Block 6
Matrix Algebra

LAND ACKNOWLEDGEMENT

The School of Public Health at the University of Minnesota Twin Cities is built within the traditional homelands of the Dakota people. Minnesota comes from the Dakota name for this region, Mni Sóta Maḳoce, which loosely translates to the land where the waters reflect the skies.

It is important to acknowledge the peoples on whose land we live, learn, and work as we seek to improve and strengthen our relations with our tribal nations. We also acknowledge that words are not enough. We must ensure that our institution provides support, resources, and programs that increase access to all aspects of higher education for our American Indian students, staff, faculty, and community members.

Matrix Algebra - Some Useful Results

$X_{r\times c}$matrix = an array of numbers with r rows and c columns.
For example:

$$X = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 5 \\ 1 & 3 & 4 \\ 1 & 8 & 6 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \end{pmatrix} = (x_{ij})$$

A **vector** is a matrix with one row or one column.

$$y = \begin{pmatrix} 2 \\ 3 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \quad \text{Column vector}$$

A **square** matrix: $r = c$
Symmetric matrix: for some square matrix Z, $Z_{ji} = Z_{ij}$
A **diagonal** matrix: square with all off-diagonal elements = 0
For example:

$$C = \begin{pmatrix} 7 & 3 & 2 & 1 \\ 3 & 4 & 1 & -1 \\ 2 & 1 & 6 & 3 \\ 1 & -1 & 3 & 8 \end{pmatrix} \leftarrow \text{square} \quad D = \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix} \leftarrow \text{diagonal}$$

Identity matrix $\mathbf{I}$ = diagonal matrix with $I_{ij} = 1 \quad$ when $i = j$
For example:

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{4x4 identity matrix}$$

Note: A scalar $\equiv 1 \times 1$ matrix $\equiv$ ordinary number

MATRIX OPERATIONS:

Addition and Subtraction

- can only perform addition/subtraction if dimensions match.

eg/ $\quad C_{r\times c} = A_{r\times c} + B_{r\times c}$

$$C = A + B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{pmatrix}$$

Commutativity: $\quad A + B = B + A$
Associativity: $\quad (A + B) + C = A + (B + C)$

Multiplication by a Scalar

$$kA_{r\times c} = (ka_{ij}) \quad (k : \text{scalar})$$

eg/ $\quad$ (σ^2: scalar, $r \times r$: recall $r = c$)

$$\sigma^2 I_{r\times r} = \begin{pmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{pmatrix}_{r\times r}$$

Matrix Multiplication

- order matters $\rightarrow A_{r \times c} B_{c \times q}$ # cols of A must match # rows of B

$$C = (c_{ij}) = \sum_{k=1}^{c} a_{ik} b_{kj}$$

For example:

$$A = \begin{pmatrix} 1 & 3 & 2 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 1 \\ -2 \\ 4 \end{pmatrix}$$

Then the product AB is:

$$AB = (1 \times 2) + (3 \times 1) + (2 \times -2) + (-1 \times 4) = -3 \quad \text{(scalar)}$$

Note:

$$A_{1\times4} B_{4\times1} \neq B_{4\times1} A_{1\times4}$$

$$BA = \begin{pmatrix} 2 & 6 & 4 & -2 \\ 1 & 3 & 2 & -1 \\ -2 & -6 & -4 & 2 \\ 4 & 12 & 8 & -4 \end{pmatrix}_{4\times4}$$

When all dimensions > 1:

$$A_{3\times3} B_{2\times2}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{pmatrix}$$

Using numbers, an example of multiplication of two matrices is:

$$\begin{pmatrix} 3 & 1 \\ -1 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 15+0 & 3+4 \\ -5+0 & -1+0 \\ 10+0 & 2+8 \end{pmatrix} = \begin{pmatrix} 15 & 4 \\ -5 & -1 \\ 10 & 10 \end{pmatrix}$$

Note: BA not defined!

Associative Law:

$$A_{r\times c}(B_{c\times q}C_{q\times p}) = (AB)C$$

Transpose of a Matrix

If $X_{r\times c}$, $X'_{c\times r}$ $\rightarrow$transpose of X

In other words,

$$X = (x_{ij}) \quad X' = (x_{ji})$$

Transpose of a column vector is a row vector.

