frac{x^2+5x-6}{x^3-7x^2+16x-12} = \frac{A_1}{x-3} + \frac{A_2}{x-2} + \frac{A_3}{(x-2)^2}.$$

Multiplying each side by the denominator,

$$x^2 + 5x - 6 = A_1(x-2)^2 + A_2(x-2)(x-3) + A_3(x-3).$$

At $x = 2$,

$$8 = A_3(2-3) \implies A_3 = -8.$$

At $x = 3$,
$$18 = A_1(3-2)^2 \implies A_1 = 18.$$
Now we'll use our results for A_1 and A_3 to find A_2 using a value for x that isn't 2 or 3 so the A_2 term doesn't become 0. A good choice is $x = 0$.
At $x = 0$,
$$-6 = 18(0-2)^2 + A_2(0-2)(0-3) + -8(0-3) \implies A_2 = -17.$$
So,
$$\frac{x^2+5x-6}{x^3-7x^2+16x-12} = \frac{18}{x-3} - \frac{17}{x-2} - \frac{8}{(x-2)^2}.$$

Quadratic Factors

If a quadratic doesn't have real roots, then we have a quadratic factor. Here, we'll assume that the quadratic factor isn't repeated. So, $Q(x) = R(x)(ax^2+bx+c)$, $b^2 - 4ac < 0$, and $R(x)$ is not evenly divisible by ax^2+bx+c. In this case, we say
$$\frac{P(x)}{R(x)(ax^2+bx+c)} = (\text{Decomposition of } R(x)) + \frac{A_1x+B_1}{ax^2+bx+c}.$$
We then solve for the constants in the numerator, possibly having to solve a system of equations or using previous results and less convenient values for x.

Example. Find the partial fraction decomposition of the following expression:
$$\frac{6x^2+21x+11}{x^3+5x^2+3x+15}.$$
Factoring,
$$x^2+5x^2+3x+15 = (x+5)(x^2+3).$$
So,
$$\frac{6x^2+21x+11}{x^3+5x^2+3x+15} = \frac{A_1}{x+5} + \frac{A_2x+B_2}{x^2+3}.$$
Multiplying each side by the denominator,
$$6x^2+21x+11 = A_1(x^2+3) + (A_2x+B_2)(x+5).$$
At $x = -5$,
$$56 = 28A_1 \implies A_1 = 2.$$
Now we'll use the previous result and another value for x. We can use $x = 0$ to not have to worry about the A_2 term. At $x = 0$,
$$11 = 2(3) + (B_2)(5) \implies B_2 = 1.$$

Now we'll use the previous 2 results to find A_2. $x = 1$ is a good choice to keep the numbers small. At $x = 1$,

$$38 = 2(1+3) + (A_2+1)(6) \implies A_2 = 4.$$

So,

$$\frac{6x^2+21x+11}{x^3+5x^2+3x+15} = \frac{2}{x+5} + \frac{4x+1}{x^2+3}.$$

Repeated Quadratic Factors

If a quadratic factor that can't be broken into linear factors is repeated, then we can write $Q(x) = R(x)(ax^2+bx+c)^k$, $k \geq 0$, and $R(x)$ is not divisible by $(ax^2+bx+c)^k$. Now we have to do a combination of what we did for repeated linear factors and quadratic factors. We say

$$\frac{P(x)}{R(x)(ax^2+bx+c)^k} = (\text{Decomposition of } R(x)) + \frac{A_1x+B_1}{ax^2+bx+c} + \ldots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}.$$

We then solve for the coefficients in the numerator.

Example. Find the partial fraction decomposition of $\frac{3x^4-2x^3+6x^2-3x+3}{x^5+3x^4+4x^3+12x^2+4x+12}$.

Factoring,

$$x^5+3x^4+4x^3+12x^2+4x+12 = (x+3)(x^2+2)^2.$$

So,

$$\frac{3x^4-2x^3+6x^2-3x+3}{x^5+3x^4+4x^3+12x^2+4x+12} = \frac{A_1}{x+3} + \frac{A_2x+B_2}{x^2+2} + \frac{A_3x+B_3}{(x^2+2)^2}.$$

Multiplying each side by the denominator,

$$3x^4-2x^3+6x^2-3x+3 = A_1(x^2+2)^2 + (A_2x+B_2)(x^2+2)(x+3) + (A_3x+B_3)(x+3).$$

At $x = -3$,

$$363 = 121A_1 \implies A_1 = 3.$$

Now, we'll use our result for A_1 and pick a value for x that minimizes the number of things we need to solve for. We'll have to solve a linear system with 4 unknowns, so we'll need up to 4 values. At $x = 0$,

$$3 = 3(2)^2 + B_2(2)(3) + B_3(3) \implies 2B_2 + B_3 = -3.$$

At $x = 1$,

$$7 = 3(3)^2 + (A_2+B_2)(3)(4) + (A_3+B_3)(4) \implies 3A_2 + A_3 + 3B_2 + B_3 = -5.$$

At $x = -1$,

$$17 = 3(3)^2 + (-A_2+B_2)(3)(2) + (-A_3+B_3)(2) \implies -3A_2 - A_3 + 3B_2 + B_3 = -5.$$

At $x = 2$,

$$53 = 3(6)^2 + (2A_2 + B_2)(6)(5) + (2A_3 + B_3)(5) \implies 12A_2 + 2A_3 + 6B_2 + B_3 = -11.$$

Now we have the following system of equations:

$$\begin{cases} 0A_2 + 0A_3 + 2B_2 + B_3 & = -3 \\ 3A_2 + A_3 + 3B_2 + B_3 & = -5 \\ -3A_2 - A_3 + 3B_2 + B_3 & = -5 \\ 12A_2 + 2A_3 + 6B_2 + B_3 & = -11 \end{cases}.$$

Solving,

$$A_2 = 0, A_3 = 0, B_2 = -2, \text{ and } B_3 = 1.$$

So,

$$\frac{3x^4 - 2x^3 + 6x^2 - 3x + 3}{x^5 + 3x^4 + 4x^3 + 12x^2 + 4x + 12} = \frac{3}{x+3} - \frac{2}{x^2+2} + \frac{1}{(x^2+2)^2}.$$

Improper Fractions

If the degree of the numerator is greater than or equal to the degree of the denominator, we have a case of improper fractions. In this case, we have to do polynomial long division to get a quotient and remainder and then decompose the remainder if necessary. So,

$$\frac{P(x)}{Q(x)} = R(x) + \frac{S(x)}{Q(x)}.$$

Example. Find the partial fraction decomposition of the following expression:

$$\frac{x^3 + 3}{x^2 - 2x - 3}.$$

First we do polynomial long division to find that

$$\frac{x^3 + 3}{x^2 - 2x - 3} = x + 2 + \frac{7x + 9}{x^2 - 2x - 3}.$$

Now that the numerator is of a lesser degree than the denominator, we can decompose it normally.

$$x^2 - 2x - 3 = (x - 3)(x + 1).$$

So,

$$\frac{7x + 9}{x^2 - 2x - 3} = \frac{A_1}{x - 3} + \frac{A_2}{x + 1}.$$

Multiplying each side by the denominator,

$$7x + 9 = A_1(x + 1) + A_2(x - 3).$$

At $x = -1$,

$$2 = -4A_2 \implies A_2 = \frac{-1}{2}.$$

At $x = 3$,

$$30 = 4A_1 \implies A_1 = \frac{15}{2}.$$

So,

$$\frac{x^3 + 3}{x^2 - 2x - 3} = x + 2 + \frac{15/2}{x-3} + \frac{-1/2}{x+1}.$$

0.2 Single Variable Calculus

0.2.1 Derivatives and Integrals

Derivatives

The derivative of a function $y = f(x)$, notated $f'(x)$, gives the slope of the tangent line to f at x.

Definition.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Below are some properties of the derivative. Let f and g be functions of x and p a scalar.

Linearity

$$(pf \pm g)' = pf' \pm g'$$

Product Rule

$$(fg)' = f'g + fg'$$

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Chain Rule

$$(f \circ g)' = (f' \circ g) \cdot g'$$

Power Rule

$$\frac{\mathrm{d}x^x}{\mathrm{d}p} = px^{p-1}, p \neq 0$$

Exponent Rule

$$\frac{\mathrm{d}x^x}{\mathrm{d}p} = p^x \ln p, p > 0$$

The Power Rule and Exponent Rule are two cases of the same rule

$$\frac{\mathrm{d}}{\mathrm{d}x} f^g = g f^{g-1} f' + f^g \ln f g'.$$

Using the definition of the derivative and these rules, we can find the derivatives to some common functions.

$$\begin{array}{r|l}
\frac{\mathrm{d}}{\mathrm{d}x} p = 0 & \frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x \\
\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x} & \frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x \\
\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x & \frac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 x
\end{array}$$

Integrals

The definite integral of a function $f(x)$ from $x = a$ to $x = b$ where $a \leq b$ is the area between $f(x)$ and the x-axis bounded by the lines $x = a$ and $x = b$ where area above the x-axis is positive, and area below the x-axis is negative.

Definition.

$$\int_a^b f(x)\mathrm{d}x = \lim_{h \to 0} \sum_{n=1}^{\frac{b-a}{h}} f(a + (n-1)h) \cdot h.$$

We also define an indefinite integral, or antiderivative of $f(x)$, notated $F(x)$ where

$$F'(x) = f(x) \implies \int f(x)\mathrm{d}x = F(x).$$

Note that there are infinitely many such functions F, since adding a constant to F does not affect its derivative. To notate this, we add a constant C to the indefinite integral. Given an initial condition for f, we can solve for C.

Below are some properties of the integral. Let f and g be functions of x and p, a, b, and c where $a < b < c$, and f and g are continuous on the closed interval $[a, c]$.

Linearity

$$\int (pf \pm g)\mathrm{d}x = p \int f\mathrm{d}x \pm \int g\mathrm{d}x$$

Flipped Bounds

$$\int_a^b f\mathrm{d}x = -\int_b^a f\mathrm{d}x$$

Union of Intervals

$$\int_a^b f\mathrm{d}x + \int_b^c f\mathrm{d}x = \int_a^c f\mathrm{d}x$$

Power Rule

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

U-Substitution

$$\int (f' \circ g)\, g' \mathrm{d}x = f \circ g + C$$

Integration by Parts

$$\int f' g \mathrm{d}x = fg - \int f g' \mathrm{d}x$$

Fundamental Theorem of Calculus

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(s)\mathrm{d}s = f(x)$$

Using the definition of the integral and the above rules, we can find the indefinite integral of some common functions.

$$\begin{aligned}
\int \frac{1}{x}\mathrm{d}x &= \ln \left|x\right| + C \\
\int \sin x \mathrm{d}x &= -\cos x + C \\
\int \cos x \mathrm{d}x &= \sin x + C \\
\int \tan x \mathrm{d}x &= -\ln \left|\cos x\right| + C
\end{aligned}$$

0.2.2 Taylor Series

A Taylor series as a way of approximating a function about a point $x = a$ using polynomials. The first approximation just keeps the same value at $x = a$, the second approximation keeps the same value and first derivative at $x = a$, etc.

Definition.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

If we approximate a function about $x = 0$, we call this a Maclaurin series. Below are some common Maclaurin series, and their radii of convergence if applicable.

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\
\frac{1}{1+x} &= 1 - x + x^2 - \dots, \text{ where } \left|x\right| < 1 \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots, \text{ where } \left|x\right| < 1
\end{aligned}
$$

Euler's Identity

Let's see what happens when we look at the Maclaurin series for e^{ix}.

$$
\begin{aligned}
e^{ix} &= 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} \dots \\
&= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \dots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right).
\end{aligned}
$$

The two expressions in parenthesis are exactly the Maclaurin series for $\cos x$ and $\sin x$. So,

$$e^{ix} = \cos x + i \sin x.$$

In the case that $x = \pi$,

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0.$$

So,

$$e^{i\pi} + 1 = 0.$$

0.3 Vectors and Matrices

0.3.1 Vectors

A vector is a quantity with both direction and magnitude. One can think of it as a directed line segment. In multivariable calculus, we mostly will work with vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, but vectors can exist in other dimensions.

Numerical (scalar) quantities have vector analogues, many of which show up in physics. Speed becomes velocity, distance becomes displacement, and mass becomes weight.

Say we have a 2D vector, $\vec{v} = \langle v_x, v_y \rangle$.

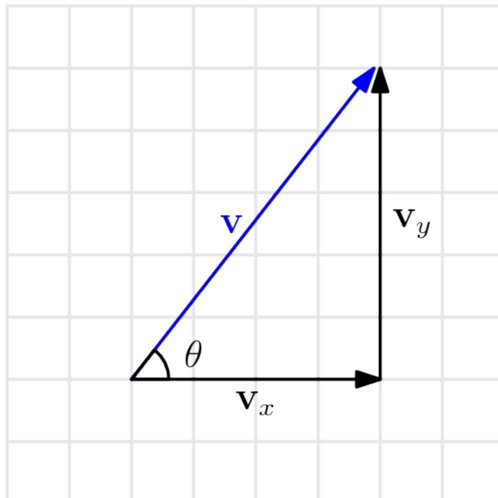


Figure 2: The x and y components of a vector v

Its length, also called magnitude or norm, is notated $\left|\left|\vec{v}\right|\right| = \sqrt{v_x^2 + v_y^2}$. This pattern of the norm being equal to the square-root of the sum of the squares of the vector's components continues into higher dimensions.

The angle a 2D vector forms with the horizontal axis is $\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$. There is not a useful version of this formula in higher dimensions. Using θ and $\left|\left|\vec{v}\right|\right|$, we can see that $v_x = \left|\left|\vec{v}\right|\right| \cos\theta$ and $v_y = \left|\left|\vec{v}\right|\right| \sin\theta$.

Vectors can be added and subtracted from each other in a way that the result is another vector. We do this numerically by adding the corresponding components of each vector. For example, if $\vec{a} = \langle 1, 3 \rangle$ and $\vec{b} = \langle 4, 7 \rangle$, then $\vec{a} + \vec{b} = \langle 1 + 4, 3 + 7 \rangle = \langle 5, 10 \rangle$ and $\vec{b} - \vec{a} = \langle 4 - 1, 7 - 3, \rangle = \langle 3, 4 \rangle$.
Visually, you can think of $\vec{v} + \vec{w}$ as the vector connecting the tail of $\vec{v}$ with the tip of $\vec{w}$ where the tail of $\vec{v}$ is on the tip of $\vec{w}$.

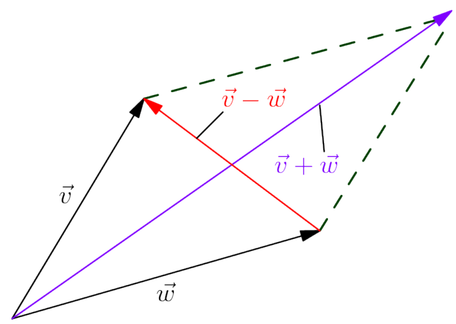


Figure 3: Visualization of $\vec{v} + \vec{w}$ and $\vec{v} - \vec{w}$

We can also multiply vectors by scalars and get another vector as a result. We do this by multiplying each component of the vector by the scalar. This has the effect of stretching or shrinking the vector and possibly changing the vector's direction if the scalar is negative.

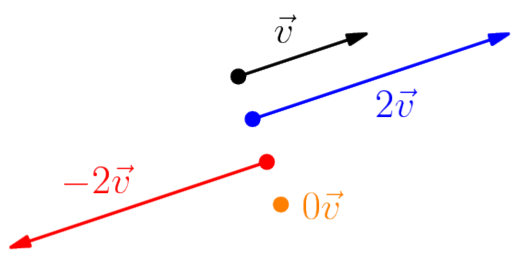


Figure 4: A vector $\vec{v}$ scaled by different constants

A unit vector is any vector with magnitude 1. Rather than using an arrow like for other vectors, unit vectors are notated with a carat ($\wedge$) over top, like $\hat{\imath}$, which is read as "i hat". We can transform any vector with non-zero magnitude into a unit vector by dividing the vector by its norm. This normalized vector will point in the same direction as the original vector.

It is common in mathematics for $\hat{\imath} = \langle 1, 0, 0 \rangle$ to be the unit vector in the x-direction, $\hat{\jmath} = \langle 0, 1, 0 \rangle$ to be the unit vector in the y-direction, and $\hat{k} = \langle 0, 0, 1 \rangle$ to be the unit vector in the z-direction. Together, $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are called the standard basis vectors because all other vectors in $\mathbb{R}^3$ can be written as linear combination of these.

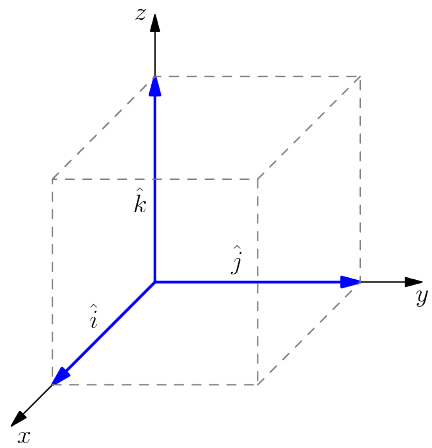


Figure 5: The standard basis vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$

0.3.2 Dot Products

A dot product is a way of multiplying two vectors so that the result is a scalar. $\vec{a} \cdot \vec{b} = ||\vec{a}||||\vec{b}|| \cos\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$. One way to think of the dot product is as a measure of how much two vectors point in the same direction. We can also show using the law of cosines that $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + ... + a_nb_n$. Knowing the lengths of two vectors and their dot product we can calculate the angle between them as

$$\theta = \arccos\left(\frac{\vec{a} \cdot \vec{b}}{||\vec{a}||||\vec{b}||}\right).$$

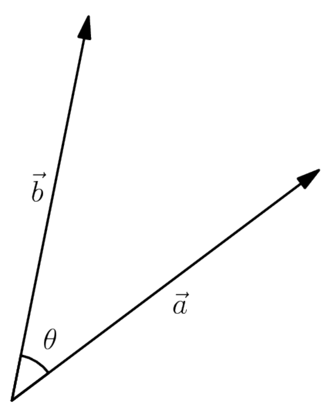


Figure 6: Two vectors and the angle between them

Although similar to scalar multiplication, dot products have some properties that set them apart.

Commutative

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

the same as scalar multiplication.

Distributive

$$\vec{a} \cdot \left(\vec{b} + \vec{c}\right) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

the same as scalar multiplication.

NOT Associative $\left(\vec{a} \cdot \vec{b}\right) \cdot \vec{c}$ is a nonsense expression. However, like scalar multiplication, dot products are scalar associative.

$$(c \cdot \vec{a}) \cdot \vec{b} = \vec{a} \cdot \left(c \cdot \vec{b}\right)$$

0.3.3 Cross Products

A cross product is a way of multiplying two vectors so that the result is a vector. Although the cross product technically only works for 3D vectors, we will first look a a "fake" 2D version to build an intuition.

$$\vec{a} \times \vec{b} = a_1 b_1 - a_2 b_2.$$

This "fake" 2D cross product gives the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = ||\vec{a}||||\vec{b}|| \sin\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$. Another way to think of the magnitude of the cross product, both in 2D and 3D, is as a measure of how perpendicular two vectors are.

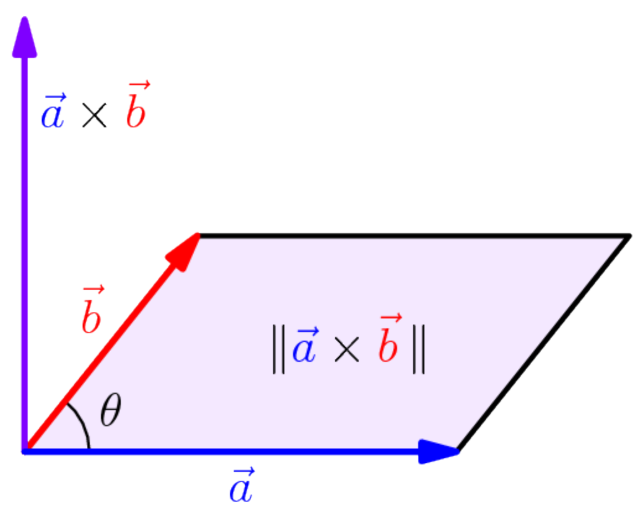


Figure 7: Visualization of the cross product

In 3D, $\vec{a} \times \vec{b}$ is a vector, and similar to the 2D case, the magnitude of $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - b_2 a_3, a_3 b_1 - b_3 a_1, a_1 b_2 - b_1 a_2 \rangle$$

and

$$||\vec{a} \times \vec{b}|| = ||\vec{a}||||\vec{b}|| \sin\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$. Each component of $\vec{a} \times \vec{b}$ gives the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$ in some plane: The x-component of $\vec{a} \times \vec{b}$ gives the area in the yz-plane (x = 0 plane). $\vec{a} \times \vec{b}$ is perpendicular, also called "normal," to the plane containing $\vec{a}$ and $\vec{b}$. It's direction, is determined by the right hand rule.

This cross product table of the standard basis vectors is useful for providing some insight into the properties of the cross product.

$\overrightarrow{\text{row}} \times \overrightarrow{\text{col}}$	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	0	$\hat{k}$	$-\hat{j}$
$\hat{j}$	$-\hat{k}$	0	$\hat{i}$
$\hat{k}$	$\hat{j}$	$-\hat{i}$	0

<u>NOT</u> Commutative, but is antisymmetric

$$\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$$

Scalar Associative

$$(c \cdot \vec{a}) \times \vec{b} = \vec{a} \times \left(c \cdot \vec{b}\right)$$

Distributive

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

One can also think of the cross product as the determinant of a matrix.

$$\vec{a} \times \vec{b} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_2 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Now that we have defined the dot product and cross product, we can put the two together as the scalar triple product, which gives the volume of the parallelepiped spanned by $\vec{a}$, $\vec{b}$, and $\vec{c}$.

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \det \begin{bmatrix} a_1 & a_2 & a_2 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

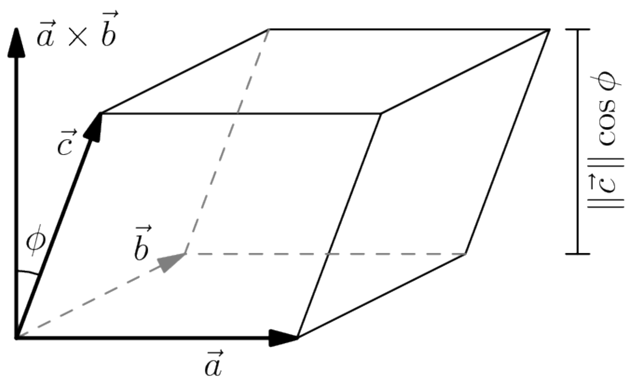


Figure 8: Scalar triple product gives volume of parallelepiped spanned by three vectors.

0.3.4 Matrices

Matrices are an array of mathematical objects, most often numbers. They are often used to represent linear transformations between two spaces and systems of linear equations. We denote the size of a matrix by saying the number of rows followed by the number of columns.

Example. Below is a 2 x 4 matrix.

$$\begin{bmatrix} 1 & 3 & 2 & -1 \\ -5 & 7 & 3 & 0 \end{bmatrix}$$

0.3.5 Types of Matrices

Below is a list of different types of matrices and their special properties.

- A square matrix has the same number of rows as columns.

$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -1 \\ 2 & 7 & 9 \end{bmatrix}$$

- Row vectors have one column. Column vectors have one row.

$$\begin{bmatrix} 1 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

- Upper triangular matrices have all 0's below the main diagonal. Lower triangular matrices have all 0's above the main diagonal.

$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -1 \\ 0 & 0 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 7 & 9 \end{bmatrix}$$

- Diagonal matrices are both upper and lower triangular. They only have non-zero entries on the main diagonal.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

- The identity matrix is one of the most common matrices. It is square, diagonal, and has all 1's on the main diagonal. It's the multiplicative identity for matrices.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The inverse matrix of A, A^{-1}, is such that

$$A^{-1}A = AA^{-1} = I.$$

- The transpose matrix of A, A^T, is where the rows and columns of A are swapped.

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ -5 & 7 & 3 & 0 \end{bmatrix} \implies A^T = \begin{bmatrix} 1 & -5 \\ 3 & 7 \\ 2 & 3 \\ -1 & 0 \end{bmatrix}$$

0.3.6 Row Reduction

Row reduction is a way of solving a system of linear equations by representing the system as a matrix and altering the rows of the matrix until we get as close as possible to an identity matrix.

Below is a list of legal row operations. Doing these does not change the solution to the system of equations.

- Multiplying or dividing each item in a row by a scalar,

$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -1 \\ 2 & 7 & 9 \end{bmatrix} \overset{R_2=R_2/2}{\rightarrow} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 1 & -1/2 \\ 2 & 7 & 9 \end{bmatrix}.$$

- Adding a multiple of one row to another row,

$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -1 \\ 2 & 7 & 9 \end{bmatrix} \overset{R_3=R_3-2R_1}{\rightarrow} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -1 \\ 0 & 1 & -5 \end{bmatrix}.$$

- Swapping two rows,

$$\begin{bmatrix}1&3&7\\0&2&-1\\0&1&-5\end{bmatrix} \overset{\text{swap } R_2,\, R_3}{\rightarrow} \begin{bmatrix}1&3&7\\0&1&-5\\0&2&-1\end{bmatrix}.$$

Using these rules, we solve a system of linear equations using a process called Gauss-Jordan Elimination.

A system may have a contradiction, meaning no solution exists. This will look like a row of 0's on the left and a non-zero term on the far right of the row.

$$\left[\begin{array}{cc|c}1&0&1\\0&1&0\end{array}\right] \implies \text{No solution}$$

A system may be underdetermined, meaning one or more variables can be any number. This will look a non-zero column on the left without a leading 1 (bolded).

$$\left[\begin{array}{ccccc|c}1&\text{-2}&0&0&\text{-3}&2\\0&\mathbf{0}&1&0&\mathbf{1}&5\\0&\mathbf{0}&0&1&\mathbf{2}&4\\0&\mathbf{0}&0&0&\mathbf{0}&0\end{array}\right] \implies \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}2\\0\\5\\4\\0\end{bmatrix} + \alpha\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix} + \beta\begin{bmatrix}3\\0\\-4\\-2\\1\end{bmatrix}, \alpha, \beta \in \mathbb{R}.$$

Example. Solve the following linear system of equations using row reduction.

$$\begin{bmatrix}1&3&0\\0&2&5\\2&7&7\end{bmatrix}\vec{x} = \begin{bmatrix}4\\13\\16\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&3&7&0\\0&2&-1&5\\2&7&9&7\end{array}\right] \overset{R_3=R_3-2R_1}{\rightarrow} \left[\begin{array}{ccc|c}1&3&7&0\\0&2&-1&5\\0&1&-5&7\end{array}\right] \overset{\text{swap } R_2,\, R_3}{\rightarrow} \left[\begin{array}{ccc|c}1&3&7&0\\0&1&-5&7\\0&2&-1&5\end{array}\right]$$

$$\overset{R_3=R_3-2R_2}{\rightarrow} \left[\begin{array}{ccc|c}1&3&7&0\\0&1&-5&7\\0&0&9&-9\end{array}\right] \overset{R_3=R_3/9}{\rightarrow} \left[\begin{array}{ccc|c}1&3&7&0\\0&1&-5&7\\0&0&1&-1\end{array}\right] \overset{R_2=R_2+5R_3}{\rightarrow} \left[\begin{array}{ccc|c}1&3&7&0\\0&1&0&2\\0&0&1&-1\end{array}\right]$$

$$\overset{R_1=R_1-7R_3}{\rightarrow} \left[\begin{array}{ccc|c}1&3&0&7\\0&1&0&2\\0&0&1&-1\end{array}\right] \overset{R_1=R_1-2R_2}{\rightarrow} \left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&2\\0&0&1&-1\end{array}\right].$$

So,

$$\vec{x} = \begin{bmatrix}1\\2\\-1\end{bmatrix}.$$

0.3.7 Determinants

The determinant of a matrix is a signed number that tells by how much the transformation represented by a matrix scales volumes in a space. The number is negative if the space was "flipped" during a transformation. The number is zero if the dimension of the output space is less than that of the input space.

The determinant is only defined for square matrices. It's easiest to understand the definition of a determinant recursively.

$$\det [a] = |a| = a$$
$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

We can define a_{ij} as the entry in the ith row and jth column of matrix A and A_{ij} as the adjudicate matrix, which is the matrix A if row i and column j were removed. This allows us to write a general formula for the determinant.

Definition.

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij} \text{ (for fixed i)} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} A_{ij} \text{ (for fixed j)}$$

This formula allows us to use any row or column to calculate the determinant, which is especially useful if a certain row contains lots of 0's.

Below are some properties of the determinant for some $n \times n$ matrix A and scalar λ.

$$\det I_n = 1$$
$$\det (A^T) = \det A$$
$$\text{If } A \text{ is invertible, } \det (A^{-1}) = \frac{1}{\det A}$$
$$\det (\lambda A) = \lambda^n \det A$$
$$\det (AB) = \det A \det B$$
$$\text{If } A \text{ is triangular, } \det A = \prod_{i=1}^{n} a_{ii}$$

Example. Find the determinant of the following 3 x 3 matrix.

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -1 \\ 2 & 7 & 9 \end{bmatrix}$$

We'll use the first column since it has only two non-zero entries.

$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -1 \\ 2 & 7 & 9 \end{bmatrix} = 1 \begin{vmatrix} 2 & -1 \\ 7 & 9 \end{vmatrix} + 2 \begin{vmatrix} 3 & 7 \\ 2 & -1 \end{vmatrix} = (18 + 7) + 2(-3 - 14) = -9.$$

0.3.8 Eigenvalues & Eigenvectors

Definition. Let A be an $n \times n$ matrix. A scalar λ and a vector $\vec{v}$ are an eigenvalue and eigenvector of A if

$$A\vec{v} = \lambda\vec{v}.$$

We call $p(\lambda) = \det(A - \lambda I)$ the characteristic polynomial of A. The eigenvalues for A are the solutions to the equation

$$p(\lambda) = \det(A - \lambda I) = 0.$$

Once we have an eigenvalue, we can find the basis vectors for the corresponding eigenspace by solving the equation

$$(A - \lambda I)\,\vec{v} = \vec{0}.$$

The basis vectors of the eigenspace for A are the union of the basis vectors of each eigenspace corresponding to each eigenvalue.

Example. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix}.$$

$$p(\lambda) = \begin{vmatrix} 2-\lambda & 1 & 3 \\ 1 & 2-\lambda & 3 \\ 3 & 3 & 20-\lambda \end{vmatrix} = -(\lambda - 21)(\lambda - 2)(\lambda - 1) = 0 \implies \lambda = 1, 2, \text{ and } 21.$$

When $\lambda = 1$,

$$A - \lambda I = \left[\begin{array}{ccc|c} 2-1 & 1 & 3 & 0 \\ 1 & 2-1 & 3 & 0 \\ 3 & 3 & 20-1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{v_1} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

When $\lambda = 2$,

$$A - \lambda I = \left[\begin{array}{ccc|c} 2-2 & 1 & 3 & 0 \\ 1 & 2-2 & 3 & 0 \\ 3 & 3 & 20-2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{v_2} = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}.$$

When $\lambda = 21$,

$$A - \lambda I = \left[\begin{array}{ccc|c} 2-21 & 1 & 3 & 0 \\ 1 & 2-21 & 3 & 0 \\ 3 & 3 & 20-21 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1/6 & 0 \\ 0 & 1 & -1/6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \vec{v_{21}} = \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}.$$

Bonus: A's diagonalization is

$$A = PDP^{-1} \implies \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 1 \\ 1 & -3 & 1 \\ 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 21 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 & 0 \\ -3/19 & -3/19 & 1/19 \\ 1/38 & 1/38 & 3/19 \end{bmatrix}.$$

0.4 Important Shapes in 2D & 3D

In 2D space ($\mathbb{R}^2$), the most common and simple shapes you will encounter are lines and circles.

- Lines have a form like $x = 1$ or $y = 2x + 1$.
- Circles have a form like $(x - 1)^2 + (y - 2)^2 = 9$.

In 3D space (R^3), these 2D shapes have higher dimensional versions: planes, cylinders, and spheres.

- Planes have forms like $x = 1$, $y = 2x + 1$, and $z = 2x + 3y + 5$.
  Note the similarity to the equation for lines. One can think of a plane as a line extruded in a direction.
- Cylinders have forms like $(x - 4)^2 + (y - 5)^2 = 36$.
  Note the similarity to the equation for circles. One can think of a cylinder as a circle extruded away from it's face.
- Spheres have forms like $(x - 7)^2 + (y - 8)^2 + (z - 9)^2 = 100$.
  Note the similarity to the equations for circles. One can think of a sphere as a circle rotated and extruded about its center.

Chapter 1

Vector-Valued Functions (VVFs)

1.1 Lines & Planes as VVFs

VVFs are parametric equations that take one input value and return one or more output values as a vector. We can draw curves in space by defining the tail of the output of the VVF be at the origin and have the tip trace out the curve. We will look at some simple VVFs that you need to be able to recognize.

1.1.1 Lines

A straight line is probably the simplest 3D VVF. We can form any straight line using a point that the line passes through and the direction vector of the line. Letting P be the point and $\vec{v}$ be the direction, a straight line has the form $\vec{r}(t) = \vec{P} + t\vec{v}$, where $\vec{P}$ is the vector with components the same as P. The function's output starts at $\vec{P}$ when $t = 0$ and moves in the direction of $\vec{v}$ as t increases.

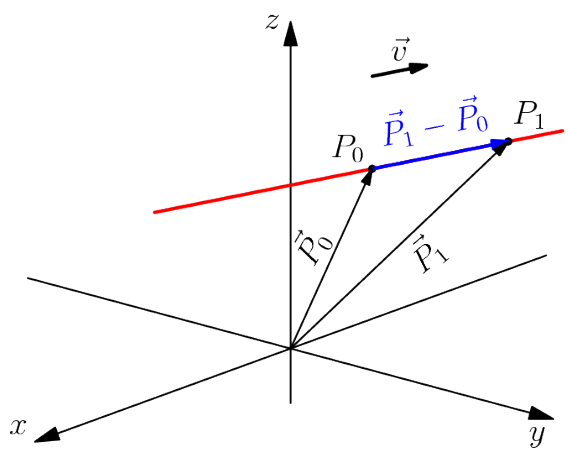


Figure 1.1: A line can be formed by a single point and a direction vector or two points.

A line can also be formed using two points. To find the equation of the line in this case, let $\vec{v}$ be the vector connecting the two points, and let $\vec{P}$ be a vector point from the origin to one of the two points. We now have an origin point and direction and can write our function as

$$\vec{r}(t) = \vec{P_0} + t\left(\vec{P_1} - \vec{P_0}\right),$$

where P_0 and P_1 are the two points on the line.

1.1.2 Planes

A plane can also be formed using a point in the plane, P, and a vector perpendicular to the plane, $\vec{n}$. All vectors $\langle x, y, z\rangle$ that originate from P and remain in the plane must be perpendicular to $\vec{n}$, so their dot product with $\vec{n}$ would be 0. So, the point-normal form of a plane is[1]

$$\vec{n} \cdot \left(\langle x, y, z\rangle - \vec{P}\right) = 0.$$

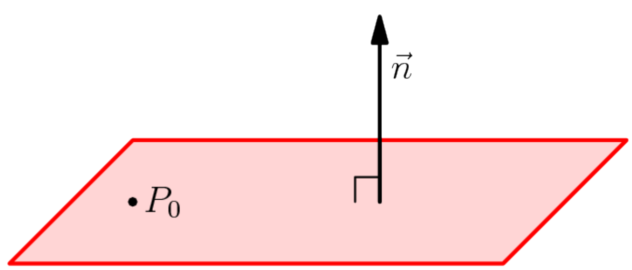


Figure 1.2: A point P_0 and a normal vector $\vec{n}$ define a plane.

One can also construct a plane from 3 non-collinear points in the plane. One can still take advantage of point-normal form here by choosing 1 point to be P_0 and drawing vectors from this point to the two other points. The cross product of these two vectors is $\vec{n}$.

$$\left(\left(\vec{P_1} - \vec{P_0}\right) \times \left(\vec{P_2} - \vec{P_0}\right)\right) \cdot \left(\langle x, y, z\rangle - \vec{P_0}\right) = 0,$$

where P_0, P_1, and P_2 are the three points in the plane.

One can also construct a plane from a point in the plane, P_0, and a line in the plane, $\vec{r}(t) = \vec{P_1} + t\vec{v}$, that doesn't pass through P_0. One can get this setup into point-normal form by choosing a an output of $\vec{r}(t)$, like $\vec{P_1}$, and constructing a vector that points from $\vec{P_1}$ to $\vec{P_0}$, $\vec{P_1} - \vec{P_0}$, and crossing this with $\vec{v}$ to find $\vec{n}$.

$$\left(\vec{v} \times \left(\vec{P_1} - \vec{P_0}\right)\right) \cdot \left(\langle x, y, z\rangle - \vec{P_0}\right) = 0$$

[1] Conventionally, $\vec{n}$ is a unit vector, $\hat{n}$.

One can also construct a plane from two intersecting lines, $\vec{r_1}(t) = \vec{P_0} + t\vec{v_1}$ and $\vec{r_2}(t) = \vec{P_0} + t\vec{v_2}$, where P_0 is where the two lines intersect. One can cross $\vec{v_1}$ with $\vec{v_2}$ to get the normal vector.

$$(\vec{v_1} \times \vec{v_2}) \cdot \left(\langle x, y, z\rangle - \vec{P_0}\right) = 0$$

1.2 Common Vector-Valued Functions

1.2.1 Circles

You should already recognize $x^2 + y^2 = r^2$ as the equation of a circle with radius r centered at the origin. A circle as a VVF in $\mathbb{R}^2$ is $\vec{r}(t) = \langle r\cos t, r\sin t\rangle$, which is identical to the parametric form of a circle. In $\mathbb{R}^3$, the z-component is some constant that tells us which plane, $z = c$, the circle is in. We can also have circles parallel to $x = 0$ and $y = 0$ planes by changing the positions of the sin, cos, and c terms. For example, $\vec{r}(t) = \langle \cos t, c, \sin t\rangle$ is a circle in the $y = c$ plane.

1.2.2 Helices

A helix looks like a spring and appears to look like a circle when viewed from the top looking down. It has the form $\vec{r}(t) = \langle r\cos t, r\sin t, ct\rangle$ where $a \in \mathbb{R}$. a defines the "tightness" between consecutive windings.

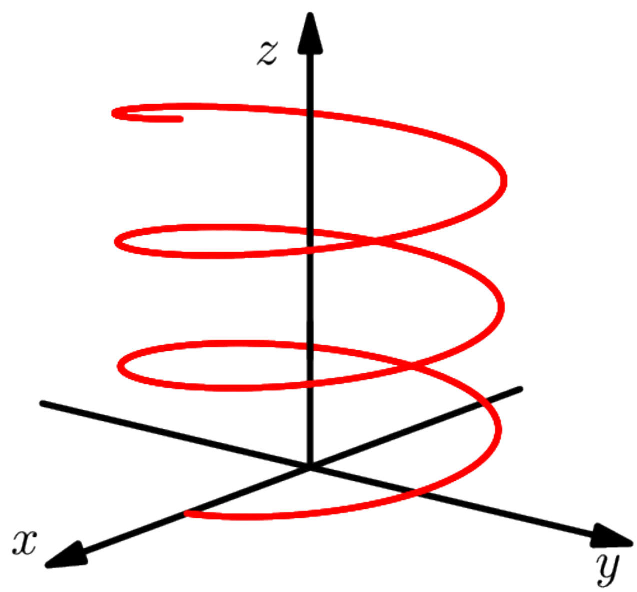


Figure 1.3: A helix

Since VVFs are essentially multiple single-input single-output functions packaged together, the domain of a VVF is the domain on which all components are defined.

For example, if $\vec{r}(t) = \langle \tan t, 6t, \ln(16 - t^2) \rangle$,

- $\tan t$ is defined for all real numbers not equivalent to $\pm\pi/2$ radians.
- $6t$ is defined for all real numbers.
- $\ln(16 - t^2)$ is defined for $t \in (-4, 4)$.

The intersection of these domains is $(-4, -\pi/2) \cup (-\pi/2, \pi/2) \cup (\pi/2, 4)$, which is the domain of $\vec{r}(t)$.

1.3 Derivatives of VVFs

Just like functions from Calc I and II, we can differentiate VVFs. In fact, the limit definitions of the derivative are nearly identical. Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.

$$\begin{aligned} \vec{r'}(t) &= \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \\ &= \lim_{h \to 0} \left\langle \frac{x(t+h) - x(t)}{h}, \frac{y(t+h) - y(t)}{h}, \frac{z(t+h) - z(t)}{h} \right\rangle. \end{aligned}$$

The limit distributes inside the vector, so

$$\vec{r'}(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

Like a position function from Calc I and II, the derivative of a VVF representing position gives a VVF representing velocity, and the 2nd derivative gives a VVF representing acceleration. The magnitude of the velocity VVF, the speed, is commonly notated $v(t)$.

There are 5 important properties of the derivatives of VVFs. These properties are similar to single-variable derivatives. Let $\vec{r}(t)$ and $\vec{s}(t)$ be VVFs, $a(t)$ be a scalar function, and c be a scalar.

Linearity

$$\frac{\mathrm{d}}{\mathrm{d}t} c\vec{r}(t) = c\vec{r'}(t)$$

Product Rule for Scalar Functions

$$\frac{\mathrm{d}}{\mathrm{d}t} a(t)\vec{r}(t) = a(t)\vec{r'}(t) + \vec{r}(t)a'(t)$$

Dot Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{s}(t) \cdot \vec{r}(t) = \vec{s}(t) \cdot \vec{r'}(t) + \vec{r}(t)\vec{s'}(t)$$

Cross Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{s}(t) \times \vec{r}(t) = \vec{s}(t) \times \vec{r'}(t) + \vec{s'}(t) \times \vec{r}(t)$$

Chain Rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{r}(a(t)) = \vec{r'}(a(t))a'(t)$$

A quotient rule doesn't make sense because we don't have an operation for dividing two vectors by each other.

Just like in single variable calculus, we can use the derivative of VVFs to find tangent lines to the curve. Similar to how $f'(a)$ represents the slope of f at a, $\vec{r'}(a)$ represents the direction of the tangent line at a. Remembering the VVF form of a line, the tangent line to $\vec{r}$ at t is

$$\vec{l}(t) = \vec{r}(t) + t\vec{r'}(t).$$

In fact, tangent lines appear so often, that we have a special unit vector representing the direction of the tangent line.

$$\hat{T}(t) = \frac{\vec{r'}(t)}{||\vec{r'}(t)||}.$$

You can remember $\hat{T}$ as the "tangent" vector.

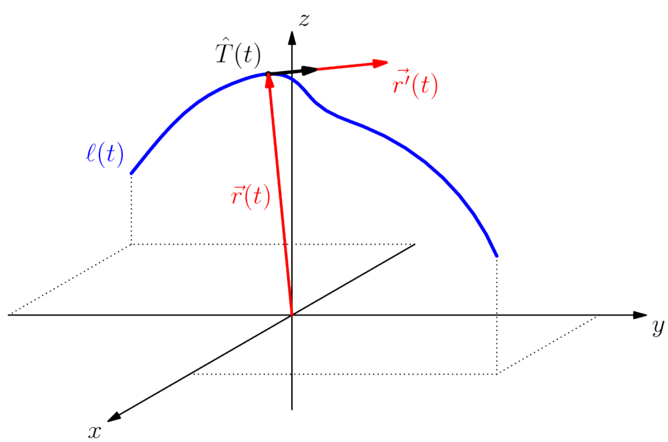


Figure 1.4: $\hat{T}$ is $\vec{r'}$ normalized.

1.4 Integrals of VVFs

VVFs can also be integrated. The integration operation also distributes inside the vector, for both definite and indefinite integrals.

$$\int \vec{r}(t)\mathrm{d}t = \left\langle \int x(t)\mathrm{d}t, \int y(t)\mathrm{d}t, \int z(t)\mathrm{d}t \right\rangle.$$

For indefinite integrals, the result will have a vector of constants will be added.

1.5 Reparameterization & Arc Length

VVFs can be reparameterized to trace out the same curve at different speeds by replacing t in $\vec{r}(t)$ with any non-decreasing function of t. This fact can come in handy to make the bounds of an integration problem more convenient.

The integral of the derivative of a VVF gives the displacement vector because

$$\int_a^b \vec{r'}(t)\mathrm{d}t = \vec{r}(b) - \vec{r}(a).$$

This is exactly like how veclvity $\cdot$ time $=$ displacment.
If we integrate the magnitude of $\vec{r'}(t)$, we can use the fact that distance $=$ speed $\cdot$ time to find the arc length of $\vec{r}(t)$ as

$$s = \int \left|\left|\vec{r'}(t)\right|\right|\mathrm{d}t = \int \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\mathrm{d}t.$$

We can also write this as an arclength function,

$$s(t) = \int_0^t \left|\left|\vec{r'}(\tau)\right|\right|\mathrm{d}\tau.$$

If we have a function

$$f(t) = s(t) = \int_0^t \left|\left|\vec{r'}(\tau)\right|\right|\mathrm{d}\tau,$$

where s is strictly increasing, then f has an inverse by the horizontal line test. That is, $t(s) = f^{-1}(s)$ exists and is also non-decreasing. If we reparameterize $\vec{r}(t)$ to $\vec{r}(t(s))$, which is called the arc length parameterization, the parameterization will have a constant speed.