Property of transpose:

$$(AB)' = B'A'$$

Other properties:

If $a_{r\times 1}$, then $a'a$ is 1×1 scalar given by

$$a'a = \sum_{i=1}^{r} a_i^2 \quad \text{[sum of squares of elements of } a\text{]}$$

If $a_{r\times 1}$ and $b_{r\times 1}$,

$$a'b = \sum_{i=1}^{r} a_i b_i = \sum_{i=1}^{r} b_i a_i = b'a$$

Useful in manipulating vectors used in regression calculations.

Another useful property:

$$(a-b)'(a-b) = a'a + b'b - 2a'b$$

Inverse of a Matrix

Recall for a scalar $c \neq 0$ and another scalar $d \neq 0$,

$$\text{if} \quad cd = 1 \quad \Longrightarrow \quad d = c^{-1}$$

Square matrices can also have inverses.

$$\Longrightarrow \text{inverse of } C \text{ is } D \text{ such that } CD = I \quad \Longrightarrow D = C^{-1} \quad \text{(unique)}$$

Not all square matrices have inverses.
Those that do are called full rank (non-singular).
We don't do calculations by hand:
- use specific decompositions in linear regression (computer)

Facts:
a) For a diagonal matrix with non-zero diagonal elements $\Rightarrow$ inverse obtained by inverting diagonal elements.
b) If any diagonal elements are zero, no inverse exists.

Orthogonality

Two vectors a and b of the same length are orthogonal if $a'b = 0$
This result can be used to extend the notion of orthogonality to matrices.
An $r \times c$ matrix Q has orthogonal columns if the set of $c \leq r$ columns (each an $r \times 1$ vector) are orthogonal and have length 1.

$$\Rightarrow Q'Q = I_{r \times r}$$

Furthermore, a square matrix A is orthogonal if $A'A = AA' = I$

$$\Rightarrow A^{-1} = A'$$

For example,

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

can be shown to be orthogonal by showing that $A'A = I$, and therefore

$$A^{-1} = A' = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

Linear dependence and rank of a Matrix

Let X be an $n \times p$ matrix with column vectors $x_1, \ldots, x_p$.
We say $x_1, \ldots, x_p$ are linearly dependent if we can find $a_1, \ldots, a_p$ (not all 0) such that

$$\sum_{j=1}^{p} a_j x_j = 0$$

If we cannot find such $a_1, \ldots, a_p$

$\Rightarrow x_1, \ldots, x_p$ are linearly independent and the matrix X is <u>full rank</u>.

eg/

$$X = \begin{pmatrix} 1 & 2 & 5 \\ 1 & 1 & 4 \\ 1 & 3 & 6 \\ 1 & 8 & 11 \end{pmatrix} = (x_1, x_2, x_3)$$

Note:

$$x_3 = 2x_1 + x_2$$

$$\therefore \text{rank}(X) = 2$$

Fact:
$X'X$ is $p \times p$
If X has rank p, so does $X'X$.
Full rank square matrices always have an inverse.

Random Vectors and Matrices

$Y_{n\times 1}$ is a random vector if each of its elements is a random variable.

$$\Longrightarrow E(Y) = \begin{pmatrix} E(y_1) \\ \vdots \\ E(y_n) \end{pmatrix}$$

$$\Longrightarrow \mathrm{var}(Y) = \begin{pmatrix} \mathrm{var}(y_1) & \cdots & \mathrm{cov}(y_1, y_n) \\ \vdots & \ddots & \vdots \\ \mathrm{cov}(y_n, y_1) & \cdots & \mathrm{var}(y_n) \end{pmatrix}$$

Note: $\mathrm{cov}(y_i, y_j) = \mathrm{cov}(y_j, y_i)$
Some properties:

$$E(a_0 + AY) = a_0 + AE(Y)$$

$$(a_0, A : \text{constants}) \quad [a_0 = \text{vector}, A = \text{matrix}]$$

$$\mathrm{var}(a_0 + AY) = A\,\mathrm{var}(Y)\,A'$$