1.6 TNB Frame & Osculating Plane/Circle

1.6.1 T-Hat $\left(\hat{T}\right)$

Arc length parameterization gives us another way to find $\hat{T}$.

$$\hat{T} = \frac{\vec{r'}(t)}{\left|\left|\vec{r'}(t)\right|\right|} = \frac{\mathrm{d}\vec{r}/\mathrm{d}t}{\mathrm{d}s/\mathrm{d}t} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}s}$$

1.6.2 Curvature

Curvature is 1 divided by the radius of the circle that best approximates the curve at a point. Tighter turns have smaller radii and higher curvature. We can use $\hat{T}$ to find the curvature at a point on $\vec{r}(t)$.

$$\kappa(t) = ||\frac{\mathrm{d}\hat{T}}{\mathrm{d}s}|| = ||\frac{\mathrm{d}\hat{T}}{\mathrm{d}t}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^{-1}|| = ||\frac{\mathrm{d}\hat{T}}{\mathrm{d}t}||\frac{1}{v(t)}$$

For example, let's find $\kappa(t)$ for the circle in the yz-plane: $\vec{r}(t) = \langle 7, R\sin t, R\cos t\rangle$.

$$\begin{aligned}
\vec{r'}(t) &= \langle 0, R\cos t, -R\sin t\rangle \text{ and } v(t) = \sqrt{0^2 + (R\cos t)^2 + (-R\sin t)^2} = R \\
\hat{T}(t) &= \frac{1}{R}\vec{r}(t) = \langle 0, \cos t, -\sin t\rangle \\
\frac{\mathrm{d}\hat{T}}{\mathrm{d}t} &= \langle 0, \sin t, -\cos t\rangle \\
||\frac{\mathrm{d}\hat{T}}{\mathrm{d}t}|| &= 1 \\
\kappa(t) &= \frac{1}{R}.
\end{aligned}$$

This relationship is true for all circles.

1.6.3 N-Hat $\left(\hat{N}\right)$

Starting with the definition of $\hat{T}(t)$,

$$\begin{aligned}
\hat{T}(t) &= \frac{\vec{r'}(t)}{v(t)} \\
\vec{r'}(t) &= v(t)\hat{T}(t) \\
\vec{r''}(t) &= v(t)\hat{T}'(t) + \hat{T}(t)v'(t).
\end{aligned}$$

We will show that $\hat{T}(t) \perp \hat{T}'(t)$.

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{T}(t)\cdot\hat{T}(t)\right) &= \hat{T}\cdot\hat{T}'(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{T}\cdot\hat{T}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}1 = 0.
\end{aligned}$$

So,

$$\hat{T}\cdot\hat{T}'(t) = 0 \implies \hat{T}(t) \perp \hat{T}'(t)$$

$$\hat{N}(t) = \frac{\hat{T}'(t)}{||\hat{T}'(t)||} \perp \hat{T}.$$

$\hat{N}$ is a unit vector perpendicular to $\hat{T}$ that points in the direction that the curve curls into. It is called the normal vector because it is perpendicular to the curve. It is in the same plane as $\vec{r}'$, $\hat{T}$, and $\vec{r}''$.

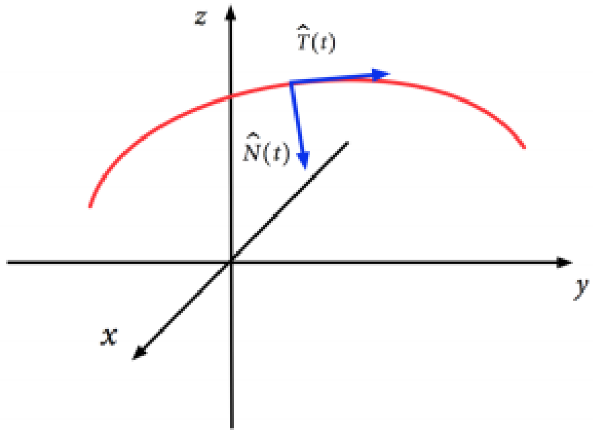


Figure 1.5: $\hat{T}$ and $\hat{N}$

$\hat{N}$ allows us to rewrite $\vec{r}''$.

$$\vec{r}''(t) = \frac{\mathrm{d}v}{\mathrm{d}t}\hat{T}(t) + v^2(t)\kappa(t)\hat{N}(t)$$

We can see that $\vec{r}''(t)$ has two parts. If $\vec{r}(t)$ represents position, then $\frac{\mathrm{d}v}{\mathrm{d}t}$ represents linear acceleration and $v^2(t)\kappa(t)$ represents centripetal acceleration. You might recognize the formula for centripetal acceleration in the 2nd part from physics. If we let $R(t) = \frac{1}{\kappa(t)}$, then the 2nd part becomes $\frac{v^2(t)}{R(t)}$, which looks exactly like the formula for centripetal acceleration for uniform circular motion: $a_c = \frac{v^2}{r}$.

1.6.4 Osculating Plane/Circle & B-Hat $\left(\hat{B}\right)$

The osculating plane in the plane containing $\vec{r}(t)$, $\hat{T}$ and $\hat{N}$. It is only defined when $\hat{N} \neq 0$. This means straight lines do not have an osculating plane. The osculating circle lives in the osculating plane, is centered at $\vec{r}(t) + \frac{\hat{N}(t)}{\kappa(t)}$, and has radius $\frac{1}{\kappa(t)}$. The tangent line at $\vec{r}(t)$ is also tangent to the osculating circle because both points have the same curvature.

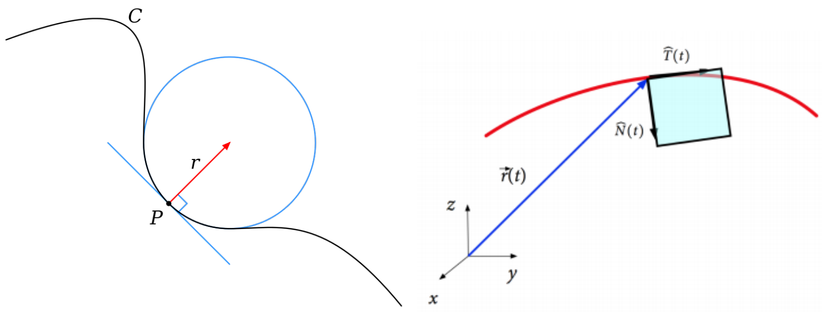


Figure 1.6: Osculating circle and plane

The vector that is normal to the plane is

$$\hat{B}(t) = \hat{T} \times \hat{N},$$

which is called the "binormal" vector because it is perpendicular to both $\hat{T}$ and $\hat{N}$. Together, $\hat{T}$, $\hat{N}$, and $\hat{B}$ form the Frenet Serret Frame, also called the TNB frame.

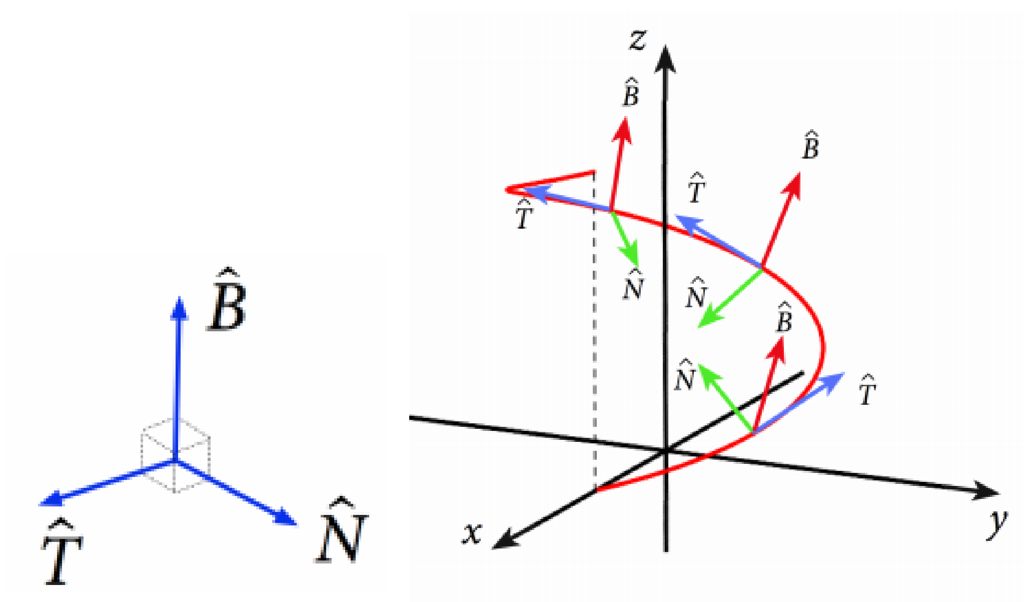


Figure 1.7: TNB frame

We can write the equation for the osculating plane as

$$\hat{B}(t) \cdot (\langle x, y, z \rangle - \vec{r}(t)) = 0.$$

Chapter 2

Differential Multivariable Calculus

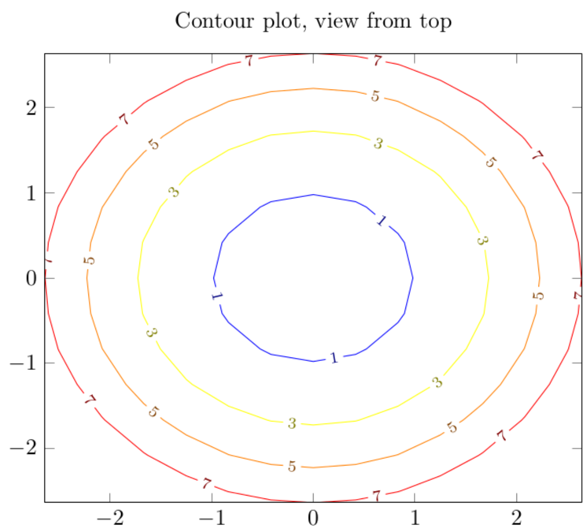
2.1 Multivariable Functions

Multivariable functions take several values as an input and return a single value as an output. For example, $z = x^2 + y^2$ takes $\mathbb{R}^2 \to$ R. Although we can only graph and fully visualize up to $\mathbb{R}^2 \to \mathbb{R}$ as a surface, we can imagine a multivariable function with 3 inputs ($\mathbb{R}^3 \to \mathbb{R}$) as a heatmap in 3D space. However, most of the mathematics we will discuss applies to functions with any number of inputs.

The domain of a multivariable function $f : \mathbb{R}^n \to \mathbb{R}$ is the largest set of points on which f is defined. For example, if $f(x, y) = \ln(9 - x^2 - y^2)$, the domain of $f(x, y)$ is $\{(x, y)|x^2 + y^2 < 9\}$.

2.2 Level Curves

We can look at different cross sections of a surface $f(x, y)$ by looking at the equation $f(x, y) = c$ where $c \in \mathbb{R}$. This curve lives in the xy-plane and is called the C-level curve. We often visualize these curves in the $z = c$ plane as part of the surface. In higher dimensions, like $f(x, y, z)$, a C-level curve becomes a C-level surface.



For example, the C-level surface of $f(x, y, z) = e^{-\left(x^2+y^2+z^2\right)}$ is the sphere centered at the origin with radius $\sqrt{-\ln c}$: $x^2 + y^2 + z^2 = -\ln c$.

2.3 Quadric Surfaces

Quadric surfaces extend parabolas and other shapes composed of at most squared terms into 3D.

2.3.1 Paraboloids

The paraboloid looks like a parabola that has been rotated and extruded about its axis of symmetry. It is radially symmetric, and its level curves are circles. Paraboloids have the form $z = ax^2 + by^2$ where $a, b \in \mathbb{R}$.

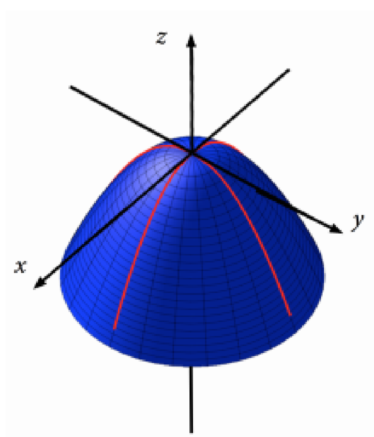


Figure 2.1: A paraboloid

2.3.2 Hyperboloids

A hyperboloid looks like a hyperbola that has been rotated and extruded about its center. It is also radially symmetric with circular level curves. Paraboloids have the form

$$d = \pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2},$$

where one sign is different from the others. Depending on the signs and the value of d, one can get a hyperboloid of one sheet, two sheets, or a cone.

Hyperboloids of One Sheet

A hyperboloid of one sheet has 2 +'s and 1 - in its equation. It is one fully connected surface.

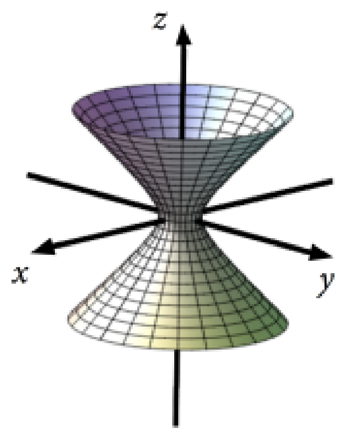


Figure 2.2: A hyperboloid of one sheet

Hyperboloids of Two Sheets

A hyperboloid of two sheets has 2 -'s and 1 + in its equation. It is made up of two disconnected, mirror image, surfaces.

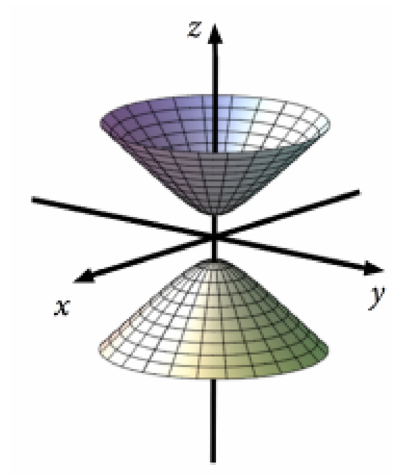


Figure 2.3: A hyperboloid of two sheets

Cone

A cone is a transition state between one-sheet and two-sheet hyperboloids. When the constant term in the hyperboloid's equation is 0, the top and bottom surfaces are only connected at a single point.

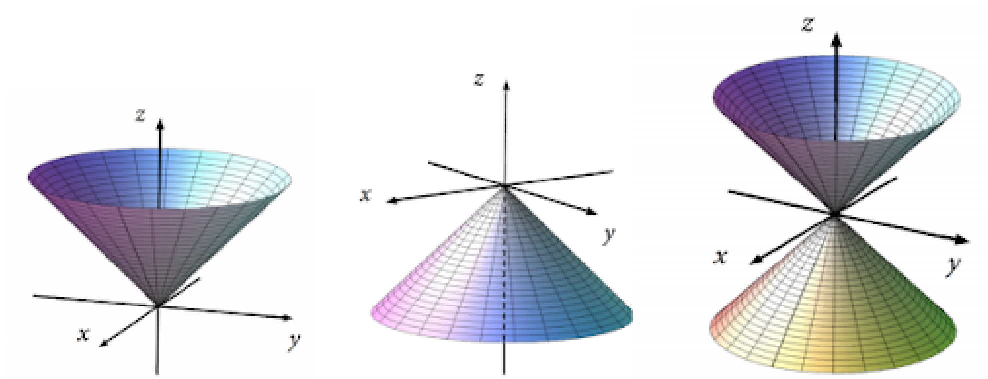


Figure 2.4: Cones

2.3.3 Hyperbolic Paraboloids

Hyperbolic paraboloids have the form

$$z = x^2 - y^2.$$

They are not radially symmetric and look like a saddle or Pringle's chip.

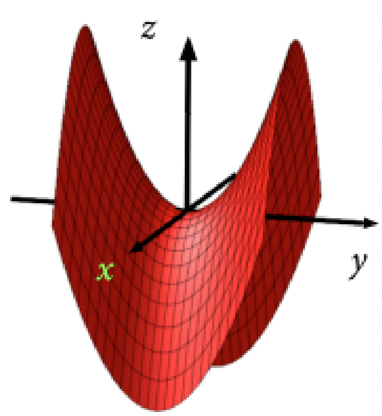


Figure 2.5: A hyperbolic paraboloid

2.3.4 Ellipsoids

Ellipsoids look like ellipses that have been rotated and extruded about their axis. They are radially symmetric about this axis. They have the general form

$$d = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}.$$

Note that the only difference in the equation between an ellipsoid and hyperboloid is the signs are all positive.

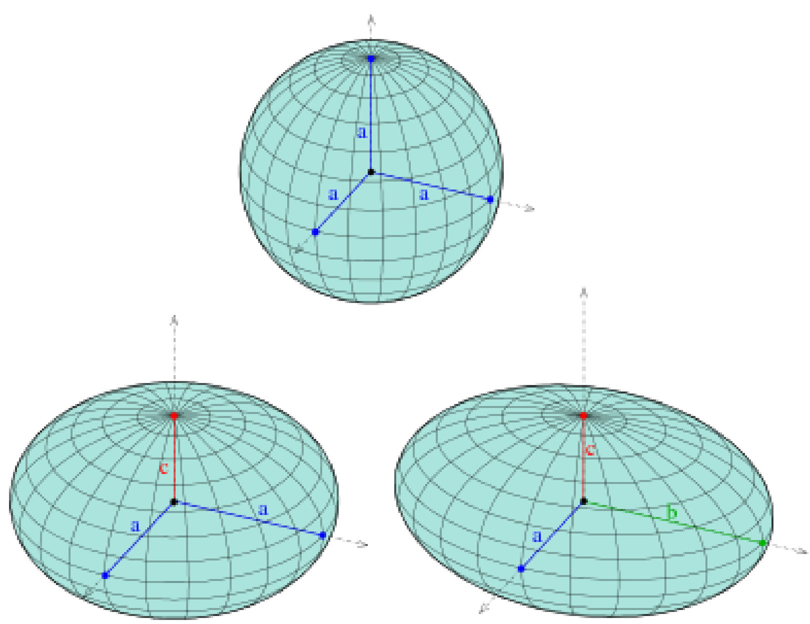


Figure 2.6: Ellipsoids

2.3.5 3D Cylinders

Although we normally think of cylinders as being extruded circles (circular cylinders), we can create other types of cylinders by extruding curves into 3D space.

Circular Cylinders

A circular cylinder is what we usually think of as a cylinder. It is a circle extruded into 3D space. One of its forms is

$$x^2 + y^2 = R^2,$$

where R is the cylinder's radius.

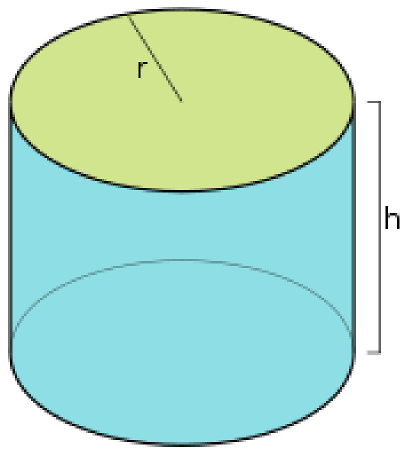


Figure 2.7: A circular cylinder

Parabolic Cylinders

A paraboloid cylinder look like a parabola that has been extruded into 3D space. One of its forms is

$$z = cy^2,$$

where $c \in \mathbb{R}$.

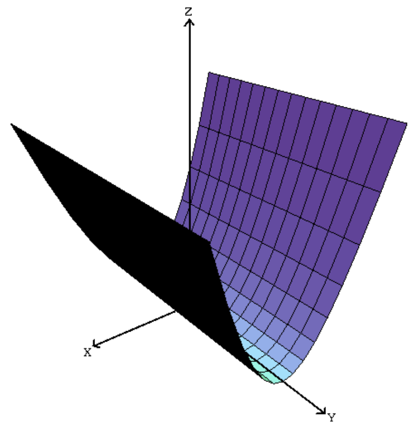


Figure 2.8: A parabolic cylinder

2.4 Parameterized Surfaces

Parameterized surfaces are a natural of VVFs that map $\mathbb{R}^n \to \mathbb{R}^m$ where usually $n < m$.

For example, a cylinder of radius 1 can be parameterized as $\vec{r}(u,v) = \langle \sin u, \cos u, v \rangle$. This particular surface maps $\mathbb{R}^2 \to \mathbb{R}^3$. The paraboloid $z = x^2 + y^2$ can be parameterized as $\vec{r}(u,v) = \langle u, v, u^2 + v^2 \rangle$.

A general trick when trying to parameterize a surface is to substitute u and v for two variables like x and y and find an expression for the third variable in terms of the u and v. Although this does not always lead to the most useful parameterization, it can be a good starting point.

For example, if we wanted to parameterize the surface $y^2 = x^2 + z^2$ from $y = 1$ to $y = 9$, we could use the general trick and get $\vec{r}(u,v) = \langle u, \sqrt{u^2 + v^2}, v \rangle$ where $1 \leq u^2 + v^2 \leq 9^2$. Although this parameterization is technically correct, it is difficult to work with because the bounds for u and v are not independent.
Instead, we can recognize that the surface we are trying to parameterize has radial symmetry about the y-axis and instead let u be and angle and v be a radius to get $\vec{r}(u,v) = \langle v\cos u, v, v\sin u \rangle$ where $0 \leq u \leq 2\pi$ and $1 \leq v \leq 9$. The parameterization now has independent bounds, which will make operations like integration much easier.

2.5 Limits & Continuity in 3D

Limits in single-variable calculus are relatively simple because thera are only two ways to approach a point on a curve: left and right. When dealing with a surface, there are infinitely many ways to approach a point. So, we need a general and more formal idea of limits that works in higher dimensions.

2.5.1 Open Delta Neighborhoods

An open delta neighborhood of a point x_0 is defined as the set

$$N(x_0, \delta) = \left\{x \in \mathbb{R}^n \mid \left|\left|x - x_0\right|\right| < \delta\right\}.$$

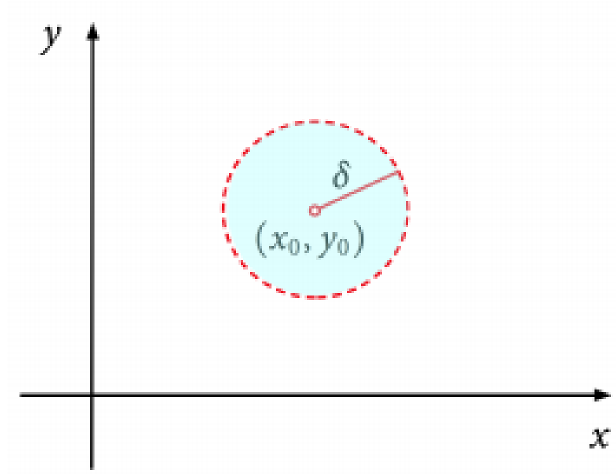


Figure 2.9: An open delta neighborhood centered at (x_0, y_0)

This simply means all points less than a distance δ away from point x_0.
For example, $N((1,2),7) = \left\{(x,y) \mid \sqrt{(x-1)^2 + (y-2)^2} < 7\right\}$, which is a ball (filled-in circle) of radius 7 centered at $(1,2)$.

2.5.2 Boundary Points, Open & Closed Sets

Given some set $\Omega \subset \mathbb{R}^n$, x is an interior point to Ω if there exists some δ such that $N(x,\delta) \subset \Omega$. That is, x is an interior point to Ω if you can draw a circle of non-zero radius around x such that the entire circle is inside of Ω. All points that are not interior points are boundary points. Formally, x is a boundary point of Ω if for all δ, $N(x,\delta) \not\subset \Omega$. Using our definitions of interior and boundary points, we can define and open set as one that doesn't contain any of its boundary points and a closed set as one that contains all of its boundary point. Note that a set that contains some of its boundary points is neither open nor closed.

2.5.3 Limit & Continuity Definitions

Finally, we have the proper tools to define a limit in higher dimensions. We say that $\lim_{p \to p_0} f(p) = L$ if for all $N(L,\epsilon)$, there exists $N(p_0,\delta)$ such that $p \in N(p_0,\delta) \implies f(p) \in N(L,\epsilon)$. That is, the limit of $f(p)$ as p approaches p_0 is equal to L if for all open delta neighborhoods around L, there exists an open delta neighborhood around p_0 such that p being in the neighborhood around p_0 means $f(p)$ must be in the neighborhood around L.

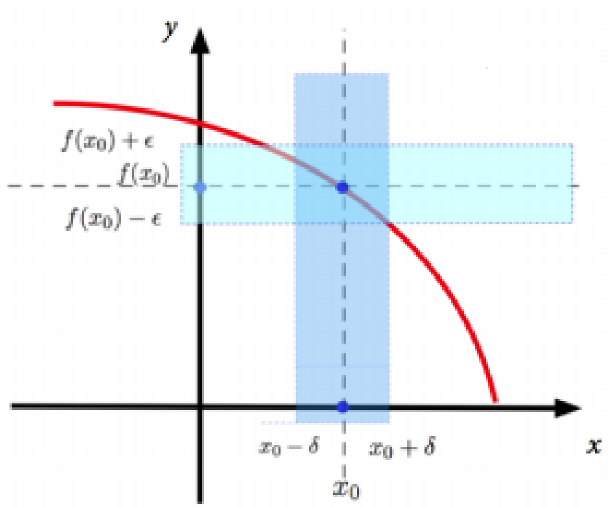


Figure 2.10: Limit definition

Now with a limit definition, we can define continuity at a point. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at p_0 if $\lim_{p\to p_0} f(p) = f(p_0)$.

Theorem. Trigonometric functions, exponentials, logarithms, and sums, products, quotients, and compositions of such functions are continuous on their domain.

Although our definitions allow us to confirm that a value is the limit of a function, they do not give us any insight into how to find the value of the limit. We'd need to approach our point of interest from every possible direction to see if the limit from that direction is the same as all the others. If any two directions give different limit values, then the limit doesn't exist. We approach a function, f, by composing it with a single variable path, $\vec{r}(t)$, that goes through the point of interest, and find the limit along the path.

If f is some surface $f(x, y)$ and $\vec{r}(t) = \langle x(t), y(t)\rangle$, then the composition of f and $\vec{r}$ is $f \circ \vec{r} = f(\vec{r}(t)) = f(x(t), y(t))$.
For example, let's try to find

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^3}{x^2 + y^2}.$$

We'll choose two paths $\vec{r_1}(t) = \langle t, 0\rangle$ and $\vec{r_2}(t) = \langle t, t\rangle$ and find the limit as $t \to 0$ in both cases.

$$\lim_{t\to 0} f(\vec{r_1}(t)) = \lim_{t\to 0} \frac{t^2}{t^2} = 1$$

$$\lim_{t\to 0} f(\vec{r_2}(t)) = \lim t \to 0\frac{t^2 - t^3}{2t^2} = \lim_{t\to 0} \frac{1}{2} - \frac{t}{2} = \frac{1}{2}$$

Since the limits on the two paths are not equal, we can say that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^3}{x^2 + y^2} = \text{ DNE.}$$

2.6 Partial Derivatives

The single-variable calculus idea of tangent lines doesn't work for higher dimensional surfaces because we can draw many different lines that are tangent to the surface, depending on which plane we use to slice the surface. That is, from which direction we approach the surface.

2.6.1 Partial Derivatives of X, Y, and Z

It's common to look at the derivative when slicing a surface in the yz, xz, and xy planes. These are called partial derivatives.

To compute $\frac{\partial}{\partial x} f(x, y)$, we take the derivative with respect to x as if y is constant. Formally,

$$\frac{\partial}{\partial x} f(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h},$$

and

$$\frac{\partial}{\partial y} f(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

We also use the shorthand $\frac{\partial}{\partial x} = f_x$ and $\frac{\partial}{\partial y} = f_y$. This shorthand can be extended to higher-order derivatives so that $\frac{\partial}{\partial y} \left(\frac{\partial}{\partial x} f(x, y) \right) = f_{xy}$.

Fubini's Theorem (also called Tonelli's or Clairaut's Theorem) says $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, and $f_{yz} = f_{zy}$. It extends into higher-order mixed partial derivatives, saying that two mixed partial derivatives of a function are equal as long as they both differentiate the same number of variables the same number of times. So, $f_{abcdab} = f_{aacdbb}$.

2.6.2 Tangent Planes

Although the tangent lines at a point on a surface can all be different depending on from which direction one approaches a point, all of these tangent lines lie in the same plane, defining the tangent plane. This means that the tangent plane to $z = f(x, y)$ at (x_0, y_0) has the following properties:

- The z-value of the tangent plane at (x_0, y_0) is the same as $f(x_0, y_0)$.
- The value of the first-order partial derivatives of the tangent plane at (x_0, y_0) should match those of $f(x_0, y_0)$.

The general form of a plane at (x_0, y_0, z_0) is

$$P(x, y) = A(x - x_0) + B(y - y_0) + z_0.$$

We want $P_x = f_x$ and $P_y = f_y$. This means that $P_x = f_x = A$ and $P_y = f_y = B$. Rewriting,

$$P(x, y) = f_x(x - x0) + f_y(y - y_0) + z_0.$$

The normal vector is $\langle \pm f_x, \pm f_y, \mp 1\rangle$. So, the point normal form of the plane is

$$\langle -f_x, -f_y, 1\rangle \cdot \langle x - x_0, y - y_0, z - f(x_0, y_0)\rangle = 0.$$

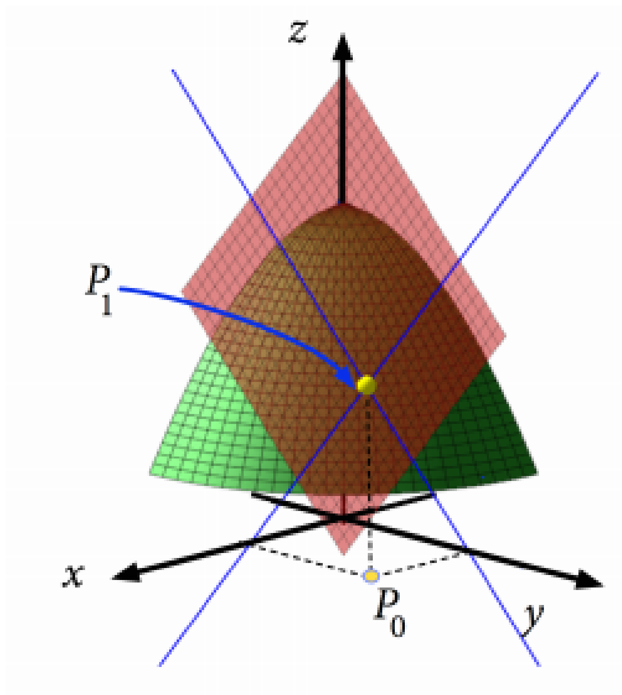


Figure 2.11: Tangent plane

2.6.3 Linear Approximations

Since $\partial z = f_x \partial x + f_y \partial y$, we can approximate Δz (the change in any function) as $\Delta z \approx f_x \Delta x + f_y \Delta y$ because values of f and the tangent plane are close. We can rewrite this approximation as a dot product:

$$\Delta z \approx \langle f_x, f_y\rangle \cdot \langle \Delta x, \Delta y\rangle.$$

For example, say a cylindrical can has a radius $r = 1$ and a height $h = 5$. If the radius is increased by 0.1 and the height is increased by 1, what is the approximate ΔV?

$$\begin{aligned}
V(r,h) &= \pi r^2 h \\
V_r &= 2\pi r h \text{ and } V_h = \pi r^2 \\
V_r(1,5) &= 10\pi \text{ and } V_h(1,5) = \pi \\
\Delta V &\approx 10\pi(0.1) + \pi(1) = 2\pi.
\end{aligned}$$

Comparing this to the actual answer of 2.26π, we see our approximation is decent.

2.7 The Gradient

If you are on a surface $f : \mathbb{R}^2 \to \mathbb{R}$, what direction $\langle \Delta x, \Delta y \rangle$ should you go to maximize the change of f?
We saw earlier that $\Delta z \approx \langle f_x, f_y \rangle \cdot \langle \Delta x, \Delta y \rangle$. To maximize a dot product, $\langle \Delta x, \Delta y \rangle$ should be in the same direction as $\langle f_x, f_y \rangle$. This directional vector is called the gradient: the direction of steepest ascent.
Notated mathematically,

$$\nabla f(x,y) = \langle f_x, f_y \rangle.$$

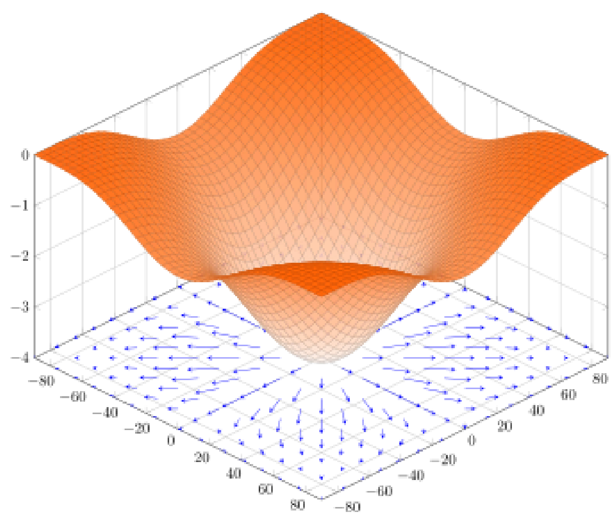


Figure 2.12: A surface and its gradient vectors

2.7.1 Gradient Properties

Let f and g be functions of multiple variables, let $\vec{r}$ be a VVF, and let $c \in \mathbb{R}$.

1. $\nabla(f \pm g) = \nabla f \pm \nabla g$
2. $\nabla(cf) = c\nabla f$

3. $\nabla(fg) = f\nabla g + g\nabla f$

4. $\nabla(f \circ \vec{r}(t)) = \nabla f \cdot \vec{r'}(t)$

The 4th gradient property can be generalized a bit further. Suppose we have $f(x, y, z)$, $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, and $g(u, v) = (f \circ \vec{r})(u, v)$.

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle = \nabla f \cdot \frac{\partial \vec{r}}{\partial u}.$$

2.7.2 Linear Approximations with the Gradient

We can rewrite our linear approximation of f at x_0 using the gradient.

$$f(x) \approx f(x_0) + (\nabla f)(x_0) \cdot (x - x_0)$$

2.7.3 The Gradient & C-Level Curves

Let $\vec{r}(t)$ be the C-level curve of $f(x, y)$.

$$\begin{aligned} & f \circ \vec{r} = C \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \vec{r}) = 0 \\ \implies & \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \vec{r}) = \nabla f \cdot \vec{r'}(t) = 0 \\ \implies & \nabla f \perp \vec{r'}(t) \\ \implies & \nabla f \text{ is perpendicular to the C-level curve of } f. \end{aligned}$$

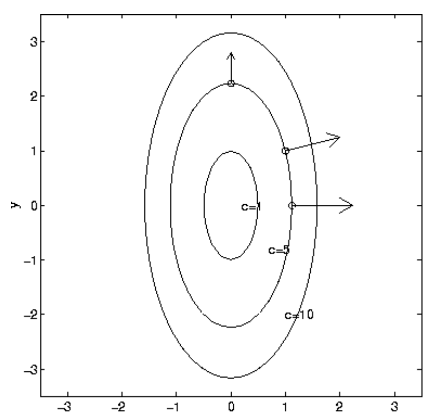


Figure 2.13: The gradient is perpendicular to C-level curves.

2.8 Directional Derivatives

We already saw partial derivatives in the x, y, and z directions. However, we can take the derivative coming from other directions. These are called directional derivatives, $D_{\hat{u}}f$, where $\hat{u}$ is the direction.

$$D_{\hat{u}}f = \lim_{h\to 0} \frac{f(\vec{p_0} + h\hat{u})}{h}$$

If $\hat{u} = \langle a, b\rangle$,

$$D_{\hat{u}}f = \lim_{h\to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h}.$$

Note that

$$D_{\hat{i}}f = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = f_x \text{ and } D_{\hat{j}}f = f_y.$$

Let's look at $D_{\hat{u}}f$.

$$\begin{aligned}
D_{\hat{u}}f &= \lim_{h\to 0} \frac{f(x+ah, y+bh) - f(x+ah,y) + f(x+ah,y) - f(x,y)}{h} \\
&= b\lim_{h\to 0} \frac{f(x+ah, y+bh) - f(x+ah,y)}{bh} + a\lim_{h\to 0} \frac{f(x+ah,y) - f(x,y)}{ah} \\
&= b\frac{\partial f}{\partial y} + a\frac{\partial f}{\partial x} \\
&= af_x + bf_y.
\end{aligned}$$

So,

$$D_{\hat{u}}f = \nabla f \cdot \hat{u}.$$

2.9 Optimization

2.9.1 Definitions

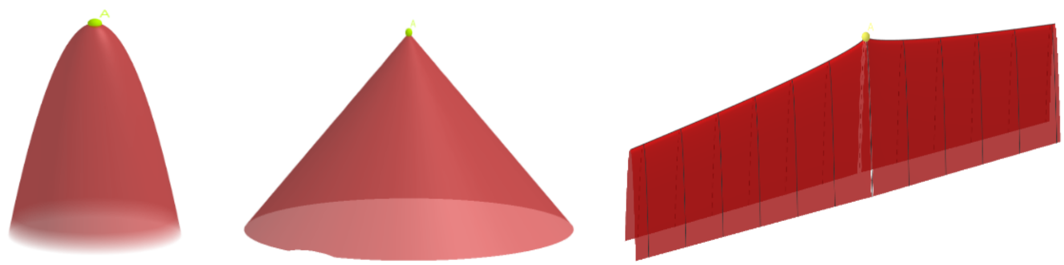
1. $f(x_0, y_0)$ is a local maximum of f if for some $\delta > 0$, $f(x_0,y_0) \geq f(x,y) \forall (x,y) \in N((x,y),\delta)$. That is, you can draw a circle in the domain of f centered at (x_0, y_0) such that the value of f at every point in the circle besides (x_0, y_0) is less than (x_0, y_0).

2. $f(x_0, y_0)$ is a local minimum of f if for some $\delta > 0$, $f(x_0,y_0) \leq f(x,y) \forall (x,y) \in N((x,y),\delta)$. That is, you can draw a circle in the domain of f centered at (x_0, y_0) such that the value of f at every point in the circle besides (x_0, y_0) is greater than (x_0, y_0).

3. $f(x_0, y_0)$ is a global max of f if $f(x_0,y_0) \geq f(x,y) \forall (x,y) \in D(f)$ where $D(f)$ is the domain of f.

4. $f(x_0, y_0)$ is a global min of f if $f(x_0,y_0) \leq f(x,y) \forall (x,y) \in D(f)$ where $D(f)$ is the domain of f.

Theorem. If (x_0, y_0) is in the domain of f and a local extrema of $f(x, y)$, then $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ is either 0 or undefined.



$z = 9 - x^2 - y^2$
Both partials 0

$z = 9 - \sqrt{x^2 + y^2}$
Both Partials undefined

$z = 9 - x^{2/3} - y^2$
One partial defined, other undefined

Figure 2.14: Critical points appear when the partial derivatives are 0 or undefined.

2.9.2 Critical Points

Critical points are those that have the possibility of being a minimum or maximum. They are an extension of critical points in single-variable calculus when the derivative is 0.

Definition. (x_0, y_0) is a critical point of f if f_x and f_y at (x_0, y_0) both are 0 or DNE.

For example, consider the function $f(x, y) = x^2/2 - y^2/2 - xy - 2x - 2y$.

$$\begin{aligned} f_x &= x - y - 2 \text{ and is } 0 \text{ when } y = x - 2 \\ f_y &= -y - x - 2 \text{ and is } 0 \text{ when } y = -x - 2 \\ x - 2 &= -x - 2 \text{ when } x = 0. \end{aligned}$$

When $x = 0$, $y = -2$, so $(0, -2)$ is a critical point of f.

Second Derivative Test & Hessian Matrix

Recall from single-variable calculus that if we have a critical point (derivative is 0), we can find whether the point is a minimum or maximum by using the second derivative test. If the second derivative is negative at the critical point, then the critical point is a maximum, and if the second derivative is positive at the critical point, then the critical point is a minimum. A similar process works in higher dimensions.

In higher dimensions:

- If f_{xx} and f_{yy} are positive at a critical point, then the critical point is a minimum.

- If f_{xx} and f_{yy} are negative at a critical point, then the critical point is a maximum.
- If f_{xx} and f_{yy} don't agree on sign, then the point is a saddle point.

Although there are only 3 options for what a critical point on a surface can be, we need a process of identifying them that works for for higher dimensional objects too.

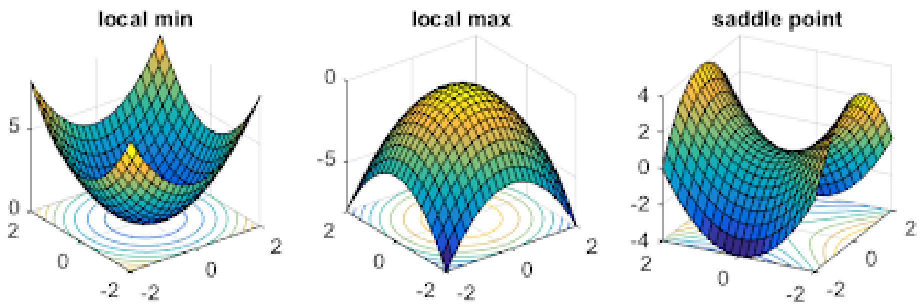


Figure 2.15: Three possibilities depending on signs of the seconds derivative

Definition. The Hessian Matrix (for $f(x, y)$):

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

Note that $\det H = f_{xx}f_{yy} - f_{xy}^2$.

If (x_0, y_0) is a critical point:

- $\det H(x_0, y_0) > 0$ means (x_0, y_0) is an extrema.
  - If $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a minima.
  - If $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a maxima.
- $\det H(x_0, y_0) < 0$ means (x_0, y_0) is a saddle point.
- $\det H(x_0, y_0) = 0$ means the 2nd derivative test is inconclusive.

2.9.3 Gradient Descent

Remember that if we have a multidimensional function, taking a step in the direction of the gradient results in the maximum possible increase of the function, and taking a step in the opposite direction of the gradient results in the maximum possible decrease of the function. Gradient descent is a method to find minima of functions.

Let's say we're trying to minimize $J(\vec{x})$ with gradient descent. Here are the steps we would take:

1. Pick (or guess) a starting point $\vec{x_0}$ and a learning rate (step size) δ.

2. $\overrightarrow{x_{n+1}} = \overrightarrow{x_n} - \delta J(\overrightarrow{x_n})$

3. Repeat step 2 until some stopping criteria is met, like $\left|\left|\delta \nabla J(\overrightarrow{x_n}) - \delta \nabla J(\overrightarrow{x_{n+1}})\right|\right| < \epsilon$.

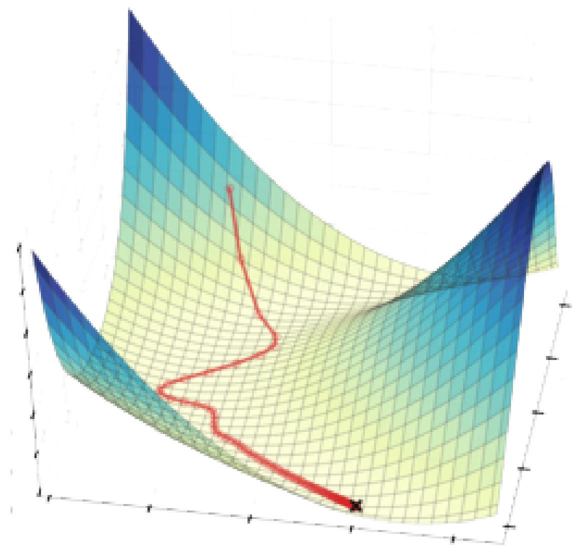


Figure 2.16: Path of several iterations of gradient descent

This method will lead you arbitrarily close to a local minimum, but does not guarantee finding the global minimum. More advanced versions of gradient descent exists that try to help with this, like giving the point "momentum" to be able to move out of local mins. This method also has a trade off between speed and accuracy. Although increasing δ means fewer iterations of gradient descent are needed to narrow in on a local minimum, one is more likely to be stuck in a local min than if they had used a smaller δ.

In the real world, the function you are trying to minimize will likely not be well defined enough to take its partial derivatives to find the gradient, so they too are approximated by doing something like

$$J_k = \frac{J(k + .0001, \ldots) - J(k, \ldots)}{.0001}.$$

2.9.4 Lagrange Multipliers

Lagrange multipliers are a method that allow us to find extrema of a function subject to domain constraints.

Toy Example

Say we wish to maximize $f(x, y) = x + y$ subject to the constraint $g(x, y) = x^2 + y^2 = 1$. We can do this y finding the C-level curve that is tangent to our constraint, as this curve will have the extrema. At this point, ∇f will be in the same direction as ∇g. That is,

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases}.$$

We also add the constraint itself, $g(x, y)$ equals some constant k, giving us a system of equations

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x, y) = k \end{cases}.$$

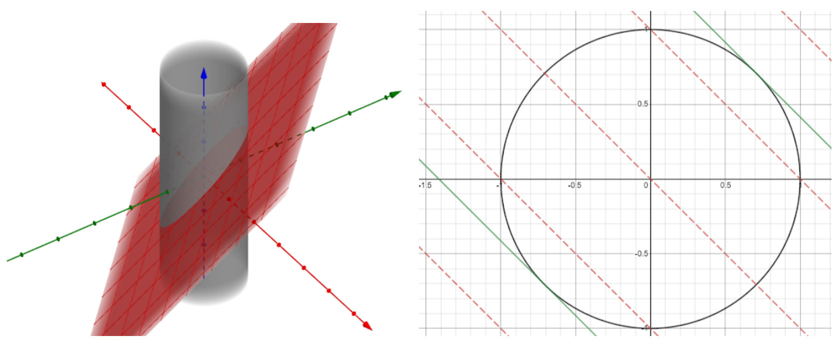


Figure 2.17: Extremas when the constraints (black) are tangent to the level curves.

In the example above, $f_x = 1$, $g_x = 2x$, $f_y = 1$, and $g_y = 2y$ with constraint $x^2 + y^2 = 1$. Giving us a system

$$\begin{cases} 1 = \lambda 2x \\ 1 = \lambda 2y \\ x^2 + y^2 = 1 \end{cases} \implies x = y = \pm\frac{1}{\sqrt{2}} \text{ and } \lambda = \frac{1}{\sqrt{2}}.$$

This means that the max value of f constrained by g is

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{2}.$$

Method of Lagrange Multipliers

Given an objective function, $f(x, y)$ and a constraint equation $g(x, y) = k$, define $F(x, y, \lambda) = f(x, y) + \lambda(k - g(x, y))$. The solution, (x, y, λ), to $\nabla F = \vec{0}$ will be the solution to the constrained optimization problem.

For example, let's maximize $f(x, y) = xy$ subject to $(x-1)^2 + (y-1)^2 = 1$.

$$F(x, y, \lambda) = xy + \lambda(1 - (x-1)^2 - (y-1)^2)$$

$$\nabla F = \langle y - 2\lambda(x-1), x - 2\lambda(y-1), 1 - (x-1)^2 - (y-1)^2 \rangle = \vec{0}$$

$$\begin{cases} y - 2\lambda(x-1) = 0 \\ x - 2\lambda(y-1) = 0 \\ 1 - (x-1)^2 - (y-1)^2 = 0 \end{cases} \implies \begin{cases} y = 2\lambda(x-1) \\ x = 2\lambda(y-1) \\ (x-1)^2 + (y-1)^2 = 1 \end{cases}$$

$$y = 2\lambda(2\lambda(y-1) - 1) = 4\lambda^2 y - 4\lambda^2 - 2\lambda = \frac{2\lambda}{2\lambda + 1}$$

$$x = 2\lambda\left(\frac{2\lambda}{2\lambda+1} - 1\right) = \frac{2\lambda}{2\lambda+1} \implies x = y$$

$$2(x-1)^2 = 1 \implies x = y = 1 \pm \frac{1}{\sqrt{2}}$$

$$\implies \text{ Max/Min of } \frac{3 \pm 2\sqrt{2}}{2}.$$

Chapter 3

Multiple Integrals

3.1 Double Integrals

Similar to how the limit of a Riemann Sum, the sum of the areas of small rectangles, is the area underneath a curve, we can find the volume underneath a surface by summing the volumes of small rectangular prisms.

[INSERT IMAGE]
In 2D (single variabe): $\Delta x = \frac{b-a}{n}$, $\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(a+i\Delta x)\Delta a$.
In 3D: $\Delta x = \frac{b-a}{n}$, $\Delta y = \frac{d-c}{n}$,

$$\int_c^d \int_a^b f(x,y)\mathrm{d}x\mathrm{d}y = \lim_{m\to\infty} \sum_{j=0}^{m-1} \left(\lim_{n\to\infty} \sum_{i=0}^{n-1} f(a+i\Delta a, c+j\Delta y)\Delta x \right) \Delta y$$

3.1.1 Fubini's Theorem & Domain Regions

Theorem (Fubini's Theorem). The order of integration on a domain where the variables of integration $(x, y,$ etc.$)$ vary independently doesn't matter.

For example, let's find the volume under $f(x,y) = 9 - x^2 - y^2$, $(x,y) \in [0,1] \times [1,2]$.
We will do so in two ways to show that they are equivalent: one with x first and then y and another with y first and then x.

$$\begin{array}{c|c}
\begin{aligned}
V &= \textstyle\int_0^1 \int_1^2 9 - x^2 - y^2 \mathrm{d}y\mathrm{d}x \\
&= \textstyle\int_0^1 \left[9y - x^2 y - \frac{y^2}{3}\right]_1^2 \mathrm{d}x \\
&= \textstyle\int_0^1 \frac{20}{3} - x^2 \mathrm{d}x \\
&= \left[\frac{20}{3}x - \frac{x^3}{3}\right]_0^1 \\
&= \frac{19}{3}
\end{aligned}
&
\begin{aligned}
V &= \textstyle\int_2^1 \int_0^1 9 - x^2 - y^2 \mathrm{d}x\mathrm{d}y \\
&= \textstyle\int_1^2 \left[9x - \frac{x^3}{3} - xy^2\right]_0^1 \mathrm{d}y \\
&= \textstyle\int_1^2 \frac{26}{3} - y^2 \mathrm{d}y \\
&= \left[\frac{26}{3}y - \frac{y^3}{3}\right]_1^2 \\
&= \frac{19}{3}
\end{aligned}
\end{array}$$

Let's look at a case where x and y are not independent. specifically, where the bounds on y are a function of x.
[INSERT IMAGE]
This is called a Type I Region. Formally, a Type I Region is a domain $D = \{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq h(x)\}$.

Theorem (Fubini's Theorem for Type I Regions). Let D be a Type I Region in $\mathbb{R}^2$.

$$\iint\limits_D f(x,y)\mathrm{d}A = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\mathrm{d}y\mathrm{d}x$$

It's also possible for y to have constant bounds and the bound for x to be a function of y. this is a Type II Region. Formally, a Type II Region is a domain $D = \{(x, y) \mid g(y) \leq x \leq h(y), a \leq y \leq b\}$.

Theorem (Fubini's Theorem for Type II Regions). Let D be a Type II Region in $\mathbb{R}^2$.

$$\iint\limits_D f(x,y)\mathrm{d}A = \int_a^b \int_{g(y)}^{h(y)} f(x,y)\mathrm{d}x\mathrm{d}y$$

Sometimes, a region can be describes as both Type I and Type II. You should pick whichever description is most convenient.

The previous two theorems can be summarized as dependent variables need to be integrated before the variables that they depend on. This core idea extends into higher dimensions where classifying regions becomes tedious and not very helpful.

One can split larger, harder to describe domains into smaller domains. Let $D_1 \cup D_2 = D$.
[INSERT IMAGE]

$$\iint\limits_D f(x,y)\mathrm{d}A = \iint\limits_{D_1} f(x,y)\mathrm{d}A + \iint\limits_{D_2} f(x,y)\mathrm{d}A - \iint\limits_{D_1 \cap D_2} f(x,y)\mathrm{d}A$$

3.1.2 Average Values

We can think of the average value of a function over some interval as the answer to the question: "If I flattened this function into a box over the interval, what would the height of the box be?"
For single-variable functions, the answer is $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \frac{\int_a^b f(x)\mathrm{d}x}{\int_a^b \mathrm{d}x}$.
This idea of summing a function over a domain and the dividing by the size of that domain holds into multivariable as

$$\bar{f} = \frac{\int\limits_D f(x,y)\mathrm{d}D}{\int\limits_D \mathrm{d}D}$$

Mean Value Theorem

Theorem (Mean Value Theorem). If a function f is continuous on a domain D, then there exists some point $p \in D$ such that $f(p) = \bar{f}$.

3.1.3 Volume Between Surfaces

Similar to how we could find the area between two curves in single-variable calculus, we can find the volume between two surfaces.

$$V_{\text{bwtn}} = \iint\limits_{D} f \mathrm{d}A - \iint\limits_{D} g \mathrm{d}A = \iint\limits_{D} (f-g)\mathrm{d}A$$

If g is below f in $D_1 \subset D$ but is above f in $D_2 \subset D$, where $D_1 \cup D_2 = D$ and $D_1 \cap D_2 = \emptyset$, we can still find the volume between the surfaces by splitting D into D_1 and D_2.

$$V_{\text{bwtn}} = \iint\limits_{D} \left|f-g\right|\mathrm{d}A = \iint\limits_{D_1} (f-g)\mathrm{d}A + \iint\limits_{D_2} (g-f)\mathrm{d}A$$

3.1.4 Plane Laminas

Definition. A plane lamina is an idealized 2D (0 thickness) object with mass that occupies a region $D \subset \mathbb{R}^2$.

Some questions one may ask about a plane lamina are "What is the total mass?" and "Where is the center of mass?".
We can think of the mass as

$$M = \iint\limits_{D} \sigma(x, y)\mathrm{d}A$$

where $\sigma(x, y)$ is the mass density of the lamina at (x, y).
The center of mass is

$$\bar{x} = \frac{M_y}{M} = \frac{\iint\limits_{D} x\sigma(x, y)\mathrm{d}A}{M} \text{ and } \bar{y} = \frac{M_x}{M} = \frac{\iint\limits_{D} y\sigma(x, y)\mathrm{d}A}{M}$$

where M_x is the moment about the x-axis and M_y is the moment about the y-axis.

3.2 Triple Integrals

Triple integrals work much the same way as single and double integrals. They are still defined by a Riemann sum, and Fubini's theorems about independent domains and the order of integration still applies.

3.2.1 Fubini's Theorem for Z-simple Regions

There is another case of Fubini's Theorem that arises in 3D.

Theorem (Fubini's Theorem for Z-simple Regions). Let $D \subset \mathbb{R}^3$ and let $\Omega = \{(x, y, z) \mid (x, y) \in D, g(x, y) \leq z \leq$ be a z-simple region.

$$\iiint_{\Omega} f(x, y, z)\mathrm{d}V = \iint_{D} \int_{g(x,y)}^{h(x,y)} f(x, y, z)\mathrm{d}z\mathrm{d}A$$

In other words, dependent variables must be integrated before the independent variables on which they rely, but the order of integration of independent variables doesn't matter.

For example, we an express the unit sphere as a z-simple region where $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ and $-\sqrt{1 - x^2 - y^2} \leq z \leq \sqrt{1 - x^2 - y^2}$. Note that D is a Type I Region.

$$V_{\text{sphere}} = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \mathrm{d}z\mathrm{d}x\mathrm{d}x = \frac{4\pi}{3}$$

3.2.2 Laminas

Similar to plane laminas:

$$M = \iiint_{\Omega} \sigma(x, y, z)\mathrm{d}V$$

$$\bar{x} = \frac{M_{yz}}{M} = \frac{\iiint_{\Omega} x\sigma(x, y, z)\mathrm{d}V}{M}, \bar{y} = \frac{M_{xz}}{M} = \frac{\iiint_{\Omega} y\sigma(x, y, z)\mathrm{d}V}{M}, \text{ and } \bar{z} = \frac{M_{xy}}{M} = \frac{\iiint_{\Omega} z\sigma(x, y, z)\mathrm{d}V}{M}$$

Chapter 4

Curvilinear Coordinates

You should already know about Cartesian (x, y) coordinates and polar (r, θ) coordinates in 2D. Cartesian extends to 3D as (x, y, z), but there are multiple ways to extend polar coordinates into 3D.

4.1 Review of Polar Coordinates

Polar coordinates represent every point in 2D space as a distance from the origin r and an angle from the horizontal θ. This means that unlike rectangular (Cartesian) (x, y) coordinates, different polar coordinates can represent the same point: $(2, -\pi/4) = (-2, 3\pi/4) = (2, 7\pi/4)$.
[INSERT IMAGE]
Polar coordinates can be transformed into rectangular coordinates by $x = r\cos\theta$ and $y = r\sin\theta$. This means that $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$.

4.1.1 Circles

A circle of radius R centered at the origin can be represented as $r = R$.
Circles not centered at the origin require using the transformation equations.

$$(x-a)^2 + (y-b)^2 = R^2 \implies (r\cos\theta - a)^2 + (r\sin\theta - b)^2 = R^2$$

$$r^2\cos^2\theta + r^2\sin^2\theta - 2ra\cos\theta - 2rb\sin\theta + a^2 + b^2 = R^2$$

$$r^2 - 2r(a\cos\theta + b\sin\theta) - R^2 + a^2 + b^2 = 0$$

$$r = (a\cos\theta + b\sin\theta) \pm \sqrt{R^2 - a^2 - b^2 + (a\cos\theta + b\sin\theta)^2}$$

4.1.2 Lines

- Lines through the origin can be represented as $\theta = \tan^{-1} m$ where m is the slope of the line.
- Lines of the form $x = a$ can be represented as $r = a\sec\theta$.

- Lines of the form $y = a$ can be represented as $r = a\csc\theta$.
- All other lines of the form $y = ax + b$ can be represented as $r = \frac{b}{\sin\theta - a\cos\theta}$. This form also covers the previous 2.

4.1.3 Integration

The line element is $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$, meaning that $s = \int_{\theta_1}^{\theta_2}\sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)}\mathrm{d}\theta$.
The area element is $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$, meaning that $A = \int_{\theta_1}^{\theta_2} r^2\mathrm{d}\theta$.

Gaussian Integral

Let's compute $\iint\limits_D e^{-x^2-y^2}\mathrm{d}A$ where D is the unit disk.

$$A = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{-x^2-y^2}\mathrm{d}y\mathrm{d}x = \int_0^1\int_0^{2\pi} e^{-r^2}r\mathrm{d}\theta\mathrm{d}r = \int_0^1 re^{-r^2}\mathrm{d}r \cdot \int_0^{2\pi}\mathrm{d}\theta.$$

Let $u = -r^2$, $\mathrm{d}u = -2r\mathrm{d}r$.

$$\begin{aligned} &= \frac{1}{2}\int_0^{-1} e^u\mathrm{d}u \cdot 2\pi \\ &= \pi\left(1 - \frac{1}{e}\right). \end{aligned}$$

Now, let's have $D = \mathbb{R}^2$. This is the famous Gaussian Integral.

$$\begin{aligned} &= \int_0^{\infty}\int_0^{2\pi} re^{-r^2}\mathrm{d}\theta\mathrm{d}r \\ &= \int_0^{\infty} re^{-r^2}\mathrm{d}r \cdot \int_0^{2\pi}\mathrm{d}\theta. \end{aligned}$$

Let $u = -r^2$, $\mathrm{d}u = -2r\mathrm{d}r$.

$$\begin{aligned} &= -\pi\int_0^{-\infty} e^u\mathrm{d}u \\ &= -\pi\left(\left(\lim_{a\to-\infty} e^a\right) - e^0\right) \\ &= -\pi(0-1) \\ &= \pi \end{aligned}$$

4.2 Cylindrical Coordinates

Cylindrical coordinates are the expansion of polar coordinates by including a third term that represents the height from the xy-plane, z. All cylindrical coordinates have the form (r, θ, z). This system is called cylindrical because it's easy to describe shapes with cylindrical symmetry because integrations have constant, independent bounds.
[INSERT IMAGE]

4.2.1 Conversions

- From Cylindrical to Cartesian

$$(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$$

- From Cartesian to Cylindrical

$$(x, y, z) = \left(\sqrt{x^2 + y^2}, \tan^{-1}\frac{y}{x}, z\right)$$

4.2.2 Integration

The volume element in cylindrical coordinates is $\mathrm{d}V = r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z$.
[INSERT IMAGE]
For example, let's evaluate $\int_0^4 \int_0^{\sqrt{16-y^2}} \int_0^{16-x^2-y^2} \mathrm{d}z\mathrm{d}x\mathrm{d}y$, the volume under the paraboloid $z = 16 - x^2 - y^2$ using cylindrical coordinates.

$$\begin{aligned}
&= \int_0^4 \int_0^{\pi/2} \int_0^{16-r^2} r\mathrm{d}z\mathrm{d}\theta\mathrm{d}r \\
&= \int_0^4 \int_0^{\pi/2} 16r - r^2\mathrm{d}\theta\mathrm{d}r \\
&= \frac{\pi}{2}\int_0^4 16r - r^3\mathrm{d}r \\
&= \frac{\pi}{2}\left[8r^2 - \frac{r^4}{4}\right]_0^4 \\
&= 32\pi
\end{aligned}$$

For another example, let's find the average value of $f(x, y, z) = z$ on the domain Ω which is bounded by $z = \sqrt{6 - x^2 - y^2}$ and $z = x^2 + y^2$ [1].

$$\bar{f} = \frac{\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{\sqrt{6-r^2}} zr\mathrm{d}z\mathrm{d}r\mathrm{d}\theta}{\int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{\sqrt{6-r^2}} r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta} = \frac{11}{12\sqrt{6} - 17}$$

[1] Work omitted for brevity

4.3 Spherical Coordinates

Unlike how cylindrical coordinates extend polar coordinates into 3D by adding a Cartesian term, spherical coordinates add an angular term, ϕ, the azimuthal angle. All spherical coordinates have the form (ρ, θ, ϕ) where ρ is the distance from the origin, θ is the polar angle in the xy-plane, and ϕ is the azimuthal angle from the $+z$-axis. Shapes with spherical symmetry have a constant bounds of integration in spherical coordinates.
[INSERT IMAGE]

4.3.1 Conversions

- From spherical to Cartesian

$$(\rho, \theta, \phi) = (r \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi)$$

- From Cartesian to Spherical

$$(x, y, z) = \left(\sqrt{x^2 + y^2 + z^2}, \arctan\left(\frac{y}{x}\right), \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)\right)$$

4.3.2 Integration

The area element is $\mathrm{d}A = \rho^2 \sin\phi \mathrm{d}\theta \mathrm{d}\phi$.
The volume element is $\mathrm{d}V = \rho^2 \sin\phi \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi$.

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Sphere Volume

Since a sphere of radius R is spherically symmetric, it should be simple to find its volume using spherical coordinates.

$$\begin{aligned} V &= \int_0^R \int_0^\pi \int_0^{2\pi} \rho^2 \sin\phi \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}\rho \\ &= \int_0^R \rho^2 \mathrm{d}\rho \cdot \int_0^\pi \sin\phi \mathrm{d}\phi \cdot \int_0^{2\pi} \mathrm{d}\theta \\ &= \frac{R^3}{3} \cdot 2 \cdot 2\pi \\ &= \frac{4\pi}{3} R^3 \end{aligned}$$

Chapter 5

Line & Surface Integrals

5.1 Vector Fields

Vector fields are a function $f : \mathbb{R}^n \to \mathbb{R}^n$. This is generally conceptualized as assigning an n-dimensional vector to every point in an n-dimensional space.
Many physics concepts can be thought of as vector fields. The electric field due to some point charge Q as some distance r from Q is $\vec{E}(x, y, z) = \frac{\epsilon_0 Q}{r^2}\hat{r}$ where ϵ_0 is a constant and $\hat{r}$ is a radial unit vector pointing away from Q.
[INSERT IMAGE]
A generic 2D vector field is

$$\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle.$$

Vector fields work similarly to VVFs in that they are added component-wise.

5.2 Line Integrals

5.2.1 Line Integrals of Scalar Functions

Let's say we have a a simple (non-self-intersecting) curve in the xy-plane C parameterized by $\vec{r}(t) = \langle x(t), y(t)\rangle$ and a surface $z = f(x, y)$ above C. We can extrude C up to f, forming a "curtain" with an area A that can be found through integration.

$$\begin{aligned}
\mathrm{d}A &= f(x, y) \cdot \mathrm{d}s \\
s &= \int \left|\left|\vec{r'}(t)\right|\right| \mathrm{d}t \\
\mathrm{d}s &= \left|\left|\vec{r'}(t)\right|\right| \mathrm{d}t \\
\implies A &= \int\limits_C f(x, y)\mathrm{d}s = \int (f \circ \vec{r})(t) \cdot \left|\left|\vec{r'}(t)\right|\right| \mathrm{d}t.
\end{aligned}$$

This is the line integral of $\vec{r}$ on f. $(f \circ \vec{r})$ is called the "pullback."

For example, let's find the line integral of $y = x^2$ for $x \leq x \leq \sqrt{2}$ in $f(x, y) = 2x$.

$$\vec{r}(t) = \langle t, t^2 \rangle, 0 \leq t \leq \sqrt{2}$$

$$\int\limits_C f(x, y)\mathrm{d}s = \int_0^{\sqrt{2}} (2x \circ \langle t, t^2 \rangle) \cdot ||\vec{r'}(t)||\mathrm{d}t = \int_0^{\sqrt{2}} 2t\sqrt{1 + 4t^2}\mathrm{d}t.$$

Let $u = 1 + 4t^2$, $\mathrm{d}u = 8t\mathrm{d}t$.

$$= \frac{1}{4}\int_1^9 \sqrt{u}\mathrm{d}u = \frac{13}{3}$$

5.2.2 Line Integrals of Vector Fields

One can think of line integral of vector fields as the total work done by the vector field as it moves along some simple path.

$$\begin{aligned} W &= \int\limits_C \left(\vec{F} \circ \vec{r}\right) \cdot \hat{T}\mathrm{d}s \\ &= \int\limits_C \frac{\vec{r'}(t)}{||\vec{r'}(t)||} \cdot ||\vec{r'}(t)||\mathrm{d}t \\ &= \int\limits_C \left(\vec{F} \circ \vec{r}\right) \cdot \vec{r'}(t)\mathrm{d}t = \int\limits_C \vec{F} \cdot \mathrm{d}\vec{r} \end{aligned}$$

For example, let's find the line integral of $\vec{r}(t) = \langle t, t^2, t \rangle$ for $0 \leq t \leq 1$ in the vector field $\vec{F}(x, y, z) = \langle e^z, \sqrt{1 - x^2}, \sin x \rangle$.

$$\vec{F} \circ \vec{r} = \langle e^t, \sqrt{1 - t^2}, \sin t \rangle$$

$$\vec{r'}(t) = \langle 1, 2t, 1 \rangle$$

$$\begin{aligned} \int_0^1 \langle e^t, \sqrt{1 - t^2}, \sin t \rangle \cdot \langle 1, 2t, 1 \rangle \mathrm{d}t &= \int_0^1 e^t + 2t\sqrt{1 - t^2} + \sin t\mathrm{d}t \\ &= e - \cos 1 + \frac{2}{3} \end{aligned}$$

Direction Matters

Lemma.

$$\int\limits_{-C} \vec{F} \cdot \mathrm{d}\vec{r} = -\int\limits_C \vec{F}\mathrm{d}\vec{r}$$

That is, the direction in which one takes a line integral matters. You can think of this as the wind helping you sail in one direction (positive line integral) but fighting against you in the opposite direction (negative line integral).

Circulations

Definition. If C is a simple, closed curve, then $\int\limits_C \vec{F} \cdot \mathrm{d}\vec{r}$ is the circulation of $\vec{F}$ on C.

We notate that we are taking a circulation as $\oint\limits_C \vec{F} \cdot \mathrm{d}\vec{r}$.
[INSERT IMAGE]
For example, let's find the circulation of $\vec{F}(x, y, z) = \langle yz, xz, xy \rangle$ on the circle of radius 1 centered at $(0, 0, 1)$ in the $z = 1$ plane in the counter-clockwise direction.

$$\vec{r}(t) = \langle \cos t, \sin t, 1 \rangle, 0 \leq t \leq 2\pi$$

$$\vec{F} \circ \vec{r} = \langle \sin t, \cos t, \sin t \cos t \rangle$$

$$\left(\vec{F} \circ \vec{r}\right) \cdot \vec{r}'(t) = \cos(2t)$$

$$\oint\limits_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_0^{2\pi} \cos(2t)\mathrm{d}t = \frac{1}{2} \sin(2t)|_0^{2\pi} = 0$$

5.2.3 Conservative Vector Fields

Definition. A vector field $\vec{F}$ is conservative if $\int\limits_C \vec{F} \cdot \mathrm{d}\vec{r}$ is the same for all C connecting the same endpoints.

It's easy to see from this definition that vector fields of constant direction and magnitude, like $\vec{F} = \langle c, c, c \rangle$ is conservative, as its line integral only depends on the curve.

Theorem. If $\vec{F}$ is conservative, then $\oint\limits_C \vec{F} \cdot \mathrm{d}\vec{r} = 0$.

Proof. We can break the simple, closed curve, C into two simple curves C_1 and C_2 that have the same endpoints and direction such that $C = C_1 - C_2$.
[INSERT IMAGE]
So,

$$\oint\limits_C \vec{F} \cdot \mathrm{d}\vec{r} = \int\limits_{C_1} \vec{f} \cdot \mathrm{d}\vec{r} - \int\limits_{C_2} \vec{F} \cdot \mathrm{d}\vec{r}.$$

Since C_1 and C_2 have the same direction and endpoints, and $\vec{F}$ is conservative, the line integrals have the same value, L.

$$\oint\limits_C \vec{F} \cdot \mathrm{d}\vec{r} = L - L = 0$$

■

5.2.4 FTC for Line Integrals

We saw earlier that $\mathrm{d}z = \nabla f \cdot \langle \mathrm{d}x, \mathrm{d}y\rangle$. This can be written as $\mathrm{d}z = \nabla f \cdot \mathrm{d}\vec{r}$ where $\vec{r}(t)$ parameterizes a simple curve C and $a \leq t \leq b$.

$$\int\limits_C \nabla f \cdot \mathrm{d}\vec{r} = \int_a^b (\nabla f \circ \vec{r}) \cdot \vec{r'}\mathrm{d}t = f \circ \vec{r}|_a^b = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$\int\limits_C \nabla f \cdot \mathrm{d}\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

the fundamental theorem of calculus for line integrals.

Let's test by computing a line integral directly and using the FTC for line integrals. Let the path be the top-half semicircle connecting $(1, 0)$ to $(-1, 0)$ and let $f(x, y) = 12 - 3x - y$.

$\vec{r}(t) = \langle \cos t, \sin t\rangle$, $0 \leq t \leq \pi$

Directly	FTC
$\vec{r'}(t) = \langle -\sin t, \cos t\rangle$	$\vec{r}(0) = \langle 1, 0\rangle$, $\vec{r}(\pi) = \langle -1, 0\rangle$
$\nabla f = \langle -3, -1\rangle$	$f(\vec{r}(0)) = 9$, $f(\vec{r}(\pi)) = 15$
$\int_0^\pi 3\sin t - \cos t\mathrm{d}t = 3 - \cos t - \sin t\|_0^\pi = 6$	$15 - 9 = 6$

Potential Functions

Note that the FTC for line integrals doesn't care about the specific path taken between two points, but only the starting and ending points. Any paths that started and ended at the same points would have the same values, and paths that start and end at the same point would have a value of 0.
So,

$$\oint\limits_C \nabla f \cdot \mathrm{d}\vec{r} = 0.$$

So, by our theorem that says all $\vec{F}$ such that $\oint\limits_C \vec{F} \cdot \mathrm{d}\vec{r} = 0$ are conservative, all vector fields $\vec{F} = \nabla f$ are conservative. f is called the potential function of $\vec{F}$.

You might recognize potential functions from physics. All conservative forces, like the force of gravity, have a potential energy function. For gravitational force, $U(r) = -\frac{GMm}{r}$ and $\vec{F}_g(r) = \frac{GMm}{r^2}\hat{r}$ where $\hat{r}$ is a radial unit vector pointing away from the object to which the force is applied.

5.2.5 Test for a Conservative Vector Field

Since we know that all conservative vector fields have potential function, we can create a test to see if a given vector field is conservative. $\vec{F}_{\text{Conservative}} = \langle f_x, f_y\rangle$, so $f_{xy} = f_{yx}$. That is, a vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is conservative if $P_y = Q_x$.

For 3D vector field, the test is a little more complicated. A vector field $\vec{F}\langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ is conservative if $P_y = Q_x$, $Q_z = R_y$, and $R_x = P_z$.

For example, let see if $\langle yz, xz, xy\rangle$ is conservative.

$$\frac{\partial}{\partial y} yz = z = \frac{\partial}{\partial x} xz$$

$$\frac{\partial}{\partial z} xz = x = \frac{\partial}{\partial y} xy$$

$$\frac{\partial}{\partial y} xy = y = \frac{\partial}{\partial z} yz.$$

So, the vector field is conservative.

5.3 Surface Integrals

We can parameterize any surface as $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ (or using some other coordinate system). This is because we are simple taking one 2D surface, the uv-plane, and transforming it into another surface in such a way that areas ear each other in the uv-plane are near each other on the surface. This fact that areas stay close to each other allows us to make statement about the complicated surface while working with the simple uv-plane.
[INSERT IMAGE]
The change of the surface in the u direction is $\frac{\partial \hat{r}}{\partial u}\mathrm{d}u$
The change of the surface in the v direction is $\frac{\partial \vec{r}}{\partial v}\mathrm{d}v$
So, the area of the surface in relation to u and v is the area of the parallelogram spanned by these two surface: a cross product.

$$\mathrm{d}s = \left|\left| \left\langle \frac{\partial \vec{r}}{\partial u}\mathrm{d}u \right\rangle \times \left\langle \frac{\partial \vec{r}}{\partial v}\mathrm{d}v \right\rangle \right|\right| = \left|\left|\vec{r_u} \times \vec{r_v}\right|\right| \mathrm{d}u\mathrm{d}v$$

Definition. For a surface S parameterized by $\vec{r}(u,v)$ and $(u,v) \subset D$, the surface area is

$$A(S) = \iint\limits_S \mathrm{d}s = \iint\limits_D \left|\left|\vec{r_u} \times \vec{r_v}\right|\right| \mathrm{d}u\mathrm{d}v$$

5.3.1 Surface Integrals of Scalar Functions

Definition. The surface integral of a scalar function $f(x,y,z)$ is

$$\iint\limits_S f(x,y,z)\mathrm{d}s = \iint\limits_D (f \circ \vec{r})\left|\left|\vec{r_u} \times \vec{r_v}\right|\right| \mathrm{d}A$$

Let's apply a surface integral to a real problem. Let the cap of the sphere or radius 8m centered at the origin between $z = 7$ and $z = 8$ have a charge density $\sigma(x, y, z) = z\ \mu\text{C/m}^2$. Find the total charge Q on the cap.
[INSERT IMAGE]
We will parameterize the cap using spherical coordinates.

$$D = \left\{(\rho, \theta, \phi) \mid \rho = 8, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \cos^{-1}\left(\frac{7}{8}\right)\right\}$$

$$\vec{r}(\theta, \phi) = \langle 8\sin\phi\cos\theta, 8\sin\phi\sin\theta, 8\cos\phi\rangle$$

$$\vec{r_\theta} = \langle -8\sin\phi\sin\theta, 8\sin\phi\cos\theta, 0\rangle,\ \vec{r_\phi} = \langle 8\cos\phi\cos\theta, 8\cos\phi\sin\theta, -8\sin\phi\rangle$$

$$\sigma \circ \vec{r} = 8\cos\phi,\ \left|\left|\vec{r_\theta} \times \vec{r_\phi}\right|\right| = 64\sin\phi$$

$$\begin{aligned} Q &= \int_0^{2\pi}\int_0^{\cos^{-1} 7/8} 64\sin\phi \cdot 8\mathrm{d}\phi\mathrm{d}\theta \\ &= 8^3 2\pi \int_0^{\cos^{-1} 7/8} \sin\phi\mathrm{d}\phi \\ &= 8^3 2\phi(-\cos\phi)\big|_0^{cos^{-1}7/8} = 8^3 2\pi\left(-7/8 + 1\right) = 128\pi\ \mu\text{C} \end{aligned}$$

5.3.2 Surface Integrals of Vector Fields

Definition. The surface integral of a vector field $\vec{F}$ through a surface S is

$$\iint\limits_S \vec{F} \cdot \hat{n}\mathrm{d}s$$

where $\hat{n}$ is a unit vector normal to the surface. This integral can also be written as

$$\iint\limits_S \vec{F} \cdot \mathrm{d}\vec{s}$$

Flux

We can rewrite the surface integral because $\hat{n} = \frac{\vec{r_u} \times \vec{r_v}}{\left|\left|\vec{r_u} \times \vec{r_v}\right|\right|}$ and $\mathrm{d}s = \left|\left|\vec{r_u} \times \vec{r_v}\right|\right|$.
[INSERT IMAGE]
So,

$$\iint\limits_S \vec{F} \cdot \mathrm{d}s = \iint\limits_D \left(\vec{F} \circ \vec{r}\right) \cdot \left(\vec{r_u} \times \vec{r_v}\right) \mathrm{d}u\mathrm{d}v$$

where D is the domain of S in uv-space (a uv-plane), and $\vec{r}(u, v)$ parameterizes the surface S. This quantity is called the "directed surface area integral" or "flux" through S.

Flux tells us how much a vector field penetrates a surface. If the field is parallel to the surface (perpendicular to the normal vector) then the flux is 0. As the field and normal vector to the surface become more aligned, the flux increases.

Flux has many practical applications. One of the fundamental equations governing electricity and magnetism talks about electric flux: the amount of an electric field that goes through a surface. We will investigate this equation later.

For example, let an electric field be $\vec{E} = \langle x, y, 0 \rangle$ N/C. Compute the electric flux, Φ_E, through the portion of the paraboloid $z = 25 - x^2 - y^2$ above the xy-plane. Assume a unit distance of 1 meter. We will take advantage of a paraboloid's radial symmetry and use cylindrical coordinates.

$$\vec{r}(u, v) = \langle u \cos v, u \sin v, 25 - u^2 \rangle \text{ where } 0 \leq u \leq 5 \text{ and } 0 \leq v \leq 2\pi$$

$$\vec{r_u} = \langle \cos v, \sin v, -2u \rangle \text{ and } \vec{r_v} = \langle -u \sin v, u \cos v, 0 \rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 2u^2 \cos v, 2u^2 \sin v, 0 \rangle$$

$$\vec{E} \circ \vec{r} = \langle u \cos v, u \sin v, 0 \rangle$$

$$\left(\vec{E} \circ \vec{r} \right) \cdot (\vec{r_u}) \times \vec{r_v}) = 2u^3$$

$$\Phi_E = \int_0^{2\pi} \int_0^5 2u^3 \mathrm{d}u \mathrm{d}v = 2\pi \left(\frac{u^4}{2} \right) |_0^5 = 625 \text{ Nm}^2/\text{C}$$

Chapter 6

Vector Analysis

6.1 Integral Curves

An integral curve, $\vec{r}$, of a vector field, $\vec{F}$, is a VVF such that $\vec{r'}(t) = \left(\vec{F} \circ \vec{r}\right)(t)$.

For example, let's find an integral curve to $\vec{F}(x, y) = \langle x, 2y \rangle$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{r}(t) = \vec{F} \circ \vec{r}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), y(t) \rangle = \langle x(t), 2y(t) \rangle \rightarrow \begin{cases} x'(t) = x(t) \\ y'(t) = 2y(t) \end{cases} \implies \vec{r}(t) = \langle c_1 e^t, c_2 e^{2t} \rangle$$

These integral curves have applications in physics and show up naturally in nature alongside vector fields. In physics, the integral curves for an electric field are called "field lines" and integral curves of the velocity field of a fluid like water or air are called "streamlines." You may recognize these curves as tracing out a slope field.
[INSERT IMAGE]

6.2 Divergence & Curl

6.2.1 Divergence

In 2D we define the divergence of a vector field $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ as $\mathrm{div}(\vec{F}) = P_x + Q_y$. This tells how the separation between particles in the vector field change over time. Positive divergence at some point means that particles tend to move away from each other, and that point is acting like a "source." Negative divergence at some point means that particles tend to move towards each other, and that point is acting like a "sink."
[INSERT IMAGE]

This operation extends into higher dimensions. We define the "del operator" as

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \dots \right\rangle \text{ so that } \mathrm{div}(\vec{F}) = \nabla \cdot \vec{F}.$$

The del operator is not coordinate system independent. The above version only works for Cartesian coordinates. For spherical coordinates,

$$\nabla \cdot \vec{F} = \frac{1}{\rho^2}\frac{\partial(\rho^2 F_\rho)}{\partial \rho} + \frac{1}{\rho \sin\theta}\frac{\partial}{\partial \theta}(F_\theta \sin\theta) + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\phi}F_\phi \text{ where } \vec{F} = \langle F_\rho, F_\theta, F_\phi\rangle.$$

Thankfully, vector fields in spherical coordinates are rare, and it's usually easier to convert to Cartesian coordinates before doing any calculations.

Definition. If $\nabla \cdot \vec{F} = 0$, then $\vec{F}$ is incompressible.

This aligns with the idea of incompressible fluids in physics and can simplify or remove the need for some calculations.

The Laplacian

The Laplacian is the higher dimension version of concavity. Let f be a scalar function, We say that $\nabla^2 f = \nabla \cdot (\nabla f) = \mathrm{div}(\mathrm{grad}(f))$. Written out more fully,

$$\nabla^2 f = f_{xx} + f_{yy} + \dots$$

6.2.2 Curl

In 2D, we define the curl of a vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ as

$$\nabla \times \vec{F} = Q_x - P_y.$$

Note that the result is a scalar for this 2D case.
If $Q_x > 0$, then the field line accelerate upwards and to the left together: a counter-clockwise rotation.
If $P_y > 0$, then the field lines accelerate upwards and to the left together: a clockwise rotation.
So, $\nabla \times \vec{F}$ tells us the net counter-clockwise rotation at a point.

We use the same del operator we introduced in divergence for curl, so $\nabla \times \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \dots \right\rangle \times \vec{F}$. This means that for 3D and higher dimensions, the curl is a vector that is perpendicular to the plane of net rotation.

For example, let's compute the curl of $\vec{F}(x,y) = \langle -y, x\rangle$.

$$\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 1 - (-1) = 2$$

Curl & Conservative Vector Fields

Theorem. $\vec{F}$ is conservative $\Leftrightarrow \nabla \times \vec{F} = \vec{0}$

Partial Proof. We will prove the 2D case and provide intuition for higher dimensions.
Let $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ be a conservative vector field.
Since $\vec{F}$ is conservative, $P_y = Q_x$.
So, $\nabla \times \vec{F} = Q_x - P_y = 0$. ■

Thinking back to what it means for a vector field to be conservative, a vector field must have path independence between all points to have a potential function and be conservative. If in some plane there is a net rotation, there cannot be path independence because one can choose one path that goes "with" the field and another that goes "against" the field.
[INSERT IMAGE]

Divergence of Curl

Theorem. Let $\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ be a twice differentiable vector field.
$\nabla \cdot (\nabla \times \vec{F}) = 0$

Proof. $\nabla \times \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$
$\nabla \cdot (\nabla \times \vec{F}) = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$
$P_{yz} = P_{zy}$, $Q_{xz} = Q_{zx}$, and $R_{xy} = R_{yx}$ by Fubini's Theorem.
So, $\nabla \cdot (\nabla \times \vec{F}) = 0$ ■

6.3 Green's Theorems

Similar to how we had the fundamental theorem of calculus (FTC) for single-variable calculus, there are several higher-dimensional versions of the FTC.
Recall that the fundamental theorem of calculus is $\int_a^b f'(x)\mathrm{d}x = f(b) - f(a)$. One way to think of this statement is "summing up the derivative on a closed interval is the same as the 'sum' of f on the boundary." We will see that many of these higher versions of the FTC deal with intervals and boundaries.

6.3.1 Green's Theorem for Circulation

Theorem (Green's Theorem for Circulation). Let C be a closed, counter-clockwise oriented curve in $\mathbb{R}^2$. For any differentiable vector field,

$$\vec{F}(x, y) = \langle Q(x, y), R(x, y)\rangle, \oint_C R\mathrm{d}x + Q\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial R}{\partial y}\right) \mathrm{d}x\mathrm{d}y$$

where D is the interior of C.

Or, in more modern notation,

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_D \nabla \times \vec{F} \mathrm{d}A.$$

[INSERT IMAGE]
This is saying that summing up the interior of the derivative is the to summing up the boundary of the function, very similar to the FTC.
One can think of this in a physical sense as saying that the work done by the vector field in moving a particle counter-clockwise on C is equal to the rotation (curl) inside of C (D).
This theorem also relates the idea of path independence and curl of a conservative vector field that we proved the 2D case for. The left side shows path independence and will be 0 for conservative vector fields, and the right side shows curl, which will also be 0 for conservative vector fields.

Partial Proof. We will prove the 2D case, but the underlying argument is easily generalized.

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \oint_c \langle P(x,y), 0\rangle \cdot \mathrm{d}\vec{r} + \oint_C \langle 0, Q(x,y)\rangle \cdot \mathrm{d}\vec{r}.$$

If D is convex, we an break C into 2 curves on the same interval, C_1 and C_2 such that $C_1 : \vec{r}\langle x, h_1(x)\rangle$ and $C_2 : \vec{r}\langle x, h_2(x)\rangle$, where $x \in [a,b]$.
[INSERT IMAGE]

$$\begin{aligned}
&= \int_a^b P(x, h_1(x))\mathrm{d}x - \int_a^b P(x, h_2(x))\mathrm{d}x \\
&= -\int_a^b P(x, h_2(x)) - P(x, h_1(x))\mathrm{d}x \\
&= -\int_a^b P(x,y)|_{y=h_1(x)}^{y=h_2(x)}\mathrm{d}x \\
&= -\int_a^b \int_{h_1(x)}^{h_2(x)} \frac{\partial P}{\partial y}\mathrm{d}y\mathrm{d}x \\
&= -\iint_D \frac{\partial P}{\partial y}\mathrm{d}A + \iint_D \frac{\partial Q}{\partial x}\mathrm{d}A \\
&= \iint_D \nabla \times \vec{F}\mathrm{d}A
\end{aligned}$$

■

For example, let's use Green's Theorem for Circulation to compute $\oint_C \vec{F} \cdot \mathrm{d}\vec{r}$ where $\vec{F} = \langle x^2, xy + x^2\rangle$ and C is the unit circle.

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_D \nabla \times \vec{F}\mathrm{d}A$$

$$\nabla \times \vec{F} = y + 2x.$$

Since C is the unit circle, we'll evaluate the integral using polar coordinates. So

$$\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$$

$$\begin{aligned}\iint_D \nabla \times \vec{F}\mathrm{d}A &= \int_0^{2\pi}\int_0^1 (r\sin\theta + 2r\cos\theta)r\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi}\left(\frac{\sin\theta}{3} + \frac{2\cos\theta}{3}\right)\mathrm{d}\theta = 0.\end{aligned}$$

Note that although this particular circulation is 0, we know that the vector field is not conservative because the curl is not 0.

Area of a Closed Region

The area inside of D is

$$\begin{aligned}A &= \iint_D \mathrm{d}x\mathrm{d}y \\ &= \iint_D \left(\nabla \times \vec{F}\right)\mathrm{d}x\mathrm{d}y\end{aligned}$$

if $\nabla \times \vec{F} = 1$. One such vector field is $\vec{F} = \langle -y/2, x/2\rangle$.

$$= \oint_C (-y/2)\mathrm{d}x + (x/2)\mathrm{d}y$$

by Green's Theorem for Circulation.

$$= \frac{1}{2}\oint_C xy' - yx'.$$

So, if we have some counter-clockwise oriented parametric function $(x(t), y(t))$ where $t_0 \leq t \leq t_1$ that parameterizes C, then

$$A = \frac{1}{2}\int_{t_0}^{t_1}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t$$

We can also choose different vector field where $\nabla \times \vec{F} = 1$ so that

$$A = \oint_C xy' = \oint_C yx'$$

For example, let's compute the area of a circle with radius R. We'll parameterize the circle as $(R\cos t, R\sin t), t \in [0, 2\pi]$.

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} ((R\cos t)(R\cos t) - (R\sin t)(-R\sin t))\mathrm{d}t \\ &= \frac{1}{2}\int_0^{2\pi} R^2 \mathrm{d}t \\ &= \frac{1}{2}R^2 2\pi \\ &= \pi R^2 \end{aligned}$$

6.3.2 Green's Theorem for Flux

Theorem (Green's Theorem from Flux). Let C be a closed, counter-clockwise oriented curve in $\mathbb{R}^2$ and let D be the region contained within C. For any differentiable vector field $\vec{F}(x, y)$,

$$\iint\limits_D \nabla \cdot \vec{f}\mathrm{d}A = \oint\limits_C \vec{F} \cdot \hat{n}\mathrm{d}s$$

[INSERT IMAGE]
This is saying that the sum of the divergence within D is equal to the flux through C.
Our intuition for this is that divergence is the tendency for integral curves of $\vec{F}$ to spread out, and flux would then be these integral curves crossing the boundary curve C.

This ability to convert between a line integral and surface integral often makes flux problems easier to solve. For example, one would have to calculate four integrals to find the flux through a rectangular region, but only a single double integral over the simple interior region.

6.4 Divergence Theorem

Theorem. Let V be a compact solid, and let S be its boundary surface. For any differentiable vector field $\vec{F}(x, y, z)$,

$$\oint\limits_S \oint \vec{F} \cdot \mathrm{d}\vec{s} = \iiint\limits_V \nabla \cdot \vec{F}\mathrm{d}V$$

where $\mathrm{d}\vec{s} = \hat{n}\mathrm{d}s = (\vec{r_u} \times \vec{r_v})\mathrm{d}u\mathrm{d}v$ and $\mathrm{d}V$ is the volume differential.

[INSERT IMAGE]
This says that the flux through a closed surface is equal to the sum of the divergence inside that surface.

Intuitively, divergence describes how much a vector field is going in or out at a point, so summing it up inside some solid would tell us the amount the vector field is going in or out on the solid's boundary, which is flux.

For example, let's find the flux through the unit sphere centered at the origin from the vector field $\vec{F}(x, y, z) = \langle x, y, z^2 \rangle$.

$$S = \{(\rho, \theta, \phi) \mid \rho = 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$$
$$V = \{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 1, 0 \leq \ theta \leq 2\pi, 0 \leq \phi \leq \pi\}$$

$$\begin{aligned}
\text{Flux} &= \oint_S \oint \vec{F} \cdot \mathrm{d}\vec{s} \\
&= \int_0^1 \int_0^\pi \int_0^{2\pi} \nabla \cdot \langle x, y, z^2 \rangle \rho^2 \sin\phi \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}\rho \\
&= \int_0^1 \int_0^\pi \int_0^{2\pi} (2 + 2z)\rho^2 \sin\phi \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}\rho \\
&= \int_0^1 \int_0^\pi \int_0^{2\pi} (2 + 2\rho \sin\phi)\rho^2 \sin\phi \mathrm{ddd} \\
&= 2\pi \int_0^1 \int_0^\pi 2\rho^2 \sin\phi + 2\rho^3 \sin\phi \cos\phi \mathrm{d}\phi \mathrm{d}\rho \\
&= 4\pi \int_0^\pi \frac{1}{3} \sin\phi + \frac{1}{4} \sin\phi \cos\phi \mathrm{d}\phi \\
&= \frac{8\pi}{3}
\end{aligned}$$

6.4.1 Gauss's Laws

Electric Fields

Gauss's Law of Electricity is an important application of Divergence Theorem in physics. It says that $\Phi_E = \frac{Q_{in}}{\epsilon_0}$. That is, the electric flux through a closed surface (real or hypothetical) is proportional to the charge contained within that surface. Rewritten more formally,

$$\oint_S \oint \vec{E} \cdot \mathrm{d}\vec{s} = \iiint_V \frac{Q_{in}}{\epsilon_0} \mathrm{d}V.$$

So,

$$\iiint_V \nabla \cdot \vec{E} \mathrm{d}V = \iiint_V \frac{\sigma}{\epsilon_0} \mathrm{d}V$$

where σ is the charge density. If we let $V \to 0$,

$$\nabla \cdot \vec{E} = \frac{\sigma}{\epsilon_0}.$$

This is the differential form of Gauss's Law and the 1st of Maxwell's Equations.

Magnetic Fields

Gauss's Law for magnetism says that that magnetic flux through a closed surface is 0. Rewritten more formally,

$$\oint_S \oint \vec{B} \cdot \mathrm{d}\vec{s} = 0.$$

So,

$$\iiint_V \nabla \cdot \vec{B}\mathrm{d}V = \iiint_V 0\mathrm{d}V.$$

If we let $V \to 0$,

$$\nabla \cdot \vec{B} = 0.$$

This means that there are no lone sources or sinks in magnetic fields. You may also hear this law summarized as "There are no magnetic monopoles." This is the 2nd of Maxwell's Equations.

6.5 Stokes's Theorem

Let C be a closed, counter-clockwise oriented curve in $\mathbb{R}^2$, and let D be the region contained within C. Let S be an open surface with opening boundary C, and let V be the region contained inside $\tilde{S} = D \cup S$.
[INSERT IMAGE]
Finding the flux of $\nabla \times \vec{F}$,

$$\oint_{\tilde{S}} \oint \nabla \times \vec{F} = \iiint_V \nabla \cdot (\nabla \times \vec{F})\mathrm{d}V$$

by Divergence Theorem. Since $\nabla \cdot (\nabla \times \vec{F}) = 0$,

$$\iint_D \nabla \times \vec{F}\mathrm{d}A = \iint_S \nabla \times \vec{F} \cdot \mathrm{d}\vec{s}$$

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S \nabla \times \vec{F} \cdot \mathrm{d}\vec{s}$$

by Green's Theorem for Circulation. This is Stokes's Theorem.

For example, let's use Stokes's Theorem to evaluate $\iint_S \nabla \times \vec{F} \cdot \mathrm{d}\vec{s}$ where S is the hemisphere $x^2 + y^2 + z^2 = 4, x \geq 0$ and $\vec{F}(x, y, z) = \langle yz, x \sin z, xyz^2 \rangle$.

$$S = \left\{(x, y, z) \mid x^2 + y^2 + z^2 = 2^2, x \geq 0\right\}$$

$$C = \left\{(y, z) \mid y^2 + z^2 = 2^2\right\} = \{(r, \theta) \mid r = 2, 0 \leq \theta \leq 2\pi\}$$

where θ is in the yz-plane.

$$C = \vec{r}(t) = \langle 0, 2\cos t, 2\sin t \rangle, 0 \leq t \leq 2\pi$$

$$\vec{F} \circ \vec{r} = \langle 4\cos t \sin t, 0, 0 \rangle$$
$$\vec{r'}(t) = \langle 0, -2\sin t, 2\cos t \rangle$$
$$\left(\vec{F} \circ \vec{r}\right) \cdot \vec{r'} = 0.$$

So,

$$\iint_S \nabla \times \vec{F} \mathrm{d}A = \oint_0^{2\pi} 0 \mathrm{d}t = 0$$

6.5.1 Faraday's Law of Induction & Ampere's Law

Faraday's Law of Induction quantifies the idea that changing magnetic flux in a coil induces a current in the coil. More precisely, it induces a voltage, the potential function of electric field $\left(\vec{E} = \nabla V\right)$, and this field will oppose the magnetic field that induced it. More formally,

$$-\frac{\partial}{\partial t} \iint_S \vec{B} \cdot \mathrm{d}\vec{s} = \oint_C \vec{E} \cdot \mathrm{d}\vec{r}$$

$$-\iint_S \frac{\partial}{\partial t} \vec{B} \cdot \mathrm{d}\vec{s} = \iint_S \nabla \times \vec{E} \cdot \mathrm{d}\vec{s}$$

by Stokes's Theorem.
As S collapses to a point,

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B}.$$

This is Faraday's Law. It is the 3rd of Maxwell's Laws.

The final of Maxwell's Equations is Ampere's Law (with Maxwell's correction). It says that

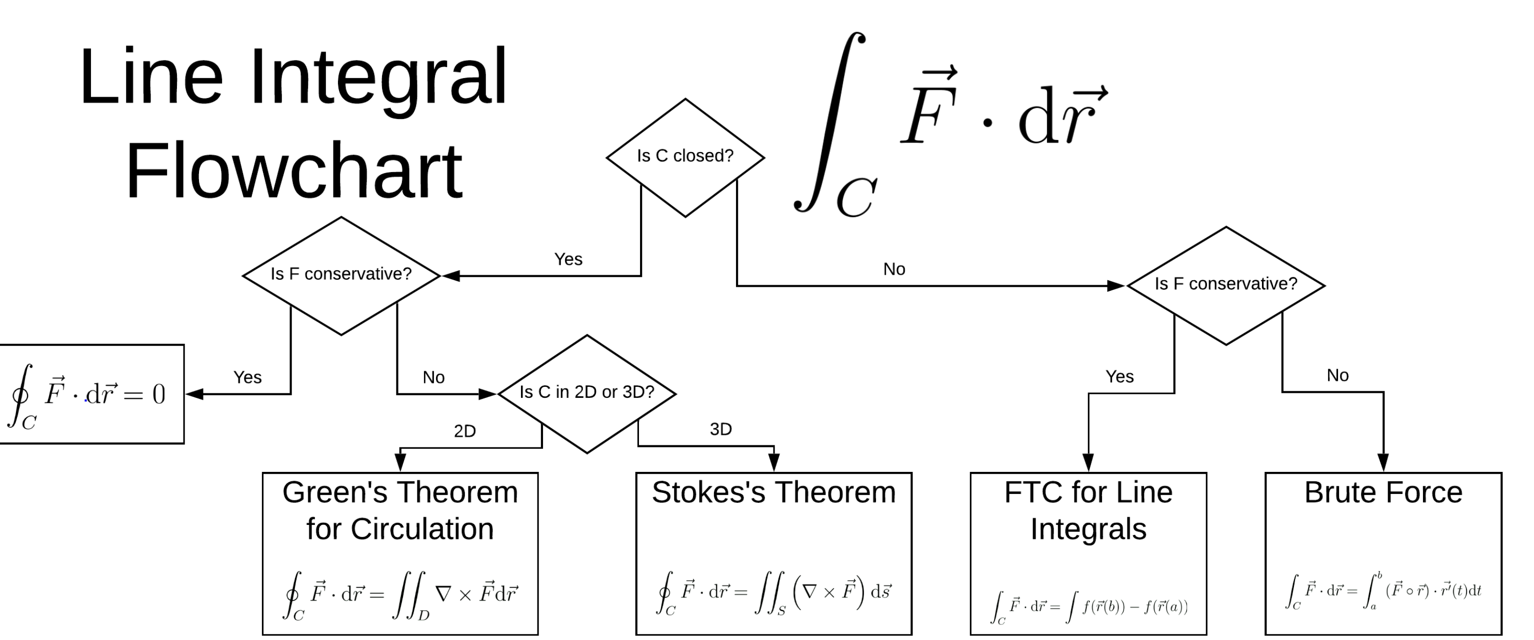
$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{e}}{\partial t} + \mu_0 J$$

where J is the current density and μ_0 is the permeability of free space. Using Maxwell's equations and some basic properties of waves, we can derive the speed of light as $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$.

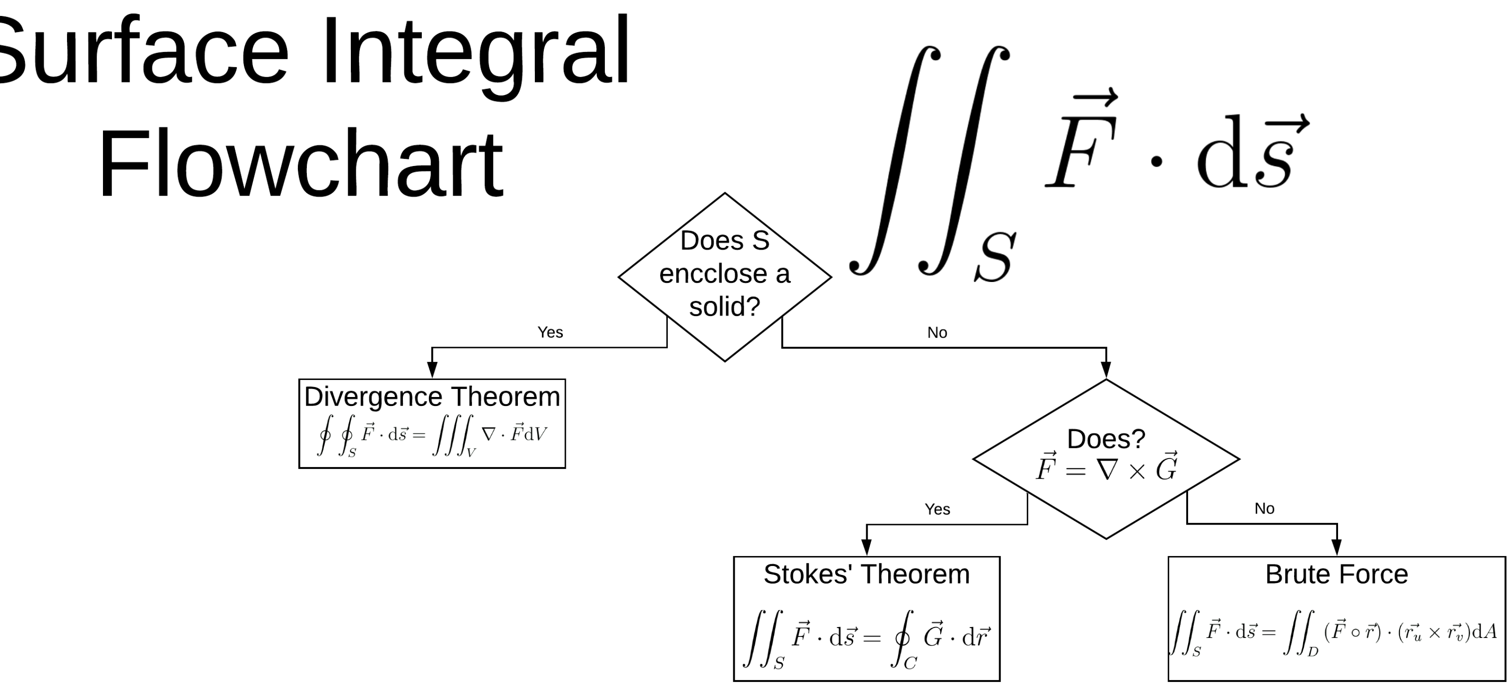
Chapter 7

Additional Materials

7.1 Line Integral Flowchart



7.2 Surface Integral Flowchart



7.3 Worked Test Questions

These questions are modeled after actual exam questions a student might face in an undergraduate multivariable calculus course. The progression of the tests roughly follows the progression of topics covered.

7.3.1 Test 1

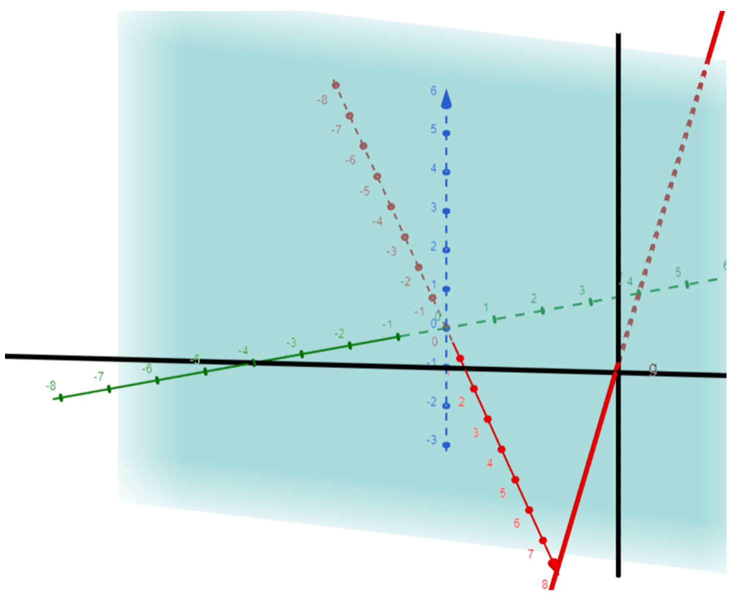
1. Consider two intersecting lines $\vec{r_1}(t) = \langle 2, 3, 4t \rangle$ and $\vec{r_2}(t) = \langle 2+t, 3+2t, 0 \rangle$. Give the direction vector of each line. Find the equation of the plane which contains both lines. Draw a diagram of the lines, the plane, and the relevant vectors.
   The direction vector of a line is the derivative of the position vector.

   - Direction 1: $\langle 0, 0, 4 \rangle$
   - Direction 2: $\langle 1, 2, 0 \rangle$

   A the normal vector of the plane is the cross product of the direction vectors.

   - $\vec{n} = \langle 0, 0, 4 \rangle \times \langle 1, 2, 0 \rangle = \langle -8, 4, 0 \rangle$

   The lines intersect when $t = 0$ at $(2, 3, 0)$. So, the plane equation is: $\langle -8, 4, 0 \rangle \cdot \langle x-2, y-3, z \rangle = 0$



2. Given the VVF $\vec{r}(t)\langle 10t, 7\cos t, 7\sin t \rangle$...

   a. Compute the unit tangent vector $\hat{T}(t)$ and the unit normal vector $\hat{N}(t)$.

$$\hat{T} = \frac{\vec{r'}(t)}{||\vec{r'}(t)||}$$

$$\vec{r'}(t) = \langle 10, -7\sin t, 7\cos t\rangle$$
$$||\vec{r'}(t)|| = \sqrt{10^2 + (-7\sin t)^2 + (7\cos t)^2} = \sqrt{149}$$
$$\hat{T}(t) = \frac{1}{\sqrt{149}}\langle 10, -7\sin t, 7\cos t\rangle$$
$$\hat{N}(t) = \frac{\mathrm{d}\hat{T}/\mathrm{d}t}{||\mathrm{d}\hat{T}/\mathrm{d}t||}$$
$$\frac{\mathrm{d}\hat{T}}{\mathrm{d}t} = \frac{1}{\sqrt{149}}\langle 0, -7\cos t, -7\sin t\rangle$$
$$||\frac{\mathrm{d}\hat{T}}{\mathrm{d}t}|| = \frac{1}{\sqrt{149}}\sqrt{(-7\cos t)^2 + (-7\sin t)^2} = \frac{7}{\sqrt{149}}$$
$$\hat{N}(t) = \langle 0, -\cos t, -\sin t\rangle$$

b. Show that $\hat{T} \perp \hat{N}$ for all t.
If $\hat{T} \perp \hat{N}$, then $\hat{T} \cdot \hat{N} = 0$ for all t.

$$\begin{aligned}\hat{T} \cdot \hat{N} &= \frac{1}{\sqrt{149}}\langle 10, -7\sin t, 7\cos t\rangle \cdot \langle 0, -\cos t, -\sin t\rangle \\ &= \frac{1}{\sqrt{149}}(0 + 7\sin t\cos t - 7\sin t\cos t) = 0 \\ &\implies \hat{T} \perp \hat{N}\end{aligned}$$

3. A cannon fires cannonballs with a speed of 20 m/s. Take acceleration due to gravity to be $g = 10\text{m/s}^2$.

a. Starting with a constant acceleration function $\vec{a} = \langle 0, -g\rangle$, find the velocity and position functions ($\vec{r'}(t)$ and $\vec{r}(t)$ respectively) of the cannonball if the cannon is fired from an angle θ with respect to the horizontal. Assume the cannonball is initially positioned at the origin.
We know that velocity is the integral of acceleration.

$$\vec{v}(t) = \vec{r'}(t) = \langle c_1, c_2 - gt\rangle.$$

We are given that the initial speed is 20 m/s at an angle θ.

$$\vec{v_0} = 20\langle\cos\theta, \sin\theta\rangle$$

$$\vec{v}(t) = \langle 20\cos\theta, 20\sin\theta - gt\rangle.$$

We know that position is the integral of velocity.

$$\vec{r}(t) = \langle 20t\cos\theta + c_1, 20t\sin\theta - \frac{1}{2}gt^2 + c_2\rangle.$$

We are given that the cannonball starts at the origin.

$$\vec{r}(t) = \langle 20t\cos\theta, 20t\sin\theta - \frac{1}{2}gt^2 \rangle.$$

Taking $g = 10\text{m/s}^2$,

$$\vec{r}(t) = \langle 20t\cos\theta, 20t\sin\theta - 5t^2 \rangle$$

b. What angle θ should the cannon be fired to hit a target on the ground at a distance 40 m away?
We want to find a point on the trajectory where $y = 0$ and $x = 40$. $y = 0$ when $t = 0, 4\sin\theta$. We can reasonably eliminate $t = 0$ because this is when the cannon first fires and $x = 0$.
Plugging in $t = 4\sin\theta$ to the x-component of position when $x = 40$,

$$\begin{aligned} 20\cos\theta \cdot 4\sin\theta &= 40 \\ 2\sin\theta\cos\theta &= 1 \\ \sin(2\theta) &= 1, 2\theta = \pi/2 \\ \implies \theta &= \pi/4 \end{aligned}$$

4. Consider the following particle trajectory: $\vec{r}(t) = \langle R\cos e^t, R\sin e^t, \frac{h}{2\pi}e^t \rangle$ for $t \geq 0$. The shape of the trajectory is a helix with radius R and vertical spacing h. Find the arc length function $s(t)$ of the trajectory starting with $s(0) = 0$. Give the arc length reparameterization of the helix.

$$\begin{aligned} s(t) &= \int_0^t ||\vec{r'}(\tau)||\mathrm{d}\tau \\ \vec{r'}(t) &= \langle -Re^t\sin e^t, Re^t\cos e^t, \frac{h}{2\pi}e^t \rangle \end{aligned}$$

$$\begin{aligned} ||\vec{r'}(t)|| &= \sqrt{(-Re^t\sin e^t)^2 + (Re^t\cos e^t)^2 + (\frac{h}{2\pi}e^t)^2} \\ &= e^t\sqrt{R^2 + \frac{h^2}{4\pi^2}} \\ s(t) &= \int_0^t e^\tau\sqrt{R^2 + \frac{h^2}{4\pi^2}}\mathrm{d}\tau = \sqrt{R^2 + \frac{h^2}{4\pi^2}}(e^t - 1). \end{aligned}$$

Solving for t,

$$t = \ln\left(\frac{s}{\sqrt{R^2 + \frac{h^2}{4\pi^2}}} + 1\right)$$

$$\vec{r}(s) = \left\langle R\cos\left(\frac{s}{\sqrt{R^2 + \frac{h^2}{4\pi^2}}} + 1\right), R\sin\left(\frac{s}{\sqrt{R^2 + \frac{h^2}{4\pi^2}}} + 1\right), \frac{h}{2\pi}\left(\frac{s}{\sqrt{R^2 + \frac{h^2}{4\pi^2}}} + 1\right)\right\rangle$$

5. Let $\vec{r}(t)$ be the position function of a particle trapped on the surface of a sphere centered at the origin. Show that $\vec{r}(t) \perp \frac{\mathrm{d}}{\mathrm{d}t}\vec{r}(t)$ for all t.
Since $\vec{r}(t)$ is on a sphere, $\left|\left|\vec{r}(t)\right|\right| = R$ and $\vec{r}(t) \cdot \vec{r}(t) = R^2$.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{r}(t) \cdot \vec{r}(t)) = 2\vec{r}(t) \cdot \vec{r'}(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{r}(t) \cdot \vec{r}(t)) = \frac{\mathrm{d}}{\mathrm{d}t}R^2 = 0.$$

So,

$$2\vec{r}(t) \cdot \vec{r'}(t) = 0$$

and

$$\vec{r}(t) \cdot \vec{r'}(t) = 0$$
$$\implies \vec{r}(t) \perp \vec{r'}(t)$$

7.3.2 Test 2

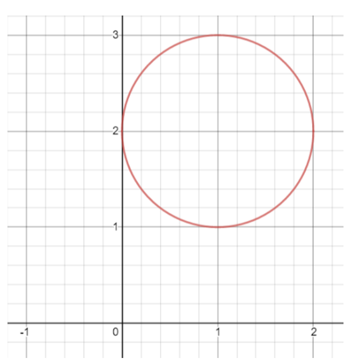
1. Consider the function $f(x, y) = x^2 - 2x + y^2 - 4y + 7$.

a. Find equations for an plot (if possible) the C-level curves of f for $C = 3$ and $C = 1$.
We will try to find the level curve for any C and then plug in 1 and 3.

$x^2 - 2x + y^2 - 4y + 7 = Cx^2 - 2x + 1 + y^2 - 4y + 4 \quad = C - 2(x-1)^2 + (y-2)^2 = C - 2.$

A circle of radius $\sqrt{C-2}$ centered at $(1, 2)$. For $C = 1$, the level curve does not exist because the circle would have a radius of $\sqrt{2-1} = \sqrt{-1}$. For $C = 3$, the level curve is

$$(x-1)^2 + (y-2)^2 = 1$$



b. Compute ∇f.

$$\nabla f = \langle f_x, f_y \rangle = \langle 2x - 2, 2y - 4 \rangle$$

c. Find the equation of the plane tangent to the surface $z = f(x, y)$ at the point $(x_0, y_0, z_0) = (2, 4, 7)$.

$$\vec{n} = \langle f_x, f_y, -1 \rangle = \langle 2x - 2, 2y - 4, -1 \rangle.$$

At $(2, 4, 7)$, $\vec{n} = \langle 2, 7, -1 \rangle$. So, the plane equation is

$$\langle 2, 7, -1 \rangle \cdot \langle x - 2, y - 4, z - 7 \rangle = 0$$

d. Perform one iteration of gradient descent on $f(x, y)$ with a learning rate $delta = 1/4$ starting from the point $(x_0, y_0) = (2, 4)$.

$$\begin{aligned}
(x_n, y_n) &= (x_{n-1}, y_{n-1}) - \delta \nabla f \\
(x_0, y_0) &= (2, 4), \delta = 1/4, \text{ and } \nabla f = \langle 2x - 2, 2y - 4 \rangle \\
(x_1, y_1) &= (2, 4) - \frac{1}{4} \langle 2(2) - 2, 2(4) - 4 \rangle \\
&= (3/2, 3)
\end{aligned}$$

2. Recall that for a differentiable function $f(x, y)$ and the unit vector $\hat{u} = \langle a, b \rangle$, we proved that $D_{\hat{u}} f = \nabla f \cdot \hat{u}$.

a. Prove the statement "The gradient is the direction of steepest ascent" by showing that the directional derivative $D_{\hat{u}} f$ is maximized when $\hat{u} \parallel \nabla f$.

$$D_{\hat{u}} f = \nabla f \cdot \hat{u} = \big|\big|\nabla f\big|\big|\big|\big|\hat{u}\big|\big| \cos\theta = \big|\big|\nabla f\big|\big| \cos\theta.$$

This value is maximized when θ is a multiple of 2π, meaning that the angle between ∇f and $\hat{u}$ is 0. This means that the maximum value of the directional derivative, the direction of steepest ascent, is in the same direction as ∇f.

b. State the limit definition of the directional derivative $D_{\hat{u}} f$. Starting from that definition, prove that $D_{\hat{u}} f = \nabla f \cdot \hat{u}$.

$$\begin{aligned}
D_{\hat{u}} f &= \lim_{h \to 0} \frac{f(x + ah, y + bh)}{h} \text{ and} \hat{u} = \langle a, b \rangle \\
&= \lim_{h \to 0} \frac{f(x + ah, y + bh) - f(x + ah, y)}{h} + \frac{f(x + ah, y) - f(x, y)}{h} \\
&= b \lim_{h \to 0} \frac{f(x, y + bh) - f(x, y)}{bh} + a \lim_{h \to 0} \frac{f(x + ah, y) - f(x, y)}{ah} \\
&= b f_y + a f_x = \langle f_x, f_y \rangle \cdot \langle a, b \rangle = \nabla f \cdot \hat{u}
\end{aligned}$$

3. Use the method of Lagrange Multipliers to find the maximum of the product of two numbers x and y given that (x, y) is a coordinate pair in the 1st quadrant located on the unit circle centered at the origin. Begin by stating the objective function $f(x, y)$ and the constraint equation $g(x, y) = k$.

- Objective Function: $f(x, y) = xy$
- Constraint Equation: $g(x, y) = x^2 + y^2 = 1$, $x \geq 0$ and $y \geq 0$

$$F(x, y, \lambda) = xy + \lambda(1 - x^2 - y^2)$$

$$\frac{\partial F}{\partial x} = y - 2\lambda x, \frac{\partial F}{\partial y} = x - 2\lambda y \text{ and } \frac{\partial F}{\partial \lambda} = 1 - x^2 - y^2$$

$$\langle y - 2\lambda x, x - 2\lambda y, 1 - x^2 - y^2 \rangle = \vec{0}$$

$$\begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ x^2 + y^2 = 1 \end{cases} \implies \begin{cases} \lambda = 1/2 \\ x = 1/\sqrt{2} \\ y = 1/\sqrt{2} \end{cases}.$$

So, the maximum product is $\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$ at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

4. The function $p(x, y) = \frac{1}{\pi} \exp\left(-(x-a)^2 - (y-b)^2\right)$ is the probability density function of a bivariate normal distribution with mean (a, b) and standard deviation $\frac{1}{\sqrt{2}}$. Show that the global maximum of $p(x, y)$ occurs at (a, b).

$$\begin{aligned} p_x &= \frac{1}{\pi}(-2(x-a)) \exp\left(-((x-a)^2 + (y-b)^2)\right) \\ p_y &= \frac{1}{\pi}(-2(y-b)) \exp\left(-((x-a)^2 + (y-b)^2)\right) \end{aligned}$$

$p_x = 0$ when $x = a$ and $p_y = 0$ when $y = b \implies (a, b)$ is a critical point.

$$\begin{aligned} p_{xx} &= \frac{1}{\pi}(-2(y-b)) \exp\left(-((x-a)^2 + (y-b)^2)\right) - 2\exp\left(-((x-a)^2 + (y-b)^2)\right) \\ &= \frac{1}{\pi}(4(x-a)^2 - 2) \exp\left(-((x-a)^2 + (y-b)^2)\right) \\ p_{yy} &= \frac{1}{\pi}(4(y-b)^2 - 2) \exp\left(-((x-a)^2 + (y-b)^2)\right) \\ p_{xy} = p_{yx} &= \frac{4}{\pi}(x-a)(y-b) \exp\left(-((x-a)^2 + (y-b)^2)\right) \end{aligned}$$

$$p_{xx}(a, b) = \frac{-2}{\pi}, p_{yy}(a, b) = \frac{-2}{\pi} \text{ and } p_{xy}(a, b) = p_{yx}(a, b) = 0$$

$$H(a, b) = \begin{bmatrix} \frac{-2}{\pi} & 0 \\ 0 & \frac{-2}{\pi} \end{bmatrix}$$

$$\det(H(a, b)) = \frac{4}{\pi^2}$$

(a, b) is an extrema because $\det(H(a, b)) > 0$. Since $f_{xx}(a, b) < 0$ and $f_{yy}(a, b) < 0$, (a, b) is a maximum. (a, b) is a global maximum because $p(x, y)$ is strictly decreasing as you move away from (a, b).

5. Let the C-level curve of the function $f(x, y)$ be parameterized by the VVF $\vec{r}(t) = \langle x(t), y(t)\rangle$. Use the chain rule to show that $\nabla f(\vec{r}(t)) \perp \vec{r'}(t)$ for all t.
Since $\vec{r}(t)$ parameterizes a C-level curve of f,

$$f \circ \vec{r}(t) = C.$$

Where C is a constant.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \vec{r}(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}C \\
\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\
\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t} \right\rangle &= 0
\end{aligned}$$

$$\begin{aligned}
\nabla f(\vec{r}(t)) \cdot \vec{r'}(t) &= 0 \\
\implies \nabla f(\vec{r}(t)) &\perp \vec{r'}(t)
\end{aligned}$$

7.3.3 Test 3

1. Evaluate each of the following integrals as they appear or by changing coordinate systems. Sketch and/or describe the region geometrically to help in choosing an appropriate coordinate system.

a.

$$\int_0^3 \int_0^2 \int_0^1 z e^{x+y+z^2} \mathrm{d}x\mathrm{d}y\mathrm{d}x.$$

We can use a simple u-substitution.

$$\begin{aligned}
u &= z^2 + x + y, \mathrm{d}u = 2z\mathrm{d}z \\
I &= \int_0^3 \int_0^2 \int_{x+y}^{x+y+1} \frac{1}{2} e^u \mathrm{d}u\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{2}\int_0^3 \int_0^2 e^{x+y+1} - e^{x+y}\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{2}\int_0^3 \left((e^{3+x} - e^{2+x}) - (e^{1+x} - e^x)\right)\mathrm{d}x \\
&= \frac{1}{2}(e^6 - e^5 - e^4 + e^2 + e - 1)
\end{aligned}$$

b.

$$\int_0^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_0^5 z\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

The region we are integrating is a half-cylinder, so we will use cylindrical coordinates.

$$\begin{aligned}
I &= \int_{-\pi/2}^{\pi/2} \int_0^{\sqrt{2}} \int_0^5 rz\mathrm{d}z\mathrm{d}r\mathrm{d}\theta \\
&= \pi \int_0^{\sqrt{2}} r\mathrm{d}r \cdot \int_0^5 z\mathrm{d}z \\
&= \pi \left(\frac{\sqrt{2}^2}{2} - \frac{0^2}{2} \right) \cdot \left(\frac{5^2}{2} - \frac{0^2}{2} \right) \\
&= \frac{25\pi}{2}
\end{aligned}$$

c.

$$\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} z\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

The region we are integrating is a quarter sphere, so we will use spherical coordinates.

$$\begin{aligned}
I &= \int_0^3 \int_0^{\pi} \int_0^{\pi/2} \rho^2 \sin\phi \cos\phi \mathrm{d}\phi\mathrm{d}\theta\mathrm{d}\rho \\
&= \int_0^3 \rho^3 \mathrm{d}\rho \cdot \int_0^{\pi} \mathrm{d}\theta \cdot \int_0^{\pi/2} \sin\phi\cos\phi\mathrm{d}\phi \\
&= \frac{3^4}{4} \cdot \pi \cdot \frac{1}{2} = \frac{81\pi}{8}
\end{aligned}$$

2. Let $\Omega \subset \mathbb{R}^3$ be a spherical ball of radius R centered at the origin. Set up and evaluate $\iiint\limits_{\Omega} \mathrm{d}V$.
   Since we are finding the volume of a ball, we'll use spherical coordinates.

$$\begin{aligned}
\mathrm{d}V &= \rho^2 \sin\phi \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi \\
I &= \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin\phi\mathrm{d}\phi\mathrm{d}\theta\mathrm{d}\rho \\
&= \int_0^R \rho^2 \mathrm{d}\rho \cdot \int_0^{2\pi} \mathrm{d}\theta \cdot \int_0^{\pi} \sin\phi\mathrm{d}\phi \\
&= \frac{R^3}{3} \cdot 2\pi \cdot 2 = \frac{4\pi}{3} R^3
\end{aligned}$$

3. A plane lamina with density $\sigma(x, y) = \sqrt{x^2 + y^2}$ occupies the region D, the region bounded by the Archimedian spiral $r = \theta$ and the half line $\theta = \alpha, r \geq 0$, where α is an unknown angle in radians. Find α such that the average density $\bar{\sigma}$ of the lamina is $\pi/2$.

$$\bar{\sigma} = \frac{\iint\limits_D \sigma\mathrm{d}A}{\iint\limits_D \mathrm{d}A} = \frac{\pi}{2}$$

$$D = \{(r,\theta) \mid 0 \le \theta \le \alpha, 0 \le r \le \theta\}, \sigma(r,\theta) = r$$

$$\begin{aligned}
\bar{\sigma} &= \frac{\int_0^\alpha \int_0^\theta r^2 \mathrm{d}r\mathrm{d}\theta}{\int_0^\alpha \int_0^\theta r\mathrm{d}r\mathrm{d}\theta} \\
&= \frac{\int_0^\alpha \frac{\theta^3}{3}\mathrm{d}\theta}{\int_0^\alpha \frac{\theta^2}{2}\mathrm{d}\theta} \\
&= \frac{\alpha^4/12}{\alpha^3/6} = \frac{\alpha}{2} \\
\frac{\alpha}{2} &= \frac{\pi}{2} \\
&\implies \alpha = \pi
\end{aligned}$$

4. Consider the 2D Gaussian function $f(x,y) = e^{-(x^2+y^2)}$. Evaluate $\iint\limits_D f(x,y)\mathrm{d}A$, where D is a disk of radius a centered at the origin. Use the results to evaluate $\iint\limits_{\mathbb{R}^2} f(x,y)\mathrm{d}A$.

$$D = \{(r,\theta) \mid 0 \le r \le a, 0 \le \theta \le 2\pi\}$$

$$f(x,y) = \exp(-x^2 - y^2) = \exp(-r^2)$$

$$\begin{aligned}
\iint\limits_D f(x,y)\mathrm{d}A &= \int_0^{2\pi} \int_0^a e^{-r^2} r\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{2\pi} \mathrm{d}\theta \cdot \frac{-1}{2} \int_0^{-a^2} e^u \mathrm{d}u \\
&= 2\pi \cdot \frac{-1}{2} \cdot \left(e^{-a^2} - 1\right) \\
&= \pi\left(1 - e^{-a^2}\right)
\end{aligned}$$

$$\iint\limits_{\mathbb{R}^2} f(x,y)\mathrm{d}A = \lim_{a\to\infty} \pi\left(1 - e^{-a^2}\right) = \pi$$

7.3.4 Test 4

1. For each of the following, determine if $\vec{F}$ is conservative. Then evaluate $\int\limits_C \vec{F} \cdot \mathrm{d}\vec{r}$.

   a. $\vec{F} = \langle xz, x^2z, xy^2z\rangle$ and C given by $\vec{r}(t) = \langle t, e^{-t}, e^t\rangle, 0 \le t \le 1$.

$$\nabla \times \vec{F} = \langle 2xyz - x^2, x - y^2z, 2xz\rangle.$$

Since $\nabla \times \vec{F} \neq \vec{0}$, $\vec{F}$ isn't conservative.

$$\begin{aligned}\vec{F} \circ \vec{r} &= \langle te^t, t^2 e^t, te^{-2t} e^t \rangle \\ &= \vec{r'}(t) = \langle 1, -e^{-t}, e^t \rangle\end{aligned}$$

$$\left(\vec{F} \circ \vec{r}\right) \cdot \vec{r'}(t) = te^t - t^2 + t$$

$$\begin{aligned}\int\limits_C \vec{F} \cdot \mathrm{d}\vec{r} &= \int_0^1 \left(te^t - t^2 + t\right) \mathrm{d}t \\ &= \frac{1}{2} - \frac{1}{3} + |te^t|_0^1 - \int_0^1 e^t \mathrm{d}t \\ &= 1 + \frac{1}{2} - \frac{1}{3} = \frac{7}{6}\end{aligned}$$

b. $\vec{F} = \left\langle \sqrt{\frac{yz}{x}}, \sqrt{\frac{xz}{y}}, \sqrt{\frac{xy}{z}} \right\rangle$ and C given by $\vec{r} = \langle \cos t, \sin t, \sin(4t) \rangle, 0 \leq t \leq 2\pi$.

$$\vec{F} = \nabla(2\sqrt{xyz}) \implies \vec{F} \text{ is conservative.}$$

$$\vec{r}(0) = \vec{r}(2\pi) \implies C \text{ is a circulation..}$$

Since $\vec{F}$ is conservative and C is a circulation,

$$\oint\limits_C \vec{F} \cdot \mathrm{d}\vec{r} = 0.$$

c. $\vec{F} = \langle yz, xz, xy \rangle$ and C given by $\vec{r}(t) = \langle 2t^2, e^{1-t^2}, \tan^{-1}(t^2/2) \rangle, 0 \leq t \leq \sqrt{2}$.

$$\vec{F} = \nabla(xyz) \implies \vec{F} \text{ is conservative.}$$

$$\int\limits_C \vec{F} \cdot \mathrm{d}\vec{r} = \int\limits_C \nabla f \cdot \mathrm{d}\vec{r} = f(\vec{r}(\sqrt{2})) - f(\vec{r}(0)) = \frac{4\pi}{4e} - 0 = \frac{\pi}{e}$$

2. Let the surface S be the portion of the paraboloid $z = 8 - \frac{x^2}{2} - \frac{y^2}{2}$ that lies above the xy-plane. Let $\vec{F}(x, y, z) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0 \right\rangle$.

   a. Parameterize $S = \vec{r}(u, v)$ with appropriate bounds for u and v. The paraboloid is above the xy-plane when $z \geq 0$.

$$\begin{aligned}8 - \frac{x^2}{2} - \frac{y^2}{2} &= \geq 0 \\ x^2 + y^2 &\leq 16.\end{aligned}$$

This a circle with radius 4. So, we'll describe the region in polar form.

$$S = \{(r,\theta) \mid 0 \leq r \leq 4, 0 \leq \theta \leq 2\pi\}$$

$$\vec{r}(u,v) = \langle u\cos v, u\sin v, 8 - \frac{1}{2}u^2\rangle, 0 \leq u \leq 4 \text{ and } 0 \leq v \leq 2\pi$$

b. Compute the surface area of S.

$$A = \iint_D \left|\left|\vec{r_u} \times \vec{r_v}\right|\right| \mathrm{d}A$$

$$\vec{r_u} = \langle \cos v, \sin u, -u\rangle$$

$$vecr_v = \langle -u\sin v, u\cos v, 0\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle u^2\cos v, u^2\sin v, u\rangle$$

$$\left|\left|\vec{r_u} \times \vec{r_v}\right|\right| = u\sqrt{1+u^2}$$

$$\begin{aligned} A &= \int_0^{2\pi} int_0^4 u\sqrt{1+u^2}\mathrm{d}u\mathrm{d}v \\ &= 2\pi \int_0^4 u\sqrt{1+u^2}\mathrm{d}u \\ &= \pi \int_1^{17} \sqrt{w}\mathrm{d}w \\ &= \pi \frac{34\sqrt{17}-2}{3} \end{aligned}$$

c. Compute the flux, Φ, of $\vec{F}$ through S.
We will assume that the surface is outward-oriented, so that Φ is positive.

$$\Phi = \iint_D \left(\vec{F} \circ \vec{r}\right) \cdot \left(\vec{r_u} \times \vec{r_v}\right) \mathrm{d}A$$

$$\vec{F} \circ \vec{r} = \langle \cos v \sin v, 0\rangle$$

$$\left(\vec{F} \circ \vec{r}\right) \cdot \left(\vec{r_u} \times \vec{r_v}\right) = u^2$$

$$\begin{aligned} \Phi &= \int_0^{2\pi} \int_0^4 u^2\mathrm{d}u\mathrm{d}v \\ &= \frac{128\pi}{3} \end{aligned}$$

3. Consider a 3D vector field $\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ and a scalar function of two variables $f(x, y)$. Determine which of the following expressions is defined. If it is defined, evaluate it. If it is not defined, explain why. If you can deduce the value of the expression from a theorem, do so and state the theorem.

   a.
   $$\nabla f \cdot \vec{F}.$$
   This operation is not defined because ∇f is a 2D vector, and the output of $\vec{F}$ is a 3D vector.

   b.
   $$\nabla \times \nabla f.$$
   If we allow the cross product in 2D to return a scalar that is the signed area spanned by the two vectors, then
   $$\nabla \times \nabla f = f_{yx} - f_{xy} = 0$$

   c.
   $$\nabla \times (\nabla \cdot \vec{F}).$$
   This operation is not defined because $\nabla \cdot \vec{F}$ results in a scalar function, and the curl of a scalar function is not defined.

   d.
   $$\nabla \cdot (\nabla \times \vec{F}).$$
   This operation is defined and always has a value of 0 if $\vec{F}$ is twice differentiable. The proof of which is below.
   Let
   $$\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$$
   $$\begin{aligned}\nabla \cdot (\nabla \times \vec{F}) &= \nabla \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle \\ &= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0\end{aligned}$$
   by Fubini's Theorem

4. Consider the vector field $\vec{F}(x, y, z) = \langle -z, 2y, x\rangle$. Find the integral curve of $\vec{F}$ with initial conditions $\vec{r}(0) = \langle 5, 1, 0\rangle$.
   We need to solve a system of differential equations.
   $$\begin{cases} x' = -z \\ y' = 2y \\ z' = x \end{cases}$$

$y(t) = Ce^{2t}$ and applying initial conditions, $y(t) = e^{2t}$

$$\begin{cases} x' = -z \\ z' = x \end{cases}$$

$z(t) = A\cos t + B\sin t$ and $x(t) = B\cos t - A\sin t$
Applying initial conditions, $z(t) = 5\sin t$ and $x(t) = 5\cos t$. So,

$$\vec{r}(t) = \langle 5\cos t, e^{2t}, 5\sin t\rangle$$

5. State and prove the Fundamental Theorem of Calculus for Line Integrals.

Theorem (FTC for Line Integrals). Let $\vec{r}(t)$ parameterize C on $a \le t \le b$.

$$\int_C \nabla f \cdot \mathrm{d}\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Proof.

$$\begin{aligned} \int_C \nabla f \cdot \mathrm{d}\vec{r} &= \int_a^b (\nabla f \circ \vec{r}) \cdot \vec{r}' \mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \vec{r}) \mathrm{d}t \\ &= f(\vec{r}(b)) - f(\vec{r}(a)) \end{aligned}$$

$\blacksquare$

7.4 Online Resources

- Khan Academy - Multivariable Calculus
- Paul's Online Notes - Calc III
- PatrickJMT - Calculus / Third Semester / Multivariable Calculus
- MIT OpenCourseWare - Multivariable Calculus
- Stewart - Single and Multivariable Calculus

7.5 Contributors

Special thanks to everyone who made contributions to this project on Github. They are listed in order of number of commits as `name (GitHub username)`.

